The Shape of the Term Structures

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Abstract

Empirical findings show that the term structures of dividend strip risk premium and volatility are downward sloping, while the term structure of interest rates is upward sloping. We investigate the general equilibrium relationships between the shape of these term structures, the dynamics of economic fundamentals, and the representative agent's preferences. We show that the observed shapes can be obtained simultaneously with simple, realistic dynamics and standard preferences. The only necessary feature is time variation in the expected economic growth rate. In this case, the observed term structures can be replicated simultaneously even when the agent has myopic or CRRA preferences.

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1 Introduction

Empirical findings have recently shown that the term structures of dividend strip risk premium and return volatility are downward sloping (van Binsbergen et al., 2012, 2013; van Binsbergen and Koijen, 2017). In addition, it is well established that the term structure of bond yields is upward sloping (Vasicek, 1977; Cox et al., 1985), while the term structure of bond yield volatility is downward sloping (Piazzesi et al., 2006).

These empirical findings are seemingly at odds with the theoretical predictions of leading asset pricing models such as, for instance, the long-run risk model of Bansal and Yaron (2004), the habit formation model of Campbell and Cochrane (1999), and the time-varying rare disaster models of Gabaix (2012) and Wachter (2013). Consequently, subsequent research has focused on building new theoretical foundations for the observed shapes of term structures (see van Binsbergen and Koijen, 2017).

In this paper, we take a step back and argue that a slight variation of the standard general equilibrium model can explain the term structures of dividend strip risk premium, dividend strip return volatility, bond yield, and bond yield volatility. The only necessary feature is a time-varying expected output growth rate that is correlated with the output growth rate. Therefore, there is little need for a drastic theoretical paradigm shift in order to reconcile empirical findings with asset pricing models. Using restrictions on the parameters of the model, we show that each of the aforementioned observed shapes can be obtained simultaneously within a model featuring 1) an agent with standard, rational preferences and 2) simple, realistic output growth dynamics.

We consider a pure-exchange economy (Lucas, 1978) with a representative agent having Epstein and Zin (1989) preferences. In equilibrium, the agent consumes the aggregate output of the economy and is fully invested in a single risky asset paying the output as a dividend. The output growth volatility is constant, while the expected output growth rate is mean reverting (Brennan and Xia, 2001). Importantly, the output growth and expected output growth rates are allowed to be correlated. Since the dynamics of the state variables are affine, we follow Eraker and Shaliastovich (2008) to solve for equilibrium asset prices and to recover the term structures of interest rates and dividend strip returns.

We show that this simple model is flexible enough to reproduce the shapes of observed term structures. In particular, the model can generate downward-sloping term structures of risk premium and volatility of dividend strips and an upward-sloping yield curve. We emphasize that the shape of the model-implied term structures depends mainly on the preference parameters and the sign of the correlation between shocks to contemporaneous output growth and shocks to expected output growth.

When this correlation is *negative*, a negative consumption/cashflow shock today is expected to be followed by a positive shock tomorrow. Since the initial negative shock will be compensated by a positive shock in the future, long-term consumption/cashflow growth rates are less risky than short-term ones. In other words, the term structure of consumption/cashflow growth risk is downward sloping. When the representative agent's elasticity of intertemporal substitution (EIS) is larger than one, the substitution effect dominates the income effect (Veronesi, 2000), which implies that the risk characteristics of cashflows are the same as those of their corresponding asset returns. Consequently, the risk premium and volatility of long-term dividend dividend strips are smaller than the risk premium and volatility of short-term dividend strips. That is, the term structures of dividend strip return volatility and risk premium are also downward sloping. Because bonds are used to hedge consumption growth risk, short-term bonds are particularly attractive and therefore their yields are smaller than those of long-term bonds: The term structure of interest rates is upward sloping.

When the EIS is smaller than one, the income effect dominates and precautionary savings motives are strong. This means that, if the agent faces high uncertainty about consumption growth at a τ-year horizon, assets that pay off precisely at the τ-year horizon are particularly attractive to her. When the correlation between consumption growth and expected consumption growth is *positive*, shocks of the same sign accumulate over time and therefore imply that long-term consumption growth rates are more risky than short-term ones. Because of the aforementioned precautionary savings motives of the agent, long-term dividend strips are particularly attractive. That is, their risk premium and volatility are smaller than those of short-term dividend strips: The term structure of dividend strip return volatility and risk premium is downward sloping. When the EIS is smaller than one and the correlation between consumption growth and expected consumption growth is positive, the term structure of interest rates tends to be upward sloping because bonds are used to hedge the increasing term structure of consumption growth risk.

We provide evidence that regardless of the parameters governing preferences and the dynamics of cashflows, the volatility of bond yields is a decreasing function of the maturity. We then show that the observed shape of the term structures of dividend strip return volatility, dividend strip risk premium, bond yields, and bond yield volatility can be obtained simultaneously within the model. Interestingly, the observed shapes can be obtained simultaneously even if the agent has time separable preferences (CRRA or log utility). Finally, we consider traditional calibrations considered in the literature, and illustrate that the observed term structures can be replicated all together using a realistic set of parameters.

This paper is related to the asset pricing literature considering a time-varying expected output growth rate. Veronesi (1999, 2000) show that when the expected output growth rate is unobservable and follows a Markov-switching process, the equilibrium stock return volatility is high, time-varying, and clusters. Brennan and Xia (2001) show that learning about an unobservable and time-varying expected output growth rate yields values for the equilibrium risk premium and stock return volatility that are close to their empirical counterparts. Bansal and Yaron (2004) show that the presence of a small, persistent component in the expected output growth rate yields a high risk premium and a high stock return volatility.

Our paper is also related to the literature providing theoretical foundations for the observed shape of the term structure of dividend strip return volatility and risk premium. In a model with an exogenously given stochastic discount factor, Lettau and Wachter (2007) show that a negative correlation between the dividend growth rate and the expected dividend growth rate yields term structures of dividend strip return volatility and risk premium that are hump-shaped and downward-sloping, respectively. Croce et al. (2015) show that limited information about the expected output growth rate together with bounded rationality generates downward-sloping term structures of dividend strip return volatility and risk premium. Other theoretical foundations for downward-sloping term structures of dividend strip returns based on leverage, recoveries following disasters, labor relations, and learning are provided in Belo et al. (2015), Hasler and Marfè (2016), Marfè (2017), and Hasler et al. (2018), respectively. We refer to van Binsbergen and Koijen (2017) for a recent survey.

We contribute to these two strands of literature by considering a model that endogenizes the stochastic discount factor, and by providing intuitive parameter conditions ensuring realistic shapes for both the term structure of interest rates and the term structure of dividend strip returns.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 describes the relationship between preferences, the dynamics of economic fundamentals, and the shape of the term structures. Section 4 illustrates the term structures obtained with traditional models and slight variations of them. Section 5 concludes. Derivations and computational details are provided in the appendix.

2 The Model

This section describes the economy and characterizes equilibrium asset prices.

2.1 The Economy

We consider an infinite horizon pure-exchange economy (Lucas, 1978) with a representative agent having the following Epstein and Zin (1989) preferences

$$U_t \equiv \left[(1-\delta^\tau)C_t^{\frac{1-\gamma}{\theta}} + \delta^\tau \mathbb{E}_t\left(U_{t+\tau}^{1-\gamma}\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}},$$

where τ is a time interval, C_t is the agent's consumption, δ is the subjective discount factor per unit of time, γ is the coefficient of relative risk aversion, ψ is the elasticity of intertemporal substitution (EIS), and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$.

There is a risky asset, in supply of one unit, that pays the aggregate output of the economy. In equilibrium, the aggregate output is equal to the agent's aggregate consumption, C_t. There is a stock, in zero net supply, that pays a dividend, D_t, and a risk-free asset, in zero net supply, that pays the endogenous risk-free rate r_{ft}. Note that, in the next section, we will introduce additional financial securities that will all be in zero net supply.

The aggregate output/consumption, C_t, and the dividend paid by the stock, D_t, satisfy

$$\begin{aligned} d\log C_t &= (\alpha_C + x_t)dt + \sigma_C dB_{1t}, \\ d\log D_t &= (\alpha_D + \phi x_t)dt + \varphi\sigma_C dB_{1t}, \end{aligned} \tag{1}$$

where x_t follows a mean reverting process

$$dx_t = -\kappa_x x_t dt + \sigma_x(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}).$$

The 2-dimensional vector $(B_{1t}, B_{2t})^\top$ is a standard Brownian motion and the parameter ρ represents the correlation between the output growth rate and the expected output growth rate. As in Abel (1999), the dividend paying stock is a levered claim on aggregate consumption, with the parameters ϕ and φ determining the extent of leverage.[1]

2.2 Equilibrium Asset Prices

To solve for the equilibrium, we follow the methodology developed by Eraker and Shaliastovich (2008), which is based on Campbell and Shiller (1988) log-linear return approximation. Campbell et al. (1997) and Bansal et al. (2012) document that the log-linearization yields low approximation errors in a Gaussian setting, while Hasler and Marfè (2016) show that the log-linearization is also accurate when state variables follow jump-diffusions. Therefore, for notational ease, we will treat the approximated results as exact in what follows.

The log wealth-consumption ratio, wc_t, of the representative agent is provided in Proposition 1 below.

Proposition 1. *The equilibrium log wealth-consumption ratio of the representative agent satisfies*

$$wc_t \equiv \log \frac{W_t}{C_t} = A + B_x x_t,$$

where the coefficient B_x governing the effect of the expected growth rate x_t on

[1] We allow ϕ and φ to be distinct, as in Bansal and Yaron (2004). Note that if $\phi = \varphi$ and $\mu_D = \phi\mu_C$ we have $D_t = C_t^\phi$, as in Abel (1999). Assuming in addition that $\phi = \varphi = 1$, yields $D_t = C_t$ as in Lucas (1978).

the wealth-consumption ratio is given by

$$B_x = \frac{1 - 1/\psi}{1 - k_1 (1 - \kappa_x)},$$

and the constants A *and* $k_1 = e^{\mathbb{E}(wc_t)} / \left(1 + e^{\mathbb{E}(wc_t)}\right)$ *solve equations* (17)–(18).

Proof. See Appendix B.1.

The logarithm of the wealth-consumption ratio is linear in the expected output growth rate in the economy x_t. The sensitivity of the equilibrium wealth-consumption ratio to changes in x_t depends only on the level of EIS and the persistence of the expected output process. k_1 is an approximating constant, as in Bansal and Yaron (2004) and Eraker and Shaliastovich (2008).

The state-price density, M_t, the risk-free rate, r_{ft}, and the 2-dimensional vector of market prices of risk, Λ, are provided in Proposition 2 below.

Proposition 2. *The state-price density* M_t *satisfies*

$$\frac{dM_t}{M_t} = -r_{ft}dt - \Lambda^\top dB_t,$$

where the risk-free rate r_{ft} *and the market price of risk* Λ *are as follows*[2]*:*

$$r_{ft} = \Phi_0 + \frac{1}{\psi} x_t, \tag{2}$$

$$\Lambda^\top = \left(\gamma\sigma_C + \rho \frac{\sigma_x (1-\theta) k_1 (1 - 1/\psi)}{1 - k_1 (1 - \kappa_x)}, \sqrt{1-\rho^2} \frac{\sigma_x (1-\theta) k_1 (1 - 1/\psi)}{1 - k_1 (1 - \kappa_x)} \right). \tag{3}$$

The coefficient $\Phi_0 \in \mathbb{R}$ *solves equation* (19).

[2]Note that, due to applying the Campbell and Shiller (1988) log-linearization, the (approximate) market price of risk vector is constant.

Proof. See Appendix B.2.

The risk-free rate given in (2) is independent of the log consumption, but depends on the expected growth rate, x_t. An increase in the expected growth rate has two effects on the risk-free rate. First, the agent increases current consumption because of consumption-smoothing motives, and therefore decreases his risk-free investment. This increases the risk-free rate. Second, higher future dividends imply that the agent invests more in the risky asset, thereby reducing his risk-free holdings and increasing the risk-free rate. Both effects show that the risk-free rate r_{ft} is increasing with the expected growth rate x_t.

The first component of the market price of risk rewards the agent for bearing both consumption growth risk and the part of expected consumption growth risk that is correlated with consumption growth risk. The second component of the market price of risk rewards the agent for bearing the part of expected consumption growth risk that is independent of unexpected consumption growth risk.

The prices of a dividend strip, $P_t^{\text{strip}}(\tau)$, and a risk-free bond, $P_t^{\text{bond}}(\tau)$, with time-to-maturity τ are provided in Proposition 3 below.

Proposition 3. *The prices of a dividend strip, $P_t^{\text{strip}}(\tau)$, and a risk-free bond, $P_t^{\text{bond}}(\tau)$, with time-to-maturity τ satisfy*

$$
\begin{aligned}
P_t^{\text{strip}}(\tau) &\equiv e^{-y_t^{\text{e}}(\tau)\tau} D_t = e^{a^{\text{strip}}(\tau)+b_x^{\text{strip}}(\tau)x_t} D_t, \\
P_t^{\text{bond}}(\tau) &\equiv e^{-y_t^{\text{b}}(\tau)\tau} = e^{a^{\text{bond}}(\tau)+b_x^{\text{bond}}(\tau)x_t},
\end{aligned}
$$

where $y_t^{\text{e}}(\tau)$ is the equity yield (van Binsbergen et al., 2013), $y_t^{\text{b}}(\tau)$ is the bond *yield, and the functions* $a^i(\tau) \in \mathbb{R}$*,* $i \in \{strip, bond\}$ *solve* (23)*, and* $b_x^{\text{strip}}(\tau)$*,* $b_x^{\text{bond}}(\tau)$ *are given by*

$$b_x^{\text{strip}}(\tau) = \frac{\left(1 - e^{-\kappa_x \tau}\right)(\phi - 1/\psi)}{\kappa_x}, \quad b_x^{\text{bond}}(\tau) = -\frac{1 - e^{-\kappa_x \tau}}{\kappa_x \psi}.$$

Proof. See Appendix B.3.

Proposition 2 shows that an increase in x_t increases interest rates, thereby decreasing the price of bonds ($b_x^{\text{bond}}(\tau) < 0$). As the expected growth rate x_t increases, dividend strip prices increase if $\phi > 1/\psi$ and decrease otherwise. The reason is that an increase in x_t is particularly good news about future dividends if ϕ is large (see equation (1)), thereby increasing prices of claims to future dividends. If ϕ is relatively low, however, the increase in future dividends is not sufficient to dominate the increase in discount rates (see equation (2)), and therefore dividend strip prices decrease. If $\phi = 1/\psi$, dividend strip prices are independent of the expected growth rate x_t because the effect of discount rates perfectly offsets the effect of future dividends.

The response of bond and dividend strip prices to a change in the expected growth rate is particularly strong for bonds and strips with long maturities because shocks in x_t are persistent. Consequently, the difference between the price responses of long maturity and short maturity claims becomes smaller when the persistence of x_t decreases, or in other words, when the mean-reversion speed κ_x increases. The next corollary summarizes these results.

Corollary 1. *Prices of bonds decrease in expected growth rate* x_t*. Prices of dividend strips increase (resp., decrease) in* x_t *if* $\phi > 1/\psi$ *(resp.,* $\phi < 1/\psi$*).*

Moreover, these effects are more pronounced for bonds and strips with longer maturities.

Proof. See Appendix B.4.

The dividend strip risk premium, $\mathrm{RP}^{\mathrm{strip}}(\tau)$, dividend strip return volatility, $\mathrm{vol}^{\mathrm{strip}}(\tau)$, and bond yield volatility, $\mathrm{vol}\left(y_t^{\mathrm{b}}(\tau)\right)$, are provided in Corollary 2 below.

Corollary 2. *The risk premium and return volatility of the dividend strip satisfy*

$$\mathrm{RP}^{\mathrm{strip}}(\tau) = \sigma^{\mathrm{strip}}(\tau)\Lambda,$$
$$\mathrm{vol}^{\mathrm{strip}}(\tau) = \left\|\sigma^{\mathrm{strip}}(\tau)\right\|,$$

where Λ is the market price of risk given in (3) and $\sigma^{\mathrm{strip}}(\tau)$ is the diffusion vector given by

$$\sigma^{\mathrm{strip}}(\tau) = \left(\varphi\sigma_C + \rho\frac{\left(1 - e^{-\kappa_x\tau}\right)\sigma_x\left(\phi - 1/\psi\right)}{\kappa_x}, \sqrt{1-\rho^2}\frac{\left(1 - e^{-\kappa_x\tau}\right)\sigma_x\left(\phi - 1/\psi\right)}{\kappa_x}\right).$$

The bond yield volatility satisfies

$$\mathrm{vol}\left(y_t^{\mathrm{b}}(\tau)\right) = \frac{1 - e^{-\kappa_x\tau}}{\tau\kappa_x\psi}\sigma_x.$$

Proof. See Appendix B.5.

3 An Investigation of the Shape of the Term Structures

In this section we analyze the shape of the term structures under the assumption that the coefficient of relative risk aversion is $\gamma \geq 1$. The case of $\gamma < 1$ has been more rarely considered in the literature and is therefore treated in the appendix.

In the next subsections, we determine the slopes of the term structures of dividend strip risk premium, dividend strip return volatility, bond yield, and bond yield volatility.

3.1 The Case of EIS $\geq$ 1

The elasticity of intertemporal substitution (EIS) is a key determinant of the shape of the term structures of dividend strip return volatility, dividend strip risk premium, and bond yields because its level determines the importance of the substitution effect relative to the income effect (Veronesi, 2000). When the EIS is larger than one, the substitution effect dominates the income effect. That is, the agent's incentive to substitute consumption intertemporally dominates her incentive to smooth it. Hence in response to a negative shock in expected future consumption, the agent increases current consumption and decreases her investment in the risky asset. This shows that, when the EIS is larger than one, a negative shock in expected future consumption/cashflow implies a negative price shock because the agent focuses primarily on the behavior of cashflows as opposed to discount rates. In other words, the impact of cashflows on asset returns dominates the impact of discount rates. As a result, asset returns and cashflows share the same risk characteristics.

Propositions 4 and 5 characterize the slopes of the term structure of dividend strip risk premium and return volatility when the EIS is larger than or equal to one.

Proposition 4 (Dividend strip premium)**.** *With the elasticity of intertemporal substitution larger than unity ($\psi > 1$), the term structure of dividend strip risk premium is decreasing for all maturities if*

$$\rho < 0 \text{ and } \frac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{RP}}, \tag{4}$$

where the threshold $\overline{\varphi}_{\mathrm{RP}}$ is defined as

$$\overline{\varphi}_{\mathrm{RP}} \equiv \frac{\gamma\psi}{1-\gamma\psi}\rho\left(\kappa_x + \frac{1-k_1}{k_1}\right). \tag{5}$$

If $\psi = 1$ and $\gamma > 1$, in addition to condition (4), to generate the negative slope the model must also feature $\phi > 1$.

If $\psi = \gamma = 1$, the negative slope obtains if $\rho < 0$ and $\phi > 1$.

Proof. See Appendix B.6.

Proposition 5 (Dividend strip volatility)**.** *With the elasticity of intertemporal substitution larger than unity ($\psi > 1$), the term structure of dividend strip return volatility is decreasing for all maturities if*

$$\rho < 0 \text{ and } \frac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}}, \tag{6}$$

where the threshold $\overline{\varphi}_{\mathrm{VOL}}$ *is defined as*

$$\overline{\varphi}_{\mathrm{VOL}} \equiv \frac{\varphi\psi}{1-\phi\psi}\rho\kappa_x. \tag{7}$$

If $\psi = 1$*, in addition to condition* (6)*, to generate the negative slope the model must also feature* $\phi > 1$*.*

Proof. See Appendix B.7.

Propositions 4 and 5 show that the sign of the correlation between the consumption growth and the expected consumption growth, ρ, is an important determinant of the shape of the term structures of dividend strip return volatility and risk premium. Since we have seen that the risk characteristics of cashflows and asset returns are the same when the EIS is larger than one, understanding the shape of the term structure of dividend growth risk allows us to understand the shape of the term structures of dividend strip return volatility and risk premium. When $\rho < 0$, a negative shock in current dividends signals high future dividend growth. That is, the negative shock incurred today is compensated by the positive shock expected tomorrow, thereby making the long term dividend growth rate less risky than the short term dividend growth rate. This shows that the term structure of dividend growth risk tends to be downward sloping when $\rho < 0$, and so are the term structures of dividend strip return volatility and risk premium when $\psi > 1$.

For the slope to be strictly negative, the ratio of expected consumption growth risk to consumption growth risk needs to be not too large. That is, changes in consumption need to be principally driven by unexpected growth shocks, as opposed to expected growth shocks. When the EIS is equal to one, the substitution and income effects perfectly offset each other. In this case, the leverage parameter ϕ needs to be larger than $1/\psi = 1$ for the impact of cashflows on asset returns to keep dominating the impact of discount rates (see Section 2.2 for a related discussion), and therefore to generate downward-sloping term structures of dividend strip return volatility and risk premium.

We next focus on the term structure of interest rates. Proposition 6 discusses the bond yield term spread as well as the sensitivity of the slope of the bond yield curve to a change in expected consumption growth. The term spread is defined as the difference between the longest-term bond yield (time to maturity $\tau \to \infty$) and the shortest-term bond yield (time to maturity $\tau \to 0$).

Proposition 6 (Bond yields)**.** *With the elasticity of intertemporal substitution larger than or equal to unity ($\psi \geq 1$), the bond yield term spread is positive if*

$$\rho < 0 \text{ and } \frac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{BY}}, \tag{8}$$

where the threshold $\overline{\varphi}_{\mathrm{BY}}$ is defined as

$$\overline{\varphi}_{\mathrm{BY}} = \frac{2\gamma\psi\kappa_x}{(1-2\gamma\psi)\kappa_x - \frac{1-k_1}{k_1}}\rho\left(\kappa_x + \frac{1-k_1}{k_1}\right). \tag{9}$$

Moreover, the slope of the bond yield curve decreases with the expected consumption growth,

$$\frac{\partial^2 y_t^b(\tau)}{\partial\tau\partial x_t} < 0.$$

Proof. See Appendix B.8.

As already mentioned, a negative consumption shock today is expected to be followed by positive consumption shocks in the future when the correlation ρ is negative. This implies that long-term consumption growth rates are less risky than short-term consumption growth rates. Since bonds are used to hedge consumption risk, short-term bonds are more attractive than long-term bonds. That is, short-term bond prices are more expensive than long-term bonds, and consequently their yields are smaller when $\rho < 0$ (Proposition 6). Furthermore, an upward-sloping (respectively, downward-sloping) yield curve becomes flatter (respectively, steeper) as the expected consumption growth rate increases. The reason is that the 0-year maturity bond yield, or in other words, the risk-free rate provided in (2) increases with the expected consumption growth rate x_t, whereas the ∞-year maturity bond yield is independent of x_t.

3.2 The Case of EIS < 1

When the elasticity of intertemporal substitution is smaller than one, the agent's incentive to smooth consumption intertemporally dominates the incentive to substitute it. In other words, the income effect dominates the substitution effect. In response to a negative shock in expected future consumption growth, the agent decreases current consumption and increases her investment in the risky asset, which yields an increase in the asset price. This shows that, when the EIS is smaller than one, risky asset returns tend to be primarily driven by discount rates as opposed to cashflows. However, we have seen in

Corollary 1 that when the leverage parameter ϕ is large enough, the impact of cashflows on prices still dominates that of discount rates. As a result, asset returns and cashflows share the same risk characteristics when the EIS is smaller than one and ϕ is large enough.

Propositions 7 and 8 characterize the shape of the term structures of dividend strip return volatility and risk premium when the EIS is smaller than one.

Proposition 7 (Dividend strip premium)**.** *With the elasticity of intertemporal substitution smaller than unity ($\psi < 1$), the term structure of dividend strip risk premium is decreasing for all maturities if either one of the following conditions holds:*

1. *The agent prefers early resolution of uncertainty ($1/\gamma < \psi < 1$), and*
   (a) $\phi > 1/\psi$ *and* $\rho \leq 0$ *and* $\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{RP}}$*, or*
   (b) $\phi < 1/\psi$ *and* $\left(\rho > 0 \text{ or } \dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{RP}}\right)$.
2. *The agent has CRRA preferences ($\psi = 1/\gamma$), and*
   (a) $\phi > \gamma$ *and* $\rho < 0$*, or*
   (b) $\phi < \gamma$ *and* $\rho > 0$.
3. *The agent prefers late resolution of uncertainty ($\psi < 1/\gamma$), and*
   (a) $\phi > 1/\psi$ *and* $\left(\rho \leq 0 \text{ or } \dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{RP}}\right)$*, or*
   (b) $\phi < 1/\psi$ *and* $\rho > 0$ *and* $\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{RP}}$,

where the threshold $\overline{\varphi}_{\mathrm{RP}}$ is defined in (5)*.*

Proof. See Appendix B.9.

Proposition 8 (Dividend strip volatility)**.** *With the elasticity of intertemporal substitution smaller than unity ($\psi < 1$), the term structure of the dividend strip return volatility is decreasing for all maturities if either one of the following conditions holds:*

1. *$\phi > 1/\psi$ and* (6) *holds, or*
2. *$\phi < 1/\psi$ and $\rho > 0$ and $\dfrac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}}$,*

where the threshold $\overline{\varphi}_{\mathrm{VOL}}$ is defined in (7)*.*

Proof. See Appendix B.10.

As mentioned previously, the risk characteristics of asset returns and cash-flow growth rates are the same when the EIS is smaller than one and the leverage parameter ϕ is large enough ($\phi > 1/\psi$). Since long-term dividend growth rates are less risky than short-term dividend growth rates when the correlation between consumption growth and expected consumption growth is negative ($\rho < 0$), dividend strips with long maturities tend to feature a smaller risk premium and lower return volatility than dividend strips with short maturities. For the slope of these term structures to be strictly negative, the ratio of expected consumption growth risk σ_x to consumption growth risk σ_C needs to be not too large.

When the EIS is smaller than one and the leverage parameter is relatively small ($\phi < 1/\psi$), the income effect dominates and precautionary savings motives are strong. In response to higher risk about future consumption, the agent increases her investments for future consumption purposes. When the correlation ρ is positive, a negative consumption shock today is expected to be followed by a negative shock tomorrow. That is, shocks of the same sign accumulate over time and imply that long-term consumption growth rates are more risky than short-term ones. Because of the agent's precautionary savings motives, long-term dividend strips are particularly attractive securities; they pay off precisely when consumption growth risk is high. Therefore, the risk premium and volatility of long-term dividend strips tend to be lower than the risk premium and volatility of short-term strips when $\psi < 1$, $\phi < 1/\psi$, and $\rho > 0$. As before, for the slope of the term structures to be strictly negative, the ratio of σ_x to σ_C needs to be not too large.

Proposition 9 describes the shape of the term structure of interest rates when the EIS is smaller than one.

Proposition 9 (Bond yields)**.** *With the elasticity of intertemporal substitution smaller than unity* ($\psi < 1$)*, the bond yield term spread (the difference between the longest-term bond yield and the shortest-term bond yield) is positive if either one of the following conditions holds:*

1. *The agent prefers early resolution of uncertainty or has CRRA preferences* ($1/\gamma \leq \psi < 1$) *and condition* (8) *holds.*

2. *The agent prefers late resolution of uncertainty* ($\psi < 1/\gamma$)*, and*

(a) $\rho < 0$ *and* $\left(\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{BY}} \text{ or } \kappa_x \geq \overline{\kappa}_{\mathrm{BY}} > 0\right)$, *or*

(b) $\rho \geq 0$ *and* $\dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{BY}}$ *and* $\kappa_x > \overline{\kappa}_{\mathrm{BY}} > 0$,

where $\overline{\sigma}_{\mathrm{BY}}$ *is defined in* (9) *and* $\overline{\kappa}_{\mathrm{BY}}$ *is*

$$\overline{\kappa}_{\mathrm{BY}} \equiv \frac{1}{1-2\gamma\psi}\frac{1-k_1}{k_1}. \tag{10}$$

Proof. See Appendix B.11.

When the correlation between consumption growth and expected consumption growth is negative ($\rho < 0$), long-term consumption growth rates tend to be less risky than short-term ones. Because bonds are used to hedge consumption growth risk, long-term bonds are less attractive than short-term bonds. Therefore the price of long-term bonds is lower than that of short-term bonds, thereby explaining the positive bond yield term spread obtained when $\rho < 0$ (Proposition 9). When the correlation between consumption growth and expected consumption growth is non-negative ($\rho \geq 0$), the bond yield term spread remains positive as long as the ratio between expected consumption growth volatility and consumption growth volatility is sufficiently large and the mean-reversion speed of expected consumption growth is sufficiently high.

3.3 Bond Yield Volatility

Proposition 10 describes the shape of the term structure of bond yield volatility.

Proposition 10 (Bond yield volatility)**.** *The term structure of bond yield volatility is decreasing for all maturities.*

Moreover, the term structure of bond yield volatility has a long-run limit of zero:

$$\lim_{\tau \to \infty} \mathrm{vol}(y_t^b(\tau)) = 0.$$

Proof. See Appendix B.12.

Regardless of the parameters governing preferences and dynamics, the term structure of bond yield volatility slopes downwards. Empirical evidence for a decreasing term structure of bond yield volatility is provided in Piazzesi et al. (2006).

3.4 Replicating Simultaneously All Observed Term Structures

Corollary 3 below provides a straightforward implication of Propositions 4, 5, 6, and 10. It shows that the observed downward-sloping term structure of dividend strip return volatility, dividend strip risk premium, and bond yield volatility together with the positive bond yield term spread can be obtained simultaneously when the EIS is larger than or equal to one. It is worth emphasizing that, even when the agent is myopic ($\psi = 1$ and $\gamma > 1$, or log utility $\psi = \gamma = 1$), the model can simultaneously replicate each of the observed term structures. It is also worth noting that, when the EIS is larger than or equal to one, the observed shapes can only be obtained if the correlation between the consumption growth rate and the expected consumption growth rate is negative ($\rho < 0$).

Corollary 3. *The term structures of dividend strip return volatility, dividend strip risk premium, and bond yield volatility are downward sloping and the bond yield term spread is positive if either one of the following conditions holds:*

1. *The elasticity of intertemporal substitution is larger than unity* ($\psi > 1$) *and*

$$\rho < 0 \text{ and } \frac{\sigma_x}{\sigma_C} < \min\left\{\overline{\varphi}_{\mathrm{BY}}, \overline{\varphi}_{\mathrm{VOL}}, \overline{\varphi}_{\mathrm{RP}}\right\}. \tag{11}$$

2. *The elasticity of intertemporal substitution is equal to one* ($\psi = 1$)*, the risk aversion is larger than one* ($\gamma > 1$)*, the leverage parameter on expected consumption growth is greater than unity* ($\phi > 1$)*, and* (11) *holds.*

3. *The agent has log utility* ($\psi = \gamma = 1$)*, the leverage parameter on expected consumption growth is greater than unity* ($\phi > 1$)*, and*

$$\rho < 0 \text{ and } \frac{\sigma_x}{\sigma_C} < \min\left\{\overline{\varphi}_{\mathrm{BY}}, \overline{\varphi}_{\mathrm{VOL}}\right\}.$$

Proof. Straightforward implication of Propositions 4, 5, 6, and 10.

Corollary 4 provides a direct implication of Propositions 7, 8, 9, and 10. It shows that when the EIS is smaller than one, the observed shapes of the term structures can be obtained simultaneously within the model. In particular, the first condition of Corollary 4 provides evidence that the observed shapes can be obtained simultaneously even when the agent has CRRA utility ($\gamma = 1/\psi$). Furthermore, when the EIS is smaller than one, all observed shapes can be replicated when the correlation between the consumption growth rate and the expected consumption growth rate is negative. When this correlation is positive, the model can replicate the observed shape of all term structures but that of bond yields.

Corollary 4.

1. *With the elasticity of intertemporal substitution smaller than unity ($\psi < 1$), the term structures of dividend strip return volatility, dividend strip risk premium, and bond yield volatility are downward sloping and the bond yield term spread is positive if the following condition holds:*

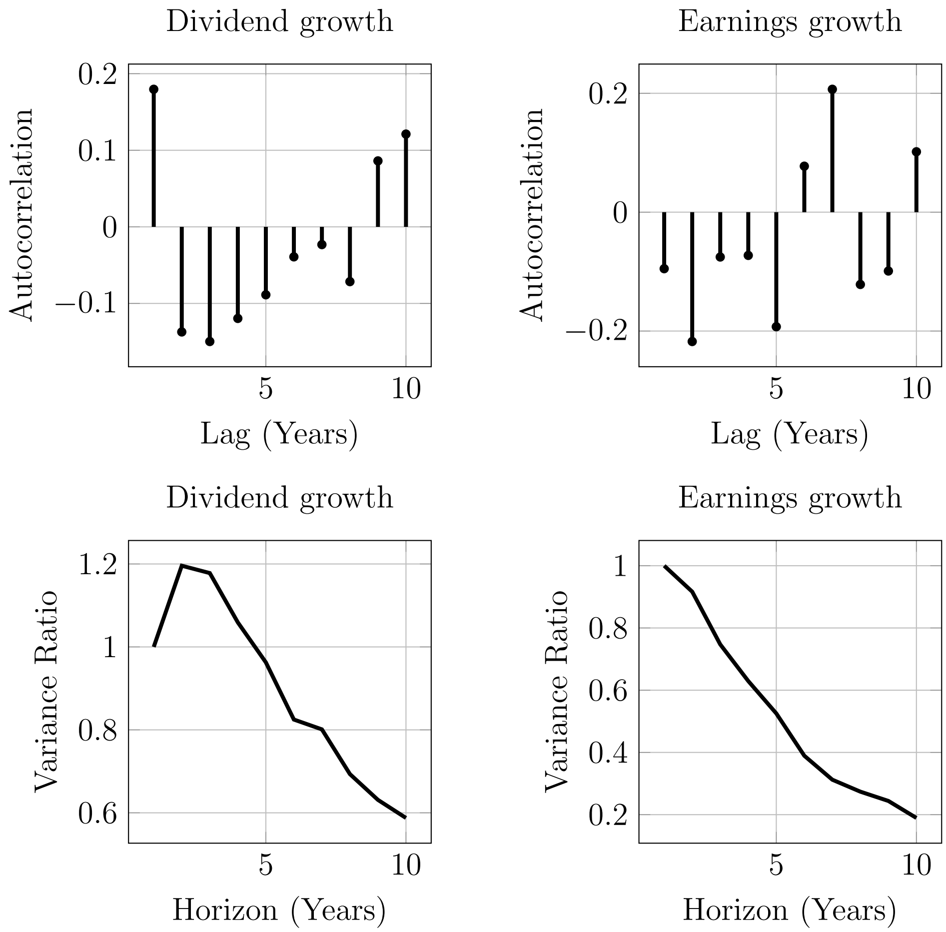
$$1/\psi < \phi \textit{ and } (11) \textit{ holds.}$$

2. *With the elasticity of intertemporal substitution smaller than unity ($\psi < 1$) and a positive correlation between the consumption growth rate and the expected consumption growth rate ($\rho > 0$), the term structures of dividend strip return volatility, dividend strip risk premium, and bond yield volatility are downward sloping but the bond yield term spread is negative if the following condition holds:*

$$1/\psi > \max\{\phi, \gamma\} \textit{ and } \frac{\sigma_x}{\sigma_C} < \min\left\{\overline{\varphi}_{\mathrm{VOL}}, \overline{\varphi}_{\mathrm{RP}}\right\}.$$

Proof. Straightforward implication of Propositions 7, 8, 9, and 10.

Figure 1: Autocorrelations and variance ratios of S&P 500 real dividend and earnings growth rates.
The data is at the yearly frequency from 1871 to 2018. It is obtained from Robert Shiller's website.

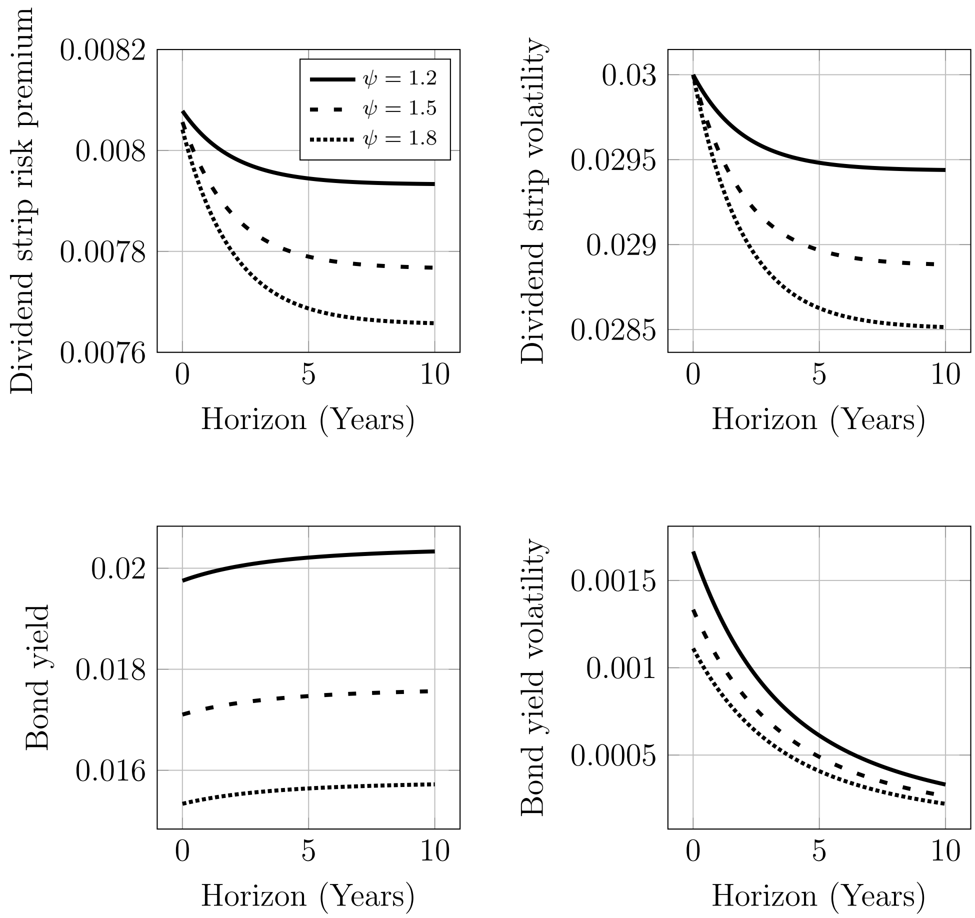


4 An Illustration with Traditional Calibrations

In this section we illustrate that traditional calibrations of the model allow it to replicate the observed term structures of dividend strip risk premium, dividend strip return volatility, bond yield, and bond yield volatility.

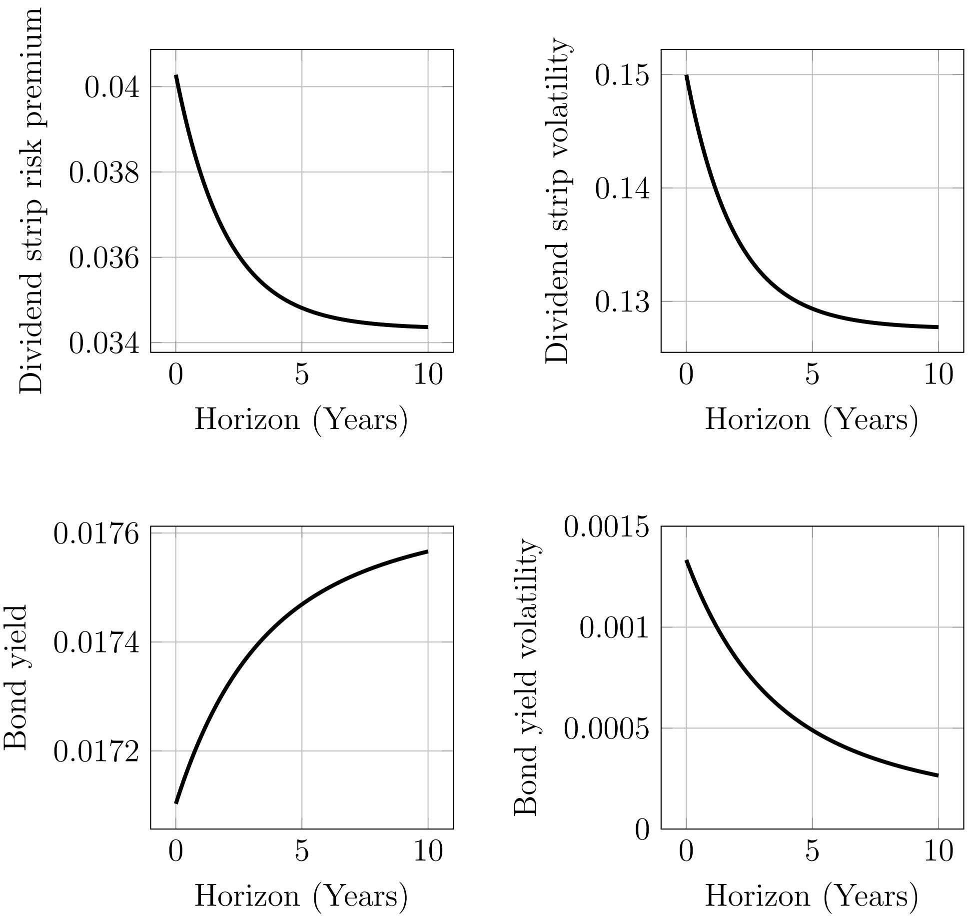
As it is standard in the literature, we assume that the long-term consump-

Figure 2: The shape of term structures with $\rho < 0$ and no leverage. Parameter values are $\alpha_C = 0.02$, $\sigma_C = 0.03$, $\alpha_D = 0.02$, $\phi = 1$, $\varphi = 1$, $\kappa_x = 0.5$, $\sigma_x = 0.002$, $\rho = -0.85$, $\delta = 0.99$, $\gamma = 10$, $\psi = 1.5$.



tion and dividend growth rates are $\alpha_C = \alpha_D = 0.02$, and that the consumption growth volatility is $\sigma_C = 0.03$. Unless specified otherwise, the leverage parameters are $\phi = 7.5$ and $\varphi = 5$. These values imply that the expected dividend growth volatility is 0.015 and the dividend growth volatility is 0.15, consistent with Lettau and Wachter (2007). The parameters $\kappa_x = 0.5$ and $\sigma_x = 0.002$ are chosen to generate a fairly flat term structure of consumption growth volatility,

Figure 3: The shape of term structures with $\rho < 0$ and leverage. Parameter values are $\alpha_C = 0.02$, $\sigma_C = 0.03$, $\alpha_D = 0.02$, $\phi = 7.5$, $\varphi = 5$, $\kappa_x = 0.5$, $\sigma_x = 0.002$, $\rho = -0.85$, $\delta = 0.99$, $\gamma = 10$, $\psi = 1.5$.



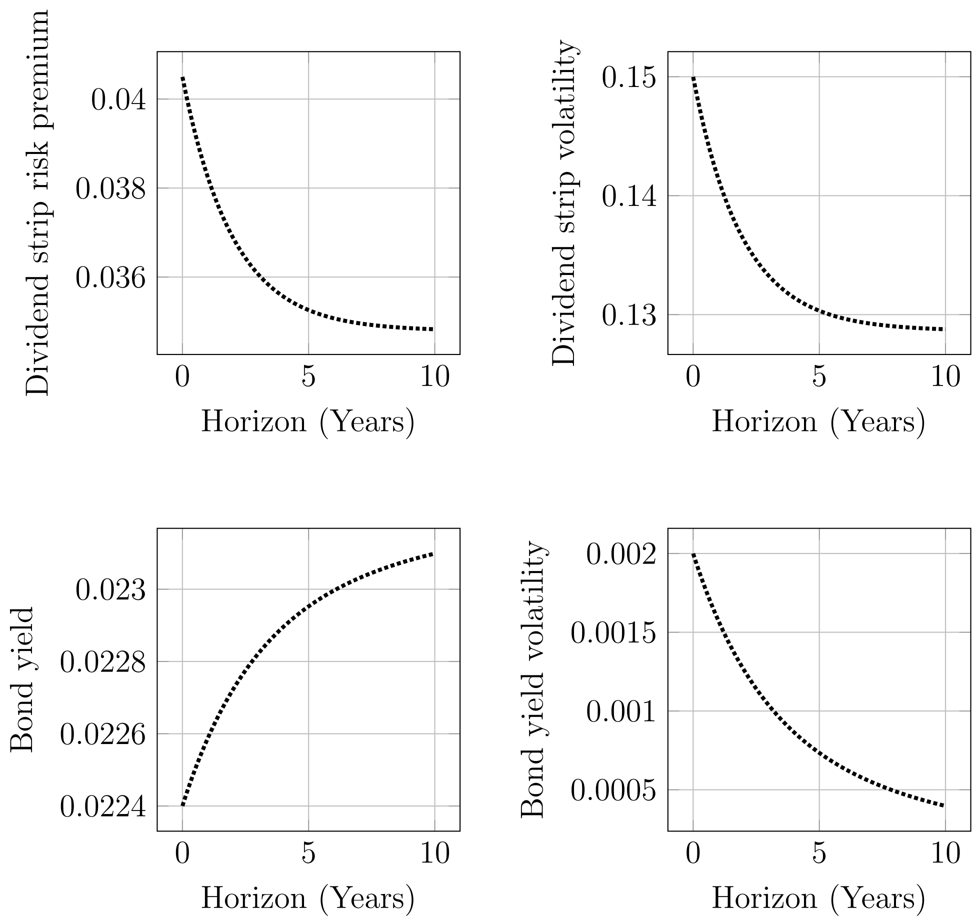
as documented by Beeler and Campbell (2012), Hasler and Marfè (2016), and Marfè (2017), irrespective of the value of the correlation between consumption growth and expected consumption growth. These values imply that the 0-year consumption growth volatility is 0.03, while the 10-year consumption growth volatility is at least 0.0268 and at most 0.0332, depending on the value of the aforementioned correlation. Our benchmark preference parameters are borrowed from the literature (e.g., Bansal and Yaron, 2004). Unless specified otherwise, the coefficient of relative risk aversion is $\gamma = 10$, the subjective discount factor is $\delta = 0.99$, and the EIS ψ is between $\frac{1}{11}$ and 1.8.

We assume that the correlation ρ between the cashflow growth rate and the expected cashflow growth rate is negative. As explained in Section 3, a negative correlation implies that the term structure of dividend growth volatility is downward sloping, consistent with the empirical findings of Beeler and Campbell (2012), Belo et al. (2015), Hasler and Marfè (2016), and Marfè (2017). To confirm these empirical findings and to further motivate our assumption that the correlation between the current and the future cashflow growth rate is negative, we depict in Figure 1 the term structure of variance ratios and the autocorrelation function of the S&P 500 real dividend and earnings growth rates. As previously documented, the correlations between the current and the future growth rates are largely negative, which implies that the long-term variance ratios are lower than the short-term ones. As in Lettau and Wachter (2007, 2011) we set the correlation $\rho = -0.85$, which yields a model-implied 7-year dividend growth variance ratio of about 0.8, consistent with empirical findings.

Figure 2 depicts the term structures of dividend strip risk premium, dividend strip volatility, bond yields, and bond yield volatility when the correlation between consumption growth and expected consumption growth is negative and there is no leverage. That is, the consumption stream is assumed to be equal to the dividend stream as in Lucas (1978). These parameter values satisfy the conditions provided in the first statement of Corollary 3. In

Figure 4: The shape of term structures with $\rho < 0$, leverage, and myopic utility.

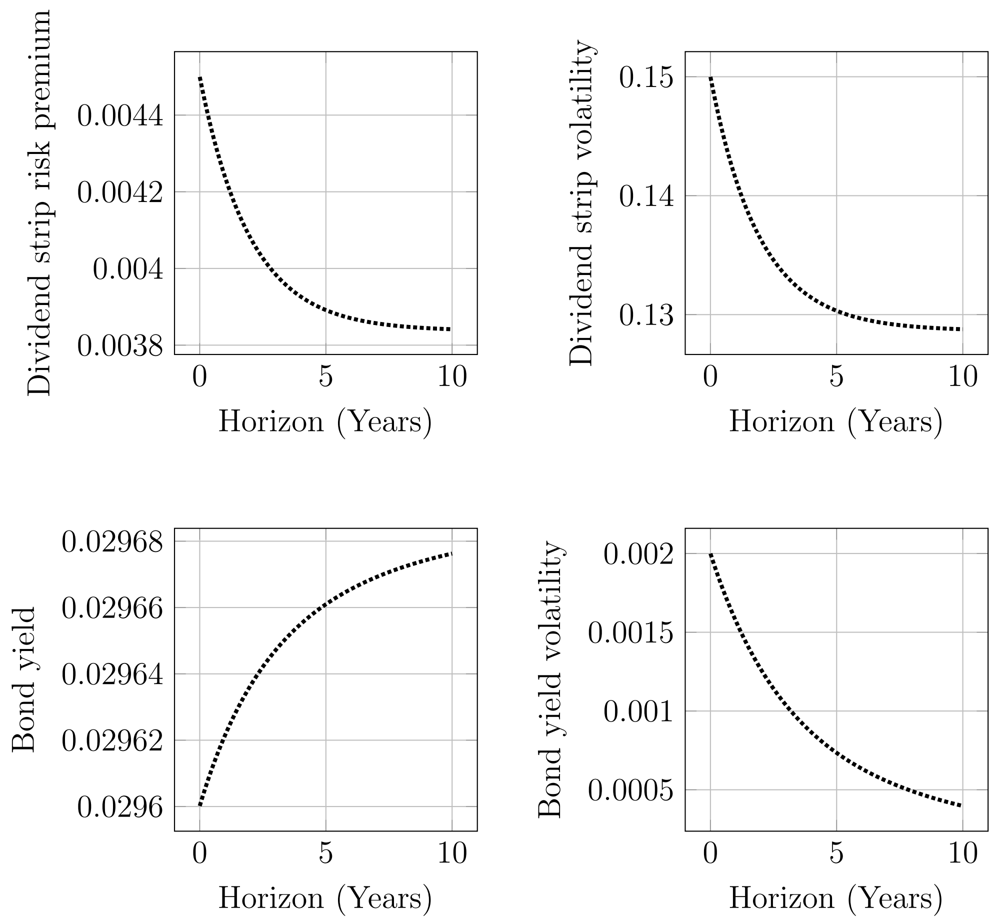
Parameter values are $\alpha_C = 0.02$, $\sigma_C = 0.03$, $\alpha_D = 0.02$, $\phi = 7.5$, $\varphi = 5$, $\kappa_x = 0.5$, $\sigma_x = 0.002$, $\rho = -0.85$, $\delta = 0.99$, $\gamma = 10$, $\psi = 1$.



this case, the model can replicate the observed shape of all term structures. The reason is that, when the EIS is larger than one, the substitution effect dominates and therefore the characteristics of asset returns coincide with the characteristics of the corresponding cashflow growth rates. Since the term structure of dividend growth return volatility is downward sloping when the correlation $\rho < 0$, the term structures of dividend strip risk premium and re-

Figure 5: The shape of term structures with $\rho < 0$, leverage, and log utility.

Parameter values are $\alpha_C = 0.02$, $\sigma_C = 0.03$, $\alpha_D = 0.02$, $\phi = 7.5$, $\varphi = 5$, $\kappa_x = 0.5$, $\sigma_x = 0.002$, $\rho = -0.85$, $\delta = 0.99$, $\gamma = 1$, $\psi = 1$.



turn volatility are also downward sloping. The term structure of bond yields is upward sloping because the term structure of consumption growth volatility is (weakly) downward sloping and the agent uses bonds to hedge consumption growth risk. Since the EIS is large, the level of bond yields is consistent with that observed in the data. The levels of the risk premium and return volatility, however, are significantly lower than their empirical counterparts because this parametrization abstracts from leverage (Lucas, 1978).

Figure 3 depicts the different term structures when the correlation between consumption growth and expected consumption growth is negative and there is leverage. This parametrization also satisfies the conditions provided in the first statement of Corollary 3. Therefore, the model replicates the observed shape of all term structures for the same reason as that detailed in the previous paragraph. In addition, this parametrization implies that the levels of the risk premium, return volatility, and bond yield are in line with the data because it assumes both leverage and a relatively high EIS.

Figures 4 and 5 depict the different term structures when the correlation between consumption growth and expected consumption growth is negative and the agent is either myopic ($\psi = 1$) or has log utility ($\gamma = \psi = 1$). These parameter values satisfy the conditions provided in the second and third statements of Corollary 3. These figures show that, even when the agent is myopic or has log utility, the observed term structures can be replicated by the model. Importantly, this holds for perfectly realistic parameter values. When the agent is myopic and has a relatively high aversion to risk, the levels of the risk premium, return volatility, and bond yield are consistent with the data. In contrast, the risk premium and return volatility are unsurprisingly much lower than their empirical counterparts when the agent has log utility.

5 Conclusion

We investigate the shape of the term structures of dividend strip return volatility, dividend strip risk premium, bond yields, and bond yield volatility in a general equilibrium model à la Lucas (1978). The representative agent has Epstein and Zin (1989) preferences and the aggregate output of the economy features a mean-reverting expected growth rate. We show that the observed shapes of the aforementioned term structures can be obtained simultaneously within a model featuring standard preferences and realistic dynamics of the economic fundamentals. We finally illustrate that standard calibrations of the model allow it to replicate simultaneously all of the observed term structures.

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Appendix

A Notation Summary

Parameter	Interpretation
α_C	Average growth in log consumption
$\sigma_C > 0$	Volatility of consumption growth
α_D	Average growth in log dividends
$\phi \geq 1$	Leverage ratio on expected consumption growth
$\varphi \geq 1$	Leverage ratio on consumption volatility
$\kappa_x > 0$	Mean reversion of expected consumption growth
$\sigma_x > 0$	Volatility of expected consumption growth
$\rho \in [-1, 1]$	Correlation between expected and unexpected consumption growth
$\delta \in (0, 1)$	Rate of time preference
$\gamma > 0$	Relative risk aversion
$\psi > 0$	Elasticity of intertemporal substitution

B Proofs of Propositions and Corollaries

B.1 Proposition 1

Proof. Following Eraker and Shaliastovich (2008), the discrete time continuously compounded log-return, $\log R_{c,t}$, on the representative agent's wealth, W_t, satisfies

$$\log R_{c,t+1} = \log \frac{W_{t+1} + C_{t+1}}{W_t} = \log \left(e^{wc_{t+1}} + 1\right) - wc_t + \log \frac{C_{t+1}}{C_t}, \quad (12)$$

where $wc_t = \log(W_t/C_t)$. Log-linearizing the first term in (12) around the mean log wealth-consumption ratio yields

$$\log R_{c,t+1} \approx k_0 + k_1 wc_{t+1} - wc_t + \log \frac{C_{t+1}}{C_t}, \quad (13)$$

where the constants k_0 and k_1 satisfy

$$k_0 = -\log \left((1-k_1)^{1-k_1} k_1^{k_1}\right), \qquad k_1 = e^{\mathbb{E}(wc_t)} / \left(1 + e^{\mathbb{E}(wc_t)}\right).$$

The continuous-time counterpart of (13) is

$$d \log R_{c,t} = k_0 dt + k_1 dwc_t - (1 - k_1) wc_t dt + d \log C_t,$$

and the Euler equation defining the state-price density, M_t, is written

$$\mathbb{E}_t \left[\exp \left(\log \frac{M_{t+\tau}}{M_t} + \int_t^{t+\tau} d \log R_{c,s} \right) \right] = 1,$$

where

$$d \log M_t = \theta \log \delta dt - \frac{\theta}{\psi} d \log C_t - (1 - \theta) d \log R_{c,t}.$$

We conjecture that the log wealth-consumption ratio is affine in the state variables $Z_t = (\log C_t, \log D_t, x_t)^\top$,

$$wc_t \equiv \log \frac{W_t}{C_t} = A + B^\top Z_t, \tag{14}$$

and use the fact that the state variables belong to the affine class, so that their dynamics can be written as:

$$\begin{aligned} dZ_t &= \mu(Z_t)dt + \Sigma(Z_t)dB_t \\ \mu(Z_t) &= \mathcal{M} + \mathcal{K} Z_t \\ \Sigma(Z_t)\Sigma(Z_t)^\top &= h + \sum_{i=1}^{3} H^i Z_t^i, \end{aligned}$$

where $\mathcal{M} \in \mathbb{R}^3$, $\mathcal{K} \in \mathbb{R}^{3\times 3}$, $h \in \mathbb{R}^{3\times 3}$, $H \in \mathbb{R}^{3\times 3\times 3}$, and $B_t = (B_{1t}, B_{2t})^\top$ is a standard Brownian motion.

The dynamics of the state-price density are

$$d \log M_t = (\theta \log \delta - (\theta - 1) \log k_1 + (\theta - 1)(k_1 - 1) B^\top (Z_t - \mu_Z) dt - \Omega^\top dZ_t, \tag{15}$$

where $Z_t = (\log C_t, \log D_t, x_t)^\top$, $\mu_Z = (0, 0, 0)^\top$, $\Omega = \gamma(1, 0, 0)^\top + (1 - \theta) k_1 B$, and the coefficients $A \in \mathbb{R}$ and $B \in \mathbb{R}^3$ are the loadings defined in (14).

The coefficients $A \in \mathbb{R}$, $B \in \mathbb{R}^3$ solve the following system of equations

$$0 = \mathcal{K}^\top \chi - \theta(1-k_1)B + \frac{1}{2}\chi^\top H \chi, \tag{16}$$

$$0 = \theta(\log\delta + k_0 - (1-k_1)A) + \mathcal{M}^\top \chi + \frac{1}{2}\chi^\top h \chi, \tag{17}$$

and the linearization coefficient $k_1 \in \mathbb{R}$ satisfies

$$\theta \log k_1 = \theta(\log\delta + (1-k_1)B^\top \mu_Z) + \mathcal{M}^\top \chi + \frac{1}{2}\chi^\top h \chi, \tag{18}$$

where $\chi = \theta\left((1-\frac{1}{\psi})(1,0,0)^\top + k_1 B\right)$.

Solving (16) for the vector of loadings $B \in \mathbb{R}^3$ gives $B^\top = \left(0, 0, \frac{1-1/\psi}{1-k_1(1-\kappa_x)}\right)$. Plugging this solution in equation (17) allows to solve for the coefficient A.

□

B.2 Proposition 2

Proof. From the arbitrage theory we know that the state-price density M_t satisfies

$$\frac{dM_t}{M_t} = -r_{ft}dt - \Lambda_t^\top dB_t.$$

where r_{ft} is the risk-free rate and Λ_t is the market price of risk vector.

Eraker and Shaliastovich (2008) show that from the expression for the state price density in (15), the risk free rate and market price of risk vector can be determined as follows:

$$r_{ft} = \Phi_0 + \Phi_1^\top Z_t,$$
$$\Lambda_t = \Sigma(Z_t)^\top \Omega,$$

where $Z_t = (\log C_t, \log D_t, x_t)^\top$ is the vector of state variables, the matrix $\Sigma(Z_t) \in \mathbb{R}^{3\times 2}$ encodes the diffusions of the state variables, vector $\Omega = \gamma(1,0,0)^\top + (1-\theta)k_1 B$ and the coefficients $\Phi_0 \in \mathbb{R}$ and $\Phi_1 \in \mathbb{R}^3$ solve the system of equations

$$\Phi_1 = (1-\theta)(k_1-1)B + \mathcal{K}^\top\Omega - \frac{1}{2}\Omega^\top H\Omega, \tag{19}$$

$$\Phi_0 = -\theta\log\delta + (\theta-1)(\log k_1 + (k_1-1)B^\top\mu_Z) + \mathcal{M}^\top\Omega - \frac{1}{2}\Omega^\top h\Omega.$$

Finally, following Eraker and Shaliastovich (2008), the dynamics of the vector of state variables Z_t under the risk neutral measure $\mathbb{Q}$ are given by

$$dZ_t = (\mathcal{M}^{\mathbb{Q}} + \mathcal{K}^{\mathbb{Q}}Z_t)dt + \Sigma(Z_t)dB_t^{\mathbb{Q}},$$

where $B_t^{\mathbb{Q}} = B_t + \int_0^t \Lambda_s ds$ is a $\mathbb{Q}$-Brownian motion and the coefficients $\mathcal{M}^{\mathbb{Q}} \in \mathbb{R}^3$ and $\mathcal{K}^{\mathbb{Q}} \in \mathbb{R}^{3\times 3}$ satisfy

$$\mathcal{M}^{\mathbb{Q}} = \mathcal{M} - h\Omega, \tag{20}$$

$$\mathcal{K}^{\mathbb{Q}} = \mathcal{K} - H\Omega. \tag{21}$$

□

B.3 Proposition 3

Proof. Prices of a dividend strip and a zero-coupon bond can be determined from

$$P_t^{\text{strip}}(\tau) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+\tau} r_{fs}ds}D_{t+\tau}\right) = e^{a^{\text{strip}}(\tau)+b^{\text{strip}}(\tau)^\top Z_t},$$

$$P_t^{\text{bond}}(\tau) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+\tau} r_{fs}ds}\right) = e^{a^{\text{bond}}(\tau)+b^{\text{bond}}(\tau)^\top Z_t}.$$

Eraker and Shaliastovich (2008) show that the functions $a^i(\tau) \in \mathbb{R}$ and $b^i(\tau) \in \mathbb{R}^3$, where $i \in \{strip, bond\}$, solve the following system of Ricatti equations

$$\frac{\partial}{\partial\tau}b^i(\tau) = -\Phi_1 + \mathcal{K}^{\mathbb{Q}^\top}b^i(\tau) + \frac{1}{2}b^i(\tau)^\top Hb^i(\tau), \tag{22}$$

$$\frac{\partial}{\partial\tau}a^i(\tau) = -\Phi_0 + \mathcal{M}^{\mathbb{Q}^\top}b^i(\tau) + \frac{1}{2}b^i(\tau)^\top hb^i(\tau), \tag{23}$$

with boundary conditions $a^{\text{strip}}(0) = 0$, $a^{\text{bond}}(0) = 0$, $b^{\text{strip}}(0) = (0,1,0)^\top$, and $b^{\text{bond}}(0) = (0,0,0)^\top$. Coefficients $\mathcal{M}^{\mathbb{Q}} \in \mathbb{R}^3$ and $\mathcal{K}^{\mathbb{Q}} \in \mathbb{R}^{3\times 3}$ are characterized in (20)–(21).

Solving (22) gives

$$
\begin{aligned}
b^{\text{strip}}(\tau)^{\top} &= \left(0, 1, \frac{\left(1-e^{-\kappa_x \tau}\right)(\phi - 1/\psi)}{\kappa_x}\right), \\
b^{\text{bond}}(\tau)^{\top} &= \left(0, 0, -\frac{1-e^{-\kappa_x \tau}}{\kappa_x \psi}\right).
\end{aligned}
$$

Using these results in (23) allows to solve for functions a^{strip} and a^{bond}.

□

B.4 Corollary 1

Proof. For dividend strips, the following holds:

$$
\begin{aligned}
\frac{\partial P_t^{\text{strip}}(\tau)}{\partial x_t} &= \frac{\left(1-e^{-\kappa_x \tau}\right)(\phi - 1/\psi)}{\kappa_x} P_t^{\text{strip}}(\tau) \\
\frac{\partial^2 P_t^{\text{strip}}(\tau)}{\partial x_t \partial \tau} &= e^{-\kappa_x \tau} (\phi - 1/\psi).
\end{aligned}
$$

Similarly, for bonds:

$$
\begin{aligned}
\frac{\partial P_t^{\text{bond}}(\tau)}{\partial x_t} &= -\frac{\left(1-e^{-\kappa_x \tau}\right)}{\kappa_x \psi} P_t^{\text{bond}}(\tau), \\
\frac{\partial^2 P_t^{\text{bond}}(\tau)}{\partial x_t \partial \tau} &= -\frac{e^{-\kappa_x \tau}}{\psi}.
\end{aligned}
$$

□

B.5 Corollary 2

Proof. Applying Itô's lemma and Girsanov's theorem gives the following bond yield volatility

$$
\operatorname{vol}\left(y_t^{\text{b}}(\tau)\right) = \left\| \frac{1}{\tau} \left[\frac{\partial}{\partial Z_t} \log\left(P_t^{\text{bond}}(\tau)\right)\right]^{\top} \Sigma(Z_t) \right\|.
$$

The dividend strip risk premium and return volatility satisfy

$$\mathrm{RP}_t^{\mathrm{strip}}(\tau) = \sigma_t^{\mathrm{strip}}(\tau)\Lambda_t,$$
$$\mathrm{vol}^{\mathrm{strip}}(\tau) = \left\|\sigma_t^{\mathrm{strip}}(\tau)\right\|,$$

where Λ_t is the market price of risk vector and $\sigma_t^{\mathrm{strip}}(\tau)$ is the diffusion vector given by

$$\sigma_t^{\mathrm{strip}}(\tau) = \frac{1}{P_t^{\mathrm{strip}}(\tau)}\left[\frac{\partial}{\partial Z_t}P_t^{\mathrm{strip}}(\tau)\right]^\top \Sigma(Z_t).$$

□

B.6 Proposition 4

Proof. The condition follows from inspecting the sign of the derivative of the dividend strip risk premium with respect to time to maturity τ under condition that $\psi \geq 1$,

$$\frac{\partial}{\partial\tau}\mathrm{RP}^{\mathrm{strip}}(\tau) = \frac{e^{-\kappa_x\tau}\sigma_x\left(-k_1\sigma_x + \gamma\rho\sigma_C\psi + k_1\gamma((-1+\kappa_x)\rho\sigma_C + \sigma_x)\psi\right)(-1+\phi\psi)}{(1+k_1(-1+\kappa_x))\psi^2}.$$

□

B.7 Proposition 5

Proof. The condition follows from inspecting the sign of the derivative of the dividend strip volatility with respect to time to maturity τ under condition that $\psi \geq 1$,

$$\frac{\partial}{\partial\tau}\mathrm{vol}^{\mathrm{strip}}(\tau) = \frac{\sigma_x e^{-2\kappa_x\tau}(\psi\phi-1)\left(\kappa_x\rho\sigma_C\psi\varphi e^{\kappa_x\tau} + \sigma_x\left(e^{\kappa_x\tau}-1\right)(\psi\phi-1)\right)}{\psi\sqrt{e^{-2\kappa_x\tau}\left(\left(\rho\sigma_x\left(e^{\kappa_x\tau}-1\right)(\psi\phi-1) + \kappa_x\sigma_C\psi\varphi e^{\kappa_x\tau}\right)^2 - \left(\rho^2-1\right)\sigma_x^2\left(e^{\kappa_x\tau}-1\right)^2(\psi\phi-1)^2\right)}}.$$

$\square$

B.8 Proposition 6

Proof. If x_t is equal to its long term mean, zero, then the following holds. The limit of the bond yield level as $\tau \to 0$ is

$$\lim_{\tau\to 0} y_t^{\mathrm{b}}(\tau) = -\frac{1}{2(\psi-1)\psi^2((\kappa_x-1)k_1+1)^2}\Big((\psi-1)\big(k_1^2\sigma_x^2(\gamma\psi-1)^2 + \gamma\psi^2((\kappa_x-1)k_1+1)^2\left(\gamma\sigma_C^2-2\alpha_C\right) + 2\gamma k_1\rho\sigma_C\sigma_x\psi(\gamma\psi-1)((\kappa_x-1)k_1+1)\big) - 2(\gamma-1)\psi^3\log(\delta)((\kappa_x-1)k_1+1)^2 + 2\psi^2(\gamma\psi-1)((\kappa_x-1)k_1+1)^2\log(k_1)\Big).$$

The limit of the bond yield level as $\tau \to \infty$ is

$$\lim_{\tau\to\infty} y_t^{\mathrm{b}}(\tau) = -\frac{1}{2\kappa_x^2(\psi-1)\psi^2((\kappa_x-1)k_1+1)^2}\Big((\psi-1)\big(\gamma\kappa_x^2\psi^2((\kappa_x-1)k_1+1)^2\left(\gamma\sigma_C^2-2\alpha_C\right) + 2\gamma\kappa_x\rho\sigma_C\sigma_x\psi((\kappa_x-1)k_1+1)(k_1(\gamma\kappa_x\psi-1)+1) + \sigma_x^2(k_1(\gamma\kappa_x\psi-1)+1)^2\big) + 2\kappa_x^2\psi^2((\kappa_x-1)k_1+1)^2((\gamma\psi-1)\log(k_1)-(\gamma-1)\psi\log(\delta))\Big).$$

The difference between the long-end and the short-end bond yields is

$$\lim_{\tau\to\infty} y_t^{\mathrm{b}}(\tau) - \lim_{\tau\to 0} y_t^{\mathrm{b}}(\tau) = \frac{\sigma_x(\sigma_x(\kappa_x k_1+k_1-1)-2\gamma\kappa_x\psi(\rho\sigma_C((\kappa_x-1)k_1+1)+k_1\sigma_x))}{2\kappa_x^2\psi^2((\kappa_x-1)k_1+1)}.$$

Condition (8) follows from inspecting the sign of this difference if $\psi \geq 1$. Moreover, we have

$$\frac{\partial^2 y_t^{\mathrm{b}}(\tau)}{\partial\tau\partial x_t} = \frac{e^{-\kappa_x\tau}\left(1-e^{\kappa_x\tau}+\kappa_x\tau\right)}{\kappa_x\tau^2\psi}.$$

$\square$

B.9 Proposition 7

Proof. Analogous to the proof of Proposition 4.

B.10 Proposition 8

Proof. Analogous to the proof of Proposition 5.

B.11 Proposition 9

Proof. Analogous to the proof of Proposition 6.

B.12 Proposition 10

Proof. The result follows immediately from inspecting the sign of the derivative of the bond yield volatility with respect to time to maturity τ,

$$\frac{\partial}{\partial \tau}\mathrm{vol}(y^{\mathrm{b}}(\tau)) = -\frac{\sigma_x e^{-\kappa_x \tau}\left(e^{\kappa_x \tau} - \kappa_x \tau - 1\right)}{\kappa_x \tau^2 \psi}.$$

□

C The Shape of Term Structures for $0 < \gamma < 1$

Proposition 11 (Dividend strip premium)**.** *The term structure of dividend strip risk premia is decreasing for all maturities if either of the following conditions holds:*

1. *The elasticity of intertemporal substitution is greater than unity and moreover*
   (a) $1 < \psi \leq 1/\gamma$ *and* $\rho < 0$, *or*
   (b) $1 < \psi \leq 1/\gamma$ *and* $\rho \geq 0$ *and* $\dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{RP}}$, *or*
   (c) $\psi > 1/\gamma$ *and* $\rho < 0$ *and* $\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{RP}}$*;*
2. *The elasticity of intertemporal substitution is equal to unity and*
   (a) $\phi > 1$ *and* $\rho \leq 0$, *or*

(b) $\phi > 1$ *and* $\dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{RP}}$;

3. *The elasticity of intertemporal substitution is less than unity and moreover*

   (a) $1/\phi < \psi < 1$ *and* $\rho \leq 0$, *or*

   (b) $1/\phi < \psi < 1$ *and* $\dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{RP}}$, *or*

   (c) $\psi < 1/\phi$ *and* $\rho > 0$ *and* $\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{RP}}$.

The threshold $\overline{\varphi}_{\mathrm{RP}}$ *is defined in* (5).

Proof. Analogous to the proof of Proposition 4.

Proposition 12 (Dividend strip volatility)**.** *The term structure of dividend strip return volatility is decreasing for all maturities if either of the following conditions holds:*

1. *The elasticity of intertemporal substitution is greater than unity and moreover*

$$\rho < 0 \textit{ and } \frac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}};$$

2. *The elasticity of intertemporal substitution is equal to unity and*

$$\phi > 1 \textit{ and } \rho < 0 \textit{ and } \frac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}};$$

3. *The elasticity of intertemporal substitution is less than unity and moreover*

   (a) $1/\phi < \psi < 1$ *and* $\phi > 1$ *and* $\rho < 0$ *and* $\dfrac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}}$, *or,*

   (b) $\psi < 1/\phi$ *and* $\rho > 0$ *and* $\dfrac{\sigma_x}{\sigma_C} \leq \overline{\varphi}_{\mathrm{VOL}}$.

The threshold $\overline{\varphi}_{\mathrm{VOL}}$ *is defined in* (7).

Proof. Analogous to the proof of Proposition 5.

Proposition 13 (Bond yields)**.** *The bond yield term spread (the difference between the longest-term bond yield and the shortest-term bond yield) is positive if either one of the following conditions holds:*

1. $\rho < 0$ *and* $\dfrac{\sigma_x}{\sigma_C} < \overline{\varphi}_{\mathrm{BY}}$, *or*

2. $\rho < 0$ *and* $\kappa_x \geq \overline{\kappa}_{\mathrm{BY}} > 0$, *or*

3. $\rho \geq 0$ *and* $\dfrac{\sigma_x}{\sigma_C} > \overline{\varphi}_{\mathrm{BY}}$ *and* $\kappa_x > \overline{\kappa}_{\mathrm{BY}} > 0$,

where $\overline{\sigma}_{\mathrm{BY}}$ *is defined in* (9) *and* $\overline{\kappa}_{\mathrm{BY}}$ *is defined in* (10).

Proof. Analogous to the proof of Proposition 6.