Search Advertising: Budget Allocation Across Search Engines

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Abstract

In this paper, we investigate advertisers' budgeting and bidding strategies across multiple search platforms. Our goal is to characterize equilibrium allocation of a limited budget across platforms and draw managerial insights for both search engines and advertisers. We develop a model with two search engines and budget-limited advertisers who compete to obtain advertising slots across search platforms. We find that the degree of asymmetry in advertisers' total budgets determines the equilibrium budget allocations. In particular, when advertisers are symmetric in their total budgets, they pursue asymmetric allocation strategies and partially differentiate: one advertiser allocates a higher share of its budget to one of the search engines, and the other allocates the same higher share of its budget to the other search engine. This partial differentiation in budget allocation strategies occurs when two forces are balanced: a demand force arising from a desire to be present on both platforms in order to obtain a greater number of clicks, and a strategic force driven by a desire to be dominant on at least one platform in order to obtain clicks at a lower cost. Thus budget allocation is a balancing act between getting more clicks and keeping costs low. When advertisers have highly asymmetric budgets, however, the second force is absent since the low-budget advertiser cannot dominate either platform. In this case, we find that the unique equilibrium for the advertisers is to allocate their budgets proportional to the traffic of each platform.

Keywords: Search Advertising, Advertising Budgets, Differentiation, Competitive Strategy, Auctions, Bid Jamming, Game Theory

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1 Introduction

Search advertising has become one the most prevalent types of online advertising. There are multiple platforms such as Google, Bing, Yahoo, Facebook, Twitter, and Amazon, etc. through which advertisers can reach their customers. Most of these platforms employ variations of second price auction to allocate advertisers to available ad spaces. These advertisers are concerned with many issues beyond the bid price: what keywords to bid on, whether to use exact match or variants of broad match, whether to target geographically or by device type, time of day/week for advertising, and start and end time for campaigns, all of which broadly constitute the theme of search engine advertising. Another important decision that advertisers should make is the budget allocated to each platform. This naturally raises questions such as: Should a limited budget be allocated all to one search engine or split across search engines? If it is to be split, what fraction of the budget should be allocated to each search engine? And, does it matter what a competing advertiser does? These questions are managerially relevant. In this paper we derive equilibrium budget allocations by competing advertisers when multiple search engines are available for them to use. Our goal is to obtain results that provide normative guidelines to managers.

Of course the presence of two search engines leads to an allocation problem for advertisers only if their budgets are limited. Every day, millions of internet users employ general search engines such as Google, Yahoo and Bing to find their desired product and services.[1] As a result, the number of clicks that these platforms are able to generate can potentially be very high. Based on Google Adwords keyword planner tool, there are about one million monthly searches for the single keyword "flowers" in the United States. This tool also predicts that, if an advertiser puts in a bid

[1] Many other *specialized* platforms such as Amazon, Expedia, Priceline, Facebook and etc. have also adopted search-based advertising to monetize their traffic. This paper's implications will also apply for those platforms.

of $5 for this keyword, it receives 57,571 daily impressions and 672 clicks, resulting in an average daily cost of $2360. Likewise, Bing Ads keyword planner tool predicts that, with the same bid of $5, an advertiser receives 9,259 impressions and 135 clicks, resulting in an average daily cost of $229. These advertising costs are way beyond the amount that most advertisers in the flower industry are able, or are willing, to spend on their online advertising campaigns on a daily basis. As a consequence, in practice, advertisers are *budget-constrained*: they have a limited amount to spend on search advertising. In turn, this means that managers must decide how to allocate their limited budget across major search engines.

It turns out that search engines, while keen on attracting advertisers, also want them to control their costs that likely weigh heavily on managers' minds. Even with pay per click, managers can measure costs more precisely than returns, if for no other reason than the challenging task of attributing revenues to one marketing instrument when consumers are certainly influenced by multiple marketing activities. Regardless of their motivations, it is a fact that all major platforms require advertisers to set a daily budget before starting their advertising campaigns. The daily budget serves as an upper bound on the amount that an advertiser would pay to a search engine for clicks it receives on a particular day.[2]

The budgets affect bidding strategies, and through that profits for both advertisers and search engines. The intuition is that after the advertiser fully exhausts its budget, its ad is not displayed anymore, and this in turn provides an opportunity for other remaining advertisers to move up and obtain better positions potentially at lower costs. This implies that budget restrictions

[2] One may wonder why platforms "force" advertisers to set the budget. Quite possibly, not doing so can cause platforms ill will. There is the possibility that advertisers complain that "we did not want to pay this much", or "clicks are fraudulent or repetitive" Therefore, by forcing advertisers to set budgets, platforms can offer "peace of mind" to advertisers, assuring them that they would not pay more than what they really intend. So, in practice, the use of a budget constraint to go along with generalized second price auction is something that, as Edelman says, "emerged in the wild".

of advertisers can have strategic effects on their bidding behavior and profits, even in the presence of only a single search engine. When an advertiser's budget is limited, a lower ranked rival (with a lower bid) may have an incentive to strategically increase its bid in order to raise the cost of the advertiser just above it. A lower-ranked advertiser, by raising its bid, can cause the higher-ranked advertiser's budget to be exhausted quickly. As a result, the lower-ranked advertiser can move up to obtain a better position and receive more clicks without raising its cost per click. This is because in second price auctions the cost that an advertiser incurs for each click is the bid of the advertiser just below it. With two search engines there is an additional strategic element. By moving some of the budget from one search engine to the other, an advertiser's profit will increase in the latter platform and will decrease in the former. The total profit *across* two platforms, however, may increase or decrease, depending on the rival's allocation of budget and bidding behavior. Thus, the allocation decision is part of competitive strategy for each advertiser. We use a game theoretic framework to analyze this competitive interaction.

1.1 Related Research

Our research falls into the broad stream of work on search engine advertising. Previous research has studied several strategic issues related to search advertising such as advertisers' bidding behaviour (Edelman et al. 2007, Varian 2007), interaction between sponsored and organic links (Katona and Sarvary 2010), search engine optimization (Berman and Katona 2013), buying competitors' keywords (Desai et al. 2014 and Sayedi et al. 2014), hybrid auctions (Zhu and Wilbur 2011) and contextual advertising (Zhang and Katona 2010). What all of these works have in common is that they consider a single platform environment. We are interested in the problem of search advertising strategies when advertisers can use more than one platform. By deriving the

equilibrium budget allocations across search engines, we add to extant work, thus obtaining new insights into advertisers' strategies.

Prior research in the auction literature has also examined different auction formats with budget-constrained bidders (Che and Gale 1998 and 2000, Benoit and Krishna 2001, Bhattacharya et al. 2010, Borgs et al. 2010, Dobzinski et al. 2012). The common assumption in this stream of literature is that if a bidder wins the item but cannot afford to pay for the item, it will incur a very high negative utility. The results of these works do not inform search engine advertising as it is possible for an advertiser to bid high, and win a high slot to enjoy clicks to the point that his budget is exhausted. Ashlagi et al. (2010) extend the interesting concept of Generalized English Auction (GEA) developed by Edelman et al. (2007) to incorporate advertisers' budget constraints and show that there exists a unique equilibrium. Although theoretically appealing, GEA is different from practice and thus their results are not readily applicable to real-world search engine auctions with budget-limited advertisers.

With respect to the way we account for the budget-limited advertisers in position auctions, our work is closest to Koh (2014), Lu et al. (2015) and Shin (2015). Their analysis shows that an equilibrium bidding outcome could result in what has come to be known as bid jamming: one advertiser bids just below its rival with intention to exhaust the rival's budget quickly. Some interesting counter-intuitive results emerge from these analyses: for example, an advertiser may bid higher than its valuation (Shin 2015); advertiser's and search engine's profit could be decreasing in budgets (Lu et al. 2015), and a search engine's revenue with budget-constrained advertisers may be larger than its revenue without budget constraints (Koh 2014).

Since our research focuses on advertisers' budgeting and bidding strategies in the presence of multiple platforms, it is also related to the stream of literature studying competing parallel

auctions and the competition among sellers in design of auction procedures (McAfee 1993, Peters and Severinov 1997, Burguet and Sákovics 1999, Gavious 2009, Haruvy et al. 2008, Ashlagi et al. 2013, Taylor 2013). This literature investigates how the design of an auction can influence bidders' choice of participation in the auction. In the context of search engine advertising, Ashlagi et al. (2011) consider a model with two simultaneous VCG advertising auctions with different CTRs where each advertiser chooses to participate in a single auction. Chen et al. (2012) consider second price auctions with different quality score mechanisms and show that the auction with a more favorable policy for less efficient bidders tends to attract more of these bidders. In this paper, we abstract away from auction design and fix the search engine's allocation and payment rule as it is practiced in real-world.

1.2 Preview of Results

We analyze a two-stage game with two search engines and budget-limited advertisers who compete to obtain advertising positions on these platforms. Our analysis confirms bid jamming to be an equilibrium strategy with limited budgets. We also obtain interesting results on advertisers' bidding and ranking outcomes. We find that the equilibrium bid is increasing in advertising budgets. Moreover, the equilibrium bid is such that a low-budget advertiser is indifferent between bidding just above the rival and just below it. We also find that advertisers' equilibrium profits are increasing in own budget but decreasing in rival's budget.

What is even more interesting is our finding about budget allocation strategies. Even if advertisers are *symmetric*, we find that they pursue *asymmetric* allocation strategies across platforms. In other words, advertisers *partially differentiate*. This differentiation results in an equilibrium such that one advertiser allocates a higher share of its budget to one of the platforms, and the other advertiser allocates the same higher share of its budget to the other platform if

platforms are also symmetric. The intuition behind partial differentiation in budget allocation strategies is that it balances two forces; (1) a demand force, arising from a desire to obtain a greater number of clicks, pulls advertisers towards each other, and (2) a strategic force, driven by a desire to obtain clicks at a lower cost, creates the differentiation in allocation strategies.

Partial differentiation remains an equilibrium allocation strategy even if we consider the real-world environment of asymmetry in platform traffic. Differentiation by symmetric advertisers in this case has them allocating higher budgets to the platform that generates higher traffic. Our analysis shows that the key determinant of equilibrium allocations is the asymmetry in advertisers' total budgets. In particular, the partial differentiation strategy does not obtain if the degree of asymmetry in advertiser budgets is sufficiently high. In this case, we find that the unique equilibrium for the advertisers is to allocate their budgets proportional to the traffic of each platform. Finally, in addition to heterogeneity in search engine traffic and advertisers' budgets, we also examine the effect of the heterogeneity in advertisers' valuation for a click. We find that budget and valuation are two sides of the same coin; advertiser partially differentiate if their valuations are close enough, and allocate proportional to each platform's traffic if their valuations are sufficiently heterogeneous. Taken together, our analysis tells an advertiser how to tailor its strategy depending on the rival's budget and valuation relative to its own.

The rest of this paper is organized as follows. We describe our general model and its sub-components in §2. To understand the main forces that are in place, we first analyze a fully symmetric model in §3 and then in §4, we investigate the role of asymmetries in platform traffic, advertiser total budgets and their valuations. Finally, we conclude in §5 with a managerial discussion of our results and suggest directions for future research.

2 The Model

We consider a market with two search engines each of which is a platform for search advertising.[3] Denote the two search engines by SE_j, $j \in \{1, 2\}$. Throughout the paper, we refer to them as "search engine *j*", SE_j or "platform *j*" interchangeably.[4] To keep the model tractable, we assume that each platform offers only one advertising slot. This means that only one of the advertisers is able to advertise at a given time and customers observe only one ad. This single-slot assumption helps us to capture the main forces in place without needlessly complicating the model.

Search engines can differ in their abilities to generate clicks for advertisers. We capture this reality by defining c_j to be the *click volume* of SE_j. The click volume is the potential number of clicks that the ad slot can receive on a daily basis. This number usually depends on the size of customer base of each platform. For example, Google has bigger user base and higher incoming traffic than the Bing network and thus it can generate more clicks for an advertiser (keeping all other thing equal). Based on Google Adwords (respectively, Bing Intelligence) keyword planner tool, we can say that an advertiser that puts a bid of $5 for the keyword "*Flowers*" can receive 672 on Google (respectively, on Bing 135) daily clicks. It is useful to define the ratio $c_j/(c_1 + c_2)$ as the "*attraction"* of platform j. We later see that this ratio plays an important role in shaping advertisers' equilibrium behavior.

We assume that there are two advertisers, denoted by A_i, $i \in \{1, 2\}$, competing for the advertising slots. Advertiser i is characterized by two dimensions; valuation v_i, and total budget

[3] This is a plausible assumption since in the United States, search engines constitute a duopoly with Google and Y!Bing network holding approximately 65% and 20% market share, respectively. It is also a duopoly in other countries such as China (*Baidu* has 55% and *Yahoo 360* has 28%), Russia (*Yandex* has 58% and *Google* has 34%) and Japan (*Google* has 57% and *Yahoo* has 40%). Source: http://goo.gl/YKkbdq.

[4] We consistently use the index "*j*" to refer to search platforms, and "*i*" to refer to advertisers.

T_i. The valuation v_i is the Advertiser i's expected value for each click. This value can be thought of as the expected net margin from a purchase, taking into account the purchase probability. The second dimension is the "total" budget T_i, which is the maximum amount of money that Advertiser i is able to spend for search advertising over two platforms on a daily basis. An advertiser is said to have *limited* (or *exhaustible* or *constrained*) total budget if its total budget T_i satisfies $T_i < c_j v_i$, for $j \in \{1,2\}$. These inequalities imply that an advertiser with limited budget is not able to pay for all potential clicks in a day at a price equal to its valuation for a click. If both of these inequities do not hold, the advertiser's total budget is said to be *unlimited* (or *inexhaustible* or *unconstrained*). We maintain the assumption that advertisers' total budgets are limited throughout the paper. We assume that advertisers' total budgets (T_i) and valuations (v_i) are exogenously given and are common knowledge. The fixed advertising budgets are a common practice in the industry and literature. [5] Moreover, with numerous online tools for keyword research, firms can obtain enough information on the amount of money their competitors assign to online advertising campaigns.[6]

We assume that search engines use second-price auction to assign ad slot to advertisers. In the beginning of the day, the advertiser with the higher bid wins the slot and starts receiving clicks.[7] For each click, it pays an amount equal to the other advertiser's bid. Depending on its budget and the bids, it is possible that the advertiser exhausts its budget before receiving all clicks in the day. According to the common practice in the industry, if the advertiser runs out of budget, it cannot participate in the auction for the remaining traffic in the day. Therefore, the other advertiser takes over the ad slot, starts receiving the remaining clicks in the day, and pays the reserve price r_j for

[5] Shin (2015), Lu et al. (2015), and Sayedi et al. (2014) also have modeled advertisers assuming exogenously given and limited budgets.
[6] For example, *www.spyfu.com* claims that it provides competitors' keywords, bids and daily budgets.
[7] In practice, search engines weight advertisers' bids by their *quality scores*, which is measure of ad relevance, landing page quality, and expected click-through rate. We abstract away from quality score mechanism for simplicity since it does not affect our result.

each click that it receives. This implies that the low bidder might be able to enjoy the ad slot at a lower price whenever the high bidder runs out of budget. We assume that reserve prices r_j satisfy $0 < r_j < v_i$. This assumption guarantees that advertisers have enough incentive to participate in search advertising and to bid for ad slots. An example will help clarify how limited budgets affect the advertisers' rankings and profits.

Example 1. Suppose SE_1 can generate $c_1 = 120$ daily clicks for its ad slot. Suppose each advertiser has a valuation $v = 1$ for each click. Moreover, assume Advertiser 1 and 2 allocate budgets of, respectively, 50 and 40, to this search engine. Finally, suppose advertisers' bids are $b_1 = 1$ and $b_2 = 0.5$, and the reserve price in that platform is $r = 0.1$. Since $b_1 > b_2$, Advertiser 1 gets the slot and starts receiving clicks. For every click it receives, Advertiser 1 should pay $b_2 = 0.5$ to the platform. Clearly, Advertiser 1's budget of 50 is depleted after receiving 100 clicks. At this time, Advertiser 2 takes over the slot, starts receiving the "remaining" 20 clicks, and pays $r = 0.1$ for each click. Thus, advertisers' profits are $\pi_1 = 100(1 - 0.5) = 50$, and $\pi_2 = 20(1 - 0.1) = 18$.

The example makes clear how both the budgets and the bids together determine the advertisers' payoffs. Therefore, a strategic decision for the advertisers is how their total budget is split across the search engines. Denote δ_i , $0 \leq \delta_i \leq 1$ to be the fraction of total budget T_i that Advertiser i allocates to SE_1. This implies that Advertiser i allocates $B_i^1 \triangleq \delta_i T_i$ to SE_1, and $B_i^2 \triangleq (1 - \delta_i)T_i$ to SE_2.[8] We call δ_i the Advertiser i's *allocation strategy*. Therefore, Advertiser i's decisions consist of δ_i and b_i^j, where the latter is the advertiser i's bid on platform j. Advertisers' objectives are to maximize their total profit summed over the two search engines.

[8] Notice that we use capital "T" to refer to *Total* budget while capital "B" to refer to the allocated budget to each platform (and hence $T_i = B_i^1 + B_i^2$). We keep this notation throughout the paper consistently.

We model the strategic interaction between the advertisers as a two-stage game. In the first stage, which we call the *allocation stage,* advertisers choose their allocation strategy δ_i. In other words, advertisers decide how to split their limited budgets of T_i across two platforms. In the second stage, which we call *bidding stage,* they choose their bids b_i^j in each search engine. In each stage, we seek a Nash equilibrium, and impose sub-game perfectness. It is important to note that advertisers take as given the search engine's second-price auction rule for allocating the slot. In other words, platforms are not strategic decision makers in our model. In the next section, we characterize the equilibrium of the two-stage game assuming that advertisers are symmetric as are the two search engines. This fully symmetric model provides key insights in a transparent way. We then consider asymmetric advertisers and/or search engines. Table 1 summarizes the notation.

Table 1

Notation	Explanation
c_j	Search Engine j's Click Volume
r_j	Search Engine j's Reserve Price
v_i	Advertiser i's Valuation for a click
T_i	Advertiser i's Total Budget
b_i^j	Advertiser i's Bid in Search Engine j
δ_i	Advertiser i's Allocation Strategy
B_i^j	Advertiser i's allocated Budget to Search Engine j $B_i^1 \triangleq \delta_i T_i$ and $B_i^2 \triangleq (1-\delta_i)T_i$

3 Equilibrium Analysis for the Fully Symmetric Model

In this section, we analyze a fully symmetric model; a model with symmetric search engines and symmetric advertisers. In other words, we assume that $T_1 = T_2 \triangleq T$, $v_1 = v_2 \triangleq v$, $c_1 = c_2 \triangleq c$ and $r_1 = r_2 \triangleq r$. Our analysis proceeds backward by first finding the equilibrium for *bidding stage* given the allocation decisions.

3.1 Bidding Stage Equilibrium

In this stage, advertisers bid for the ad slot in the each search engine, conditioned on their own and their rival's limited budgets allocated to each platform in the first stage of the game. In light of symmetry, the analysis would be similar for both search engines. Once budgets have been chosen in the allocation stage, bids in one search engine do not affect the bids on the other platform. So we can analyze the bidding stage as though there is just one platform. Therefore, in this section we drop the superscript j referring to platforms, and thus denote advertiser bids and budgets simply by b_i and B_i.

To determine the equilibrium bids, first consider the case where the sum of advertisers' budgets are small enough such that they satisfy $B_1 + B_2 < cr$. In this situation, advertisers' equilibrium bids must be equal to the reserve price r. At $b_1^* = b_2^* = r$, each advertiser has a 50% chance of getting the slot in the beginning of the day. Suppose Advertiser 1 gets the slot first. It then receives B_1/r number of clicks, exhausts its budget, and leaves remaining $c - B_1/r$ number of clicks for Advertiser 2. Advertiser 2 then also cannot afford to pay for all of the remaining clicks because $c - B_1/r > B_2/r$. As a result, when bids are at the reserve price level, Advertiser i receives B_i/r number of clicks, regardless of whether it gets the slot first or second. Therefore, deviating to higher bids does not increase an advertiser's profit. This is because both number of clicks it receives, B_i/r, and the margin on each click, $v - r$, remain unchanged.

Consider now the case when the reserve price is relatively small, i.e. $r < (B_1 + B_2)/c$. In this case, bidding at reserve price cannot be NE since the advertiser who gets the slot second will be left with some extra budget at the end of the day. Therefore, it will have an incentive to increase its bid from reserve price in order to be the first advertiser who is assigned to the ad slot. Of course, if an advertiser deviates to a bid slightly higher than r, its rival will also respond by increasing its

bid. As a result, one could conjecture that the equilibrium bids will be higher than the reserve price. We construct these bids in two steps.

First, we note that it is always a weakly dominant strategy for the advertiser with lower bid to increase its bid to just below its rival's. Intuitively, by increasing its bid, lower-bid advertiser increases the cost for its rival. Consequently, higher-bid advertiser's budget will be depleted faster and lower-bid advertiser will receive more remaining clicks. This type of strategic bidding to exhaust the rival's budget has been referred to in the literature as *bid jamming* or *aggressive bidding*. For expository purposes, henceforth we call the advertiser who has the lower bid, just below the bid of higher-bid advertiser, the *jammer*. The higher-bid advertiser, who bids higher and gets the advertising slot initially but is jammed by the jammer, is denoted as *jammee*.

Second, we find that the High-budget advertiser jams the Low-budget one in equilibrium. To see this, it is useful to consider an advertiser's revenue and cost separately. In terms of revenue, both advertisers' incentives are identical. In fact, the *difference* between an advertiser's revenues when being a jammer versus a jammee does *not* depend on the advertiser's type (High- or Low-Budget). The difference in costs, however, does. A jammer pays a smaller cost to platform (since reserve price is small) whereas a jammee exhaust its budget fully. Consequently, a High-Budget advertiser has greater incentive to be the jammer as it pays less compared to the Low-budget advertiser. Given these two facts, we are able to compute equilibrium bid level. We summarize our results of this section in the following Lemma.

Lemma 1. *Suppose advertisers have limited budgets* $(B_1, B_2 < cv)$*, and let* $B_H = Max(B_1, B_2)$ *and* $B_L = Min(B_1, B_2)$*,*

(i) If reserve price is relatively high, $r > (B_1 + B_2)/c$*, then equilibrium bids and profits are* $b_i^* = r$ *and* $\pi_i^* = \tilde{\pi}(B_i)$*, where,* $\tilde{\pi}(B_i) \triangleq B_i(v - r)/r$*.*

(ii) If reserve price is relatively low, $r \leq (B_1 + B_2)/c$*, then equilibrium bids are* $b_L^* = b^*$*,* $b_H^* = b^* - \epsilon$*, and equilibrium profits are* $\pi_L^* = \underline{\pi}$ *and* $\pi_H^* = \overline{\pi}$*, where,* $b^*(B_1, B_2) \triangleq \frac{(B_H+B_L)v-B_H r}{c(v-r)+B_L}$, $\underline{\pi}(B_1, B_2) \triangleq \frac{B_L(cv-B_H)(v-r)}{(B_H+B_L)v-rB_H}$ *and* $\overline{\pi}(B_1, B_2) \triangleq \frac{\{cvB_H-B_L^2-cr(B_H-B_L)\}(v-r)}{(B_H+B_L)v-rB_H}$.

Lemma 1 fully characterizes the equilibrium bids and profits of advertisers, given their allocated budgets to a platform. One interpretation of search engine's reserve price r can be the level of outside competition. In other words, there might be other advertisers who are not strategic and their bids for the ad slot is always fixed at r. If the level of outside competition is high and so the reserve price r is high, then advertisers' bid will also be r. In this case, each advertiser's profit is a linear function of its budget and does not depend on the rival's budget. On the other hand, when r is relatively low, both equilibrium bids and profits are influenced by advertisers' budgets. In particular, High-budget advertiser jams the Low-budget advertiser and the bid level decreases with budgets. In fact, advertisers shade more (from bidding their valuations) when their budgets become more constrained. Furthermore, advertiser profits are increasing in their own budget but decreasing in the rival's budget, i.e. budgets are strategic substitutes.

The analysis of this section demonstrates that the budget in a search engine determines both equilibrium bid and advertisers' profits in that search engine. In particular, an advertiser's allocation of budget across search engine platforms has strategic effects on its own and the rival's profit. By moving the budget from one platform to the other, its effect on the profits across search engines is in opposite directions, and so the net effect is not obvious but must be analyzed explicitly. We do this next by working backwards to characterize the equilibrium allocation strategies.

3.2 Allocation stage equilibrium

Denote advertisers' equilibrium allocation strategies by δ_1^* and δ_2^*. Recall that in the fully symmetric model the total budgets of the two advertisers are equal, $T_1 = T_2 = T$. Moreover, A_i allocates $B_i^1 \triangleq \delta_i T$ to search engine 1, and $B_i^2 \triangleq (1-\delta_i)T$ to search engine 2. To obtain equilibrium allocation strategies, we characterize $\rho_i(\delta)$, the advertiser i's best response to competitor's allocation strategy δ. The point of intersection of advertisers' best response functions corresponds to a Nash equilibrium.

In deriving best response functions, we exploit two types of symmetry that exist in the fully symmetric model in order to simplify exposition. First, note that full symmetry implies: advertisers' budgets are equal as are their valuations. Hence, the advertisers' best response functions should be identical. In other words, $\rho_1(\delta)$ and $\rho_2(\delta)$ are exactly the same. Therefore, we derive only Advertiser 1's best response to Advertiser 2's choice of δ_2. Second, platforms are also fully symmetric. In essence, we can swap the platform names since they are identical. Thus, it is sufficient for us to derive the best response function only for $0 \leq \delta \leq 0.5$. The second part of the graph would be a mirror image of the first part.

The explicit formula for best response function $\rho_1(\delta)$ and the details of the derivation can be found in the Appendix. Our procedure for deriving $\rho_1(\delta)$, briefly outlined is: we first write the Advertiser 1's total profit over both platforms $\pi_1(\delta_1, \delta_2)$. We note that this profit function takes on different forms, depending on the values of δ_1, δ_2. For example, if both δ_1, δ_2 are small, then the sum of budgets would be small in platfirm 1 and large in platform 2. Then according to Lemma 1, advertisers' profits in platform 1 will be of $\tilde{\pi}(.)$ form and of the form of $\overline{\pi}(.,.)$ or $\underline{\pi}(.,.)$ in platform 2. We identify five different cases covering the entire $\delta_1 - \delta_2$ space, with $\pi_1(\delta_1, \delta_2)$ differing in each case. Next, we examine how $\pi_1(\delta_1, \delta_2)$ changes with Advertiser 1's allocation

strategy δ_1 by using derivatives with respect to δ_1 in each case. Finally, we construct the best response function by comparing profit maximizing allocation strategy across cases. Advertisers' best response functions and their intersections (NE) are illustrated in Figure 1.

Let us now discuss the rationale and intuition behind Advertiser 1's best response function. Suppose Advertiser 2 has assigned all its budget to platform 2, i.e. $\delta_2 = 0$. One may think that Advertiser 1's best response then could be to maximally differentiate by assigning all of its budget to platform 1. By doing so, each advertiser can get all the clicks in their platform at the cheap reserve price. However, this will leave some part of the budget unused when reserve price is not too high. Therefore, Advertiser 1 can do better by moving the unused portion of its budget to platform 2 and earn more profit. Consequently, the best response to $\delta_2 = 0$ will be to allocate just enough budget to platform 1 to obtain all the potential clicks (at reserve price), and allocate the remaining budget to platform 2. This implies that $\rho_1(0) = cr/T$, as shown in Figure 1.

When $\delta_2 > 0$ and Advertiser 2 allocates some part of its budget to platform 1, Advertiser 1's best response will be such as to keep the price at a minimum in platform 1. Recall that based on Lemma 1, bid level will be at reserve price if and only if the sum of budgets is not larger than cr. Thus, Advertiser 1 decreases its budget in platform 1 so that the sum of advertisers' budget in platform 1 does not exceed this threshold. Since Advertiser 2 has allocated most of its budget to platform 2, the marginal benefit of budget increase for Advertiser 1 is lower in that platform. Consequently, Advertiser 1 wants to put its budget in platform 1 as long as the price is at reserve price. This implies $\rho_1(\delta_2) = cr/T - \delta_2$, which is the negatively sloped line shown in Figure 1.

With even higher δ_2, keeping the bid level at reserve price requires more reduction in budget allocated to platform 1 by Advertiser 1. This in turn makes advertisers' budgets closer to each other in *both* platforms, making competition fiercer. As a result, Advertiser 1's response

function starts to *increase* in δ_2 at some point. This is the *turning point* (kink point) in $\rho_1(\delta)$. At this point, the bid level in platform 1 starts to rise from its reserve price minimum. Moreover, Advertiser 1 now becomes jammer in platform 1 (and jammee in platform 2). This is another reason for Advertiser 1 to increase its allocation to platform 1 when δ_2 increases.

Finally, consider $\delta_2 = 0.5$, when Advertiser 2 splits its budget equally across platforms. In the Appendix, we show that $\rho_1(\delta_2 = 0.5) \neq 0.5$ and there is a discontinuity at this point (as shown in Figure 1). Thus, Advertiser 1's best response is *not* to split its budget equally across platforms. Intuitively, advertisers are willing to differentiate, at least to some degree, in order to mitigate competition. Therefore, Advertiser 1 has an incentive to move part of its budget from one platform to the other. This is because the marginal benefit of becoming a jammer in one platform outweighs the marginal loss of becoming a jammee in the other platform. This partial differentiation equilibrium is a key result of ours, and it is summarized in the following proposition.

Proposition 1: *In the fully symmetric model (symmetric platforms and symmetric advertisers) with* $cr < T < cv$*, there is a unique, up to renaming the advertisers, asymmetric pure strategy Nash equilibrium in which advertisers partially differentiate; one advertiser allocates* $\delta^* > 1/2$ *fraction of its budget to platform 1 whereas the other allocates* $\delta^* > 1/2$ *fraction of its budget to platform 2, where* $\delta^* = \frac{1}{2} + \frac{1}{4T}\{2cv - cr - \sqrt{4(cv)^2 - 4T^2 + cr(cr + 4T - 4cv)}\}$.

Proposition 1 shows how symmetric advertisers allocate their limited budgets across symmetric platforms. In particular, advertisers differentiate by focusing more on different platforms. One advertiser allocates a higher share of its budget to platform 1 and the other advertiser allocates a higher share to platform 2. However, they do not fully specialize. We can think of this as partial differentiation by platform. Since the outside competition is not too high

($cr < T$), full specialization ($\delta^* = 0\ or\ 1$) cannot be a Nash equilibrium because the extra remaining budget can be allocated profitably to the other platform. This can be thought of as a *demand force*. Advertisers are willing to allocate their budget equally across symmetric platforms in order obtain higher total number of clicks. What is interesting is that $\delta^* = 0.5$ is not equilibrium. This is because advertisers can benefit from differentiation and mitigate competition by moving a fraction of their budget from one platform to the other. The advertiser's tendency to differentiate is a *strategic force*. Differentiation can decrease the average price per click and increase the advertiser's total profit. In fact, the total adverting expenditure is minimized when one advertiser allocates its entire budget to platform 1 and the other advertiser allocates none of its budget to platform 1. The strategic force motivates advertisers to differentiate while the demand force incentivizes them to spread their budget more evenly across platforms. The outcome is partial differentiation. This is analogous to the classical differentiation literature where sellers benefit from differentiation by being able to increase price and mitigate competition. The critical difference, however, is that in classical differentiation sellers rely on heterogeneity in customer valuation or preference. However, the differentiation in our context arises through advertisers' bidding and budgeting mechanisms.

It is also useful to define the *degree of differentiation (D)* as the difference between advertisers' equilibrium allocation strategies, $D = |\delta_1^* - \delta_2^*|$, and examine how it is affected by advertiser budgets. We note that $D = 1$ when advertisers fully differentiate, and $D = 0$ when advertisers follow identical allocation strategies. The following corollary establishes how D changes with budgets.

Corollary 1. *The degree of differentiation $D = |\delta_1^* - \delta_2^*|$ is increasing in advertiser budgets T.*

What is the intuition behind corollary 1? Note that the difference between bid level and reserve price, which is the benefit of differentiation, is increasing in budget. This is because bid price is an increasing function of budgets as shown in Lemma 1. Alternatively, we can interpret this result using reserve price. When outside competition increases (r rises), the benefit from differentiation gets smaller. Therefore, advertisers differentiate less as the reserve price goes up.

4 Asymmetric Platforms and/or advertisers

In practice search engines are not identical. They differ in their ability to generate clicks for advertisers. We analyze this case in §4.1. Furthermore, advertisers may also differ in their budget constraints. Big firms such as Amazon spend millions of dollars on online advertising every year whereas small, local advertisers can afford much less. In §4.2, we take up an important extension of our model by incorporating asymmetric budgets of advertisers. Finally, we consider the consequences of advertiser valuation for click being different in §4.3. For the sake of exposition, we normalize the reserve price to zero in all three sub-sections that follow.

4.1 Asymmetry in Platform Traffic

What is the effect of asymmetry in search engines' click volumes on advertisers' allocation strategies? One may think that the platforms with higher ability to generate leads for advertisers are naturally more attractive, and hence advertisers will allocate more resources to these platforms. On the other hand, the greater the advertisers' investment in a search engine, the higher the competition and the higher the bid price in that platform. Suppose platform 1 and platform 2 can generate c_1 and c_2 click volume, respectively. Recall that the ratio $c_j/(c_1 + c_2)$ is the platform j's attractiveness. This ratio reflects the ability of search engine j to produce leads for an advertiser,

relative to the other platform. It is not clear a priori that a platform with higher attractiveness will attract a higher share of advertisers' limited budgets. This is due to fact that advertisers' valuation for a click is v, no matter which search engine it comes from. Hence, it might seem that the platforms' attractiveness should not have any effect on the allocation strategies of the advertisers. However, it turns out that the advertiser on average invests more on a platform that has higher ability to generate clicks.

To formalize the analysis, suppose advertisers' budgets are limited such that $T < c_j v$ for $j \in \{1,2\}$. This condition implies that advertiser budgets are not enough to cover all potential clicks at a price equal to their valuations in either platform, and hence budgets are exhaustible. Let us first consider the competition in a single platform. Based on Lemma 1, it can be seen that given the budgets, the equilibrium bid is decreasing in traffic. Advertiser profits, however, are increasing in a platform's traffic. This is intuitive because with more number of clicks available, the jammer will receive more number of remaining clicks at the same bid level. This will create incentive for the low-budget advertiser to deviate and undercut the equilibrium bid, effectively lowering the equilibrium bid after consequent response by the high-budget advertiser.

Therefore, the platform with more number of potential clicks seems to be more attractive for advertisers, leading to higher budget investments. The interesting result is that the partial differentiating still exists in equilibrium and advertisers follow an asymmetric strategy. If both advertisers allocate the same amount to platform 1 (and thus to platform 2), then each of them would be a jammer with probability 0.5 and a jammee with probability 0.5 in either platform. To reduce the cost per click, advertisers differentiate by deviating from symmetric strategies. The full differentiation, on the hand, is not optimal since each advertiser will have some extra budget to invest in the platform in which rival is present.

Proposition 2. *(i) advertisers together allocate higher share of budgets to the platform with higher attractiveness, i.e.* $\delta_1^* + \delta_2^* >=< 1$ *iff* $c_1 >=< c_2$. *(ii) advertisers partially differentiate; one advertiser allocates a share of its budget to platform 1 that is greater than the attractiveness of platform 1 and a share of its budget to platform 2 that is less than the attractiveness of platform 2, while the other advertiser does the opposite. Mathematically, allocation strategies* $\delta_i^*, \delta_{-i}^*$ *are not equal and satisfy* $\delta_i^* < c_1/(c_1 + c_2) < \delta_{-i}^*$. *(iii) Allocation strategies* $\delta_i^*, \delta_{-i}^*$ *solve* $\left(\frac{\delta_1^*+\delta_2^*}{2-\delta_1^*-\delta_2^*}\right)^2 = \frac{(c_1 v-\delta_i^* T)\delta_i^*}{(c_2 v+(1-\delta_i^*)T)(1-\delta_i^*)} = \frac{(c_1 v+\delta_{-i}^* T)\delta_{-i}^*}{(c_2 v-(1-\delta_{-i}^*)T)(1-\delta_{-i}^*)}$.

Proposition 2 illustrates how platform asymmetry in their ability to produce clicks influences advertisers' decision via budget allocation across platforms. First, total spending by advertisers is higher in the platform that generates more traffic. Intuitively, other things being equal, the higher number of clicks decreases the bid price in a platform, incentivizing advertisers to move their budget to that platform. This is a demand force that incentivizes advertisers' budget allocations to be correlated with the platforms' click volume. However, advertisers can still benefit by deviating from the symmetric equilibrium. This is the strategic force. Partial differentiation, through allocating relatively lower share of budget to one platform and higher share to the other, helps advertisers to benefit from dampened price competition and reduced average cost per clicks. The following example illustrates this partial differentiation.

Example 2. Suppose $c_1 = 3, c_2 = 1, T = 5, r = 0, v = 10$. Therefore, Platform 1 and Platform 2 produce respectively 75% and 25% of the total click volume in the market. The solution of the equations in Proposition 2 is, $\delta_i^* = 0.71$, $\delta_{-i}^* = 0.80$.Therefore, advertiser partially differentiate. One of them allocates 80% to platform 1 (higher than platforms 1's attractiveness of 75%). The other advertiser allocates 29% to platform 2 (higher than platforms 2's attractiveness of 25%).

Another interesting question is how a platform's attractiveness can influence the bid level. Suppose platform 1's traffic (c_1) increases. The bid level first decreases since the jammee finds it attractive to decrease its bid. But, the bid level might then increase since advertisers allocate more of their budgets to platform 1, which in turn intensifies the competition and increases the bid. Hence, the net effect of an increase in platform 1's traffic on its bid level is not very clear. Our analysis shows that the bid levels in both platforms are decreasing in platform traffic. However, the equilibrium bid level is higher in the platform that is more attractive, i.e. has higher traffic. Note that a platform's revenue is equal to the budget of the jammee. Therefore, Proposition 2 predicts that a platform's revenue increases when its traffic increases, while its rival loses revenue.

4.2 Asymmetry in Advertiser's Total Budget

In practice, advertisers' budgets may not be equal. The total amount of budget assigned for online advertising, and in particular search engine advertising, might vary depending on the firm's total sales or profit, or other considerations and so we ask: how does asymmetry in advertiser total budgets affect their allocation strategies in equilibrium?

To understand the effect of budget heterogeneity, it is useful to start from the single platform results in Lemma 1. In particular, from the profit functions of jammee and jammer, it can be observed that the sum of the advertisers' profit is increasing in *budget asymmetry*. To be more specific, suppose $B_H = B + \epsilon$ and $B_L = B - \epsilon$, where the variable ϵ captures the budget heterogeneity. If ϵ is zero, budgets are equal. The budget heterogeneity increases when ϵ increases. Since in the single platform setting with zero reserve price we have $\pi_H + \pi_L = cv - B_L$, we can conclude that the advertisers' total profits (industry profit) increase when their budget heterogeneity increases. In fact, this is the main reason that symmetric allocation strategy is not a

Nash Equilibrium in the full symmetric setting in section 3 noting that a small deviation from the symmetric equilibrium increases both advertisers' total profit across two platforms.

We now return to the allocation strategies across platforms when advertiser total budgets are heterogeneous. In particular suppose $T_H = T + \epsilon$ and $T_L = T - \epsilon$. When the heterogeneity (ϵ) is small, the symmetric strategy will result in advertiser budgets close to each other in each platform ($B_1^j \approx B_2^j$). In this situation, a reallocation of the total budget which would result in one advertiser to become jammee in one platform and a jammer in the other platform can benefit both advertisers. This deviation increases the heterogeneity *within* each platform resulting in Pareto efficiency. However, when the heterogeneity in total budgets (ϵ) is high, this may not be the case.

Suppose now that the heterogeneity in total budgets (ϵ) is high. In particular, assume that, $\frac{T_H}{T_L} > \frac{c_1+c_2}{Min(c_1,c_2)}$. In this case, we see in proposition 3 that the symmetric allocation strategy is indeed a Nash equilibrium.

Proposition 3. *If asymmetry in advertises' budgets is high enough,* $\frac{T_H}{T_L} > 1 + \frac{Max(c_1,c_2)}{Min(c_1,c_2)}$, *then there exists a unique symmetric pure strategy Nash equilibrium where both advertisers split their budgets proportional to each platform's attractiveness, i.e.* $\delta_H^* = \delta_L^* = c_1/(c_1 + c_2)$.

This proposition sheds light on the effect of budget asymmetry on allocation strategies. When the budget asymmetry is high enough, then advertisers allocate their budget proportional to the platform's click volume. In this case, the high-budget advertiser becomes a jammer in both platforms and enjoys the click at the low reserve price. The condition in Proposition 3 guarantees a high enough budget asymmetry such that the low-budget advertiser can never reallocate its budget so as to become a jammer in one of the platforms when the high-budget advertiser employs its equilibrium strategy. The high-budget advertiser, on the other hand, does not want to lose the

benefit of being the jammer in either platform. Therefore, partial differentiation does not occur when advertisers' total budgets are sufficiently heterogeneous. In fact, the strategic force that we described earlier is absent here. The reason is that industry profit cannot be improved by reallocation of the budgets since the low-budget advertiser will remain the jammee in both platforms. As a result, the demand force will be the only driver of equilibrium strategies and hence advertisers split their budgets proportional to the platform's attractiveness.

4.3 Heterogeneity in Advertiser Valuation

Having examined the effect of asymmetric platform attractiveness and asymmetric advertiser budget on allocation strategy, we next ask: what is the effect of heterogeneity in advertiser valuation,v_i, on equilibrium allocation? Recall that the valuation reflects the advertisers' margin on the product, as well as the conversion rate. So advertisers might value a click differently if, for example, one of them has a marginal cost advantage over its rival. Another reason might be the different conversion rates. For example, one website can have a better design and layout than its rival's which leads to a higher purchase rate, effectively increasing the valuation for a click.

Assume without loss of generality, $v_1 \geq v_2$. Hence, Advertiser 1 values each click more than Advertiser 2. We refer to advertisers 1 and 2, respectively, as High-value and Low-value. In the first step, we derive the equilibrium outcomes of the second stage of the game- the bidding stage- in presence of heterogeneity in advertiser valuation. These equilibrium strategies and outcomes of the bidding stage have been fully characterized in appendix, Lemma 2. We discuss the most important results below.

Our analysis shows that higher valuation for clicks, just as higher budget, can be exploited as a strategic mechanism in search advertising. Advertisers' profits are increasing in their valuations. One important and interesting finding is that the ratio of budget to valuation B_i/v_i

determines which advertiser will be the jammer. The advertiser with lower B_i/v_i gets the slot first and is jammed by the rival. Therefore, if budgets are equal, higher-value advertiser gets the slot first. This result is similar to the traditional one in the literature (Varian 2007, Edelman et al. 2007) of generalized second price auction and is replicated here: advertisers are ranked based on their valuations. On the other hand, if valuations are equal, the low-budget advertiser gets the ad slot first and is jammed by high-budget advertiser. Another important result is that the low-value advertiser may bid higher than its valuation, thus replicating in our model the result in Shin (2015). This happens in our model when the low-value advertiser has a sufficiently higher budget than its rival. In this situation, low-value advertiser, who has a large budget, leverages its budget strength to jam the high-value, low-budget advertiser at a bid even higher than its own valuation and gets the remaining clicks at reserve price.

Now we are in a position to explore the effect of heterogeneity in valuation on budget allocation across search engines. The following proposition shows that the result is similar to the one in proposition 3.

Proposition 4. *If asymmetry in advertises' valuation is high enough,* $\frac{v_H}{v_L} > 1 + \frac{Max(c_1,c_2)}{Min(c_1,c_2)}$, *then there exists a unique symmetric pure strategy Nash equilibrium where both advertisers split their budgets proportional to each platform's attractiveness, i.e.* $\delta_H^* = \delta_L^* = c_1/(c_1 + c_2)$.

Proposition 4 sheds light on the role of advertiser valuation in equilibrium allocation strategies across platforms. The similarity between propositions 3 (effect of budget heterogeneity) and proposition 4 (valuation heterogeneity) has an intuitive explanation: budget and valuation are two sides of the same coin. If heterogeneity in valuations is sufficiently high, then in equilibrium

advertisers allocate their budgets proportional to the platform's attractiveness. What is more relevant, the high-value advertiser will be the jammee in both platforms and gets the ad slot first.

5 Managerial Insights, Conclusions and Future Research

Our results provide guidance to managers that are concerned with the rising cost of search advertising. The first question we posed at the outset is: Should a limited budget be allocated all to one search engine or split across search engines? Our finding is that a firm should always split its budget across search engines. Next, what fraction of the budget should be allocated to each search engine? We show that an advertiser should consider concentrating more resources in one search engine while its rival concentrates on the other search engine. In particular, a firm facing a rival with an advertising budget that is close to its budget should consider differentiating by focusing on one platform more than its rival. Thus, we find that indeed how a firm allocates its budget does depend on what its competitor does. Our analysis has provided answers to the questions we wanted to answer.

Of-course the advertiser should also take into account the attractiveness of each search engine. What that means is relative to attractiveness it should allocate a higher share of its budget to one platform while its rival focuses on the other platform. This sort of differentiation is key to keeping costs of search advertising low. If the platforms have identical attractiveness, the firm should actually spend more on one platform while the rival spends more on the other platform. For this sort of differentiation to be profitable, managers must ensure that those in charge of the bidding decision understand that the strategy is to be not the high bidder on the platform that the firm is focusing on in the budget allocation. It is useful to think of the focus on platform through budget and bids are negatively correlated.

A firm that has a huge budget advantage should allocate its budget proportional to each platform's attractiveness. And this allocation should be followed up by a not too aggressive bidding strategy. On the other hand, a firm that has a disadvantage in its budget should also allocate its budget proportional to the platform's attractiveness but follow it up with a more aggressive bidding strategy.

Managers also need to understand the role of valuation of a click by their competitor relative to their own. These valuations are usually connected to the profit margins of selling products. Keeping every other thing equal, an advertiser who can sell its product at a higher margin will value a click more than its rival advertiser. The valuations for each click together with budget constraints jointly determine the equilibrium behavior of advertisers. For example, when facing a rival whose valuation for each click is sufficiently higher than its own, a firm should allocate its budget proportional to a platform's attractiveness and not differentiate. Moreover, it should bid aggressively with the intention to deplete its rival's budget, if it is exhaustible, of course.

By focusing on a multi-platform environment and on advertisers' budgeting decisions we have analyzed an aspect of search engine advertising that has received relatively less attention in prior work. Our analysis is based on a game theoretic model of advertisers in which a firm faces a rival who may be asymmetric in its budget or valuation. And the firm must allocate its budget across platforms that may be asymmetric in their traffic. We exploit the fact that bid jamming occurs in equilibrium: high-budget advertiser bids infinitesimally below low-budget advertiser and then use it to analyze the budget allocation strategy. The bidding allows an advertiser to strategically leverage its budget power to earn clicks at lower costs.

An important result emerging form our analysis is that firms can profitably differentiate through their budget allocation strategy especially when competitors have approximately equal

budgets. Indeed, we identify the equilibrium to be one of partial differentiation. This occurs by advertisers focusing on different platforms: focusing consisting of allocating a share of budget higher than a platform's attractiveness, even while allocating greater amount to the platform with a higher attractiveness. This sophisticated differentiation strategy helps reduce the bid competition, lowering cost of search advertising and in turn boosting profits.

We also find that the simple rule of allocating a share of budget equal to a platform's attractiveness is the equilibrium strategy when firms differ sufficiently in their budgets. In other words, differentiation is worthwhile when competitors are similar, an intuitive finding.

Our paper could inform empirical work. First, we expect budget constrained advertisers to bid close to each other. Second, we expect that advertisers' differentiation strategy across platforms to be negatively correlated with the degree of asymmetry in their budget-to-valuation ratio. We think these predictions could be tested with appropriate data though we also realize the challenges involved in that because many factors in addition to budgets, valuations and platform attractiveness may influence advertisers' bidding and budgeting decisions.

Our work can be extended in different ways. While we have assumed only two advertisers, in practice there are usually multiple advertisers for each keyword. Moreover, there are multiple ad spots sometimes approaching as many as ten. So a more elaborate model could be appropriate depending on the goal of the research. Another interesting aspect of search engine advertising is the variation in the auctions employed, such as automatic bidding, targeted ads, quality score mechanism, and hybrid auctions to name a few.

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Figures

Figure 1. Best Response Functions in Fully Symmetric Model
$(T = 5, r = 2, c = 1,\ v = 10)$

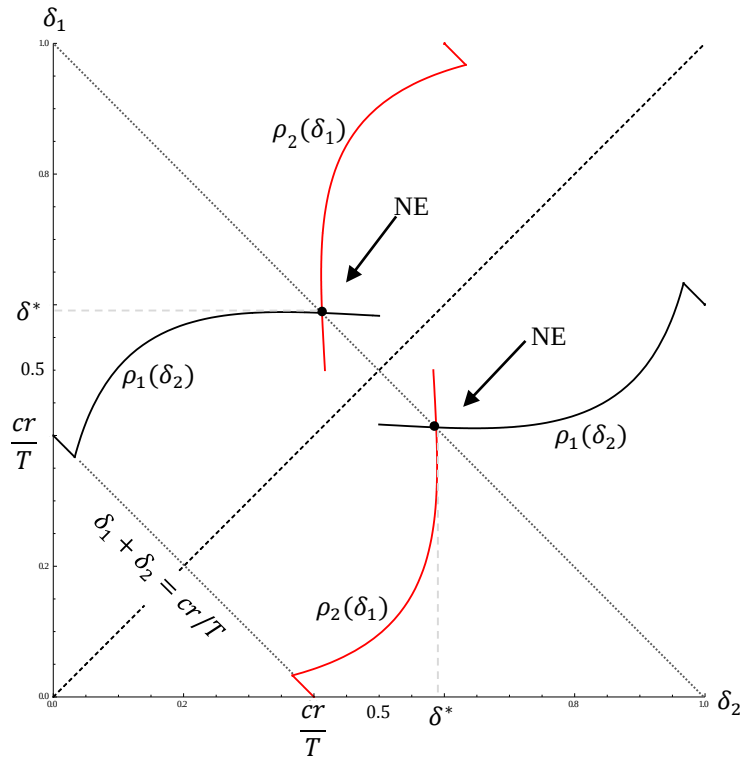
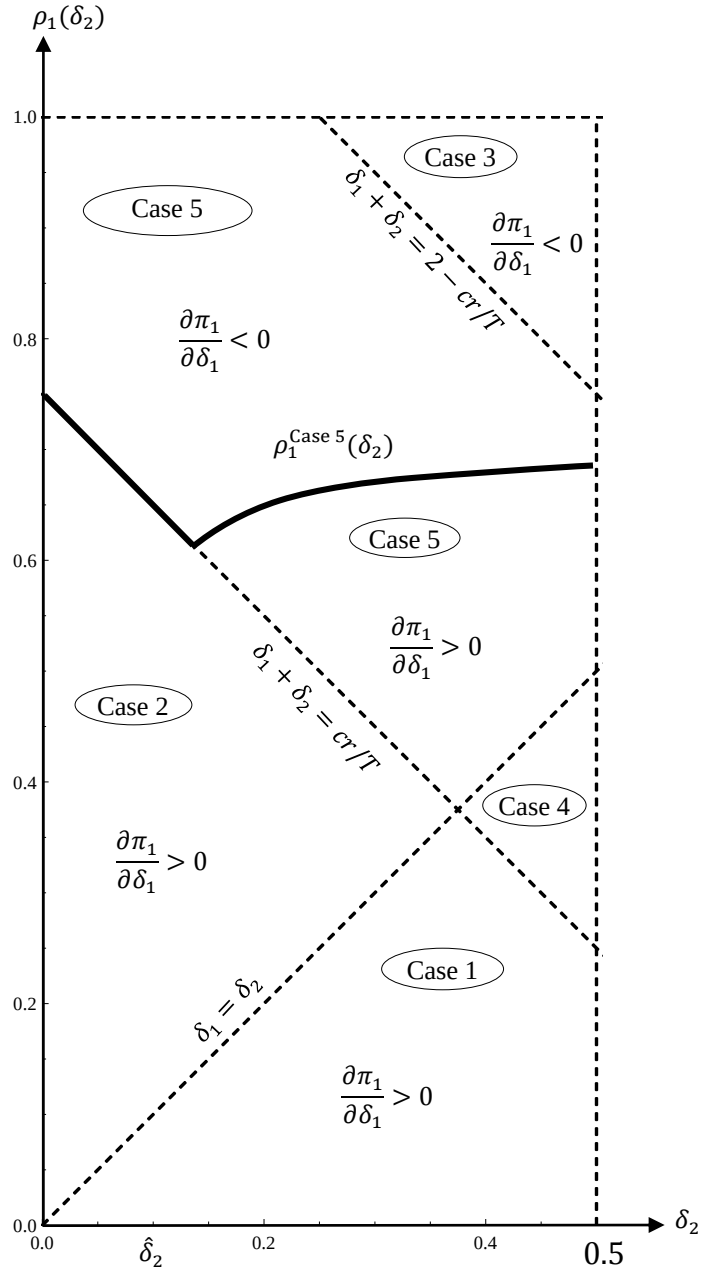


Figure 2. Best Response Function (Solid Line) and Different Regions (Dashed Lines) in Fully Symmetric Model ($T = 4, r = 3, c = 1, v = 10$)



Appendix

Proof of Lemma 1

We first characterize advertisers' profit function (in a single platform) given their bids and budgets. In general, there are five possibilities; (1) advertiser i gets the ad slot first and remains there for the entire day, (2) advertiser i gets the ad slot first and stays there only for a part of the day, because its budget is limited, (3) advertiser i gets the ad slot after its rival in the middle of the day, and stays there until the end of the day, (4) advertiser i gets the ad slot after its rival in the middle of the day, but it cannot afford to stay until the end of the day, and (5) advertiser i never gets the ad slot. Hence, advertiser i's profit π_i as a function of its bid given the rival's bid can be written as,

$$\pi_i(b_i|b_{3-i}) = \begin{cases} c(v - b_{3-i}) & b_i > b_{3-i}, \quad B_i > cb_{3-i} \\ \frac{B_i}{b_{3-i}}(v - b_{3-i}) & b_i > b_{3-i}, \quad B_i \leq cb_{3-i} \\ \left(c - \frac{B_{3-i}}{b_i}\right)(v - r) & b_i < b_{3-i}, \quad B_{3-i} < c\, b_i, \quad B_i > \left(c - \frac{B_{3-i}}{b_i}\right) r \\ \frac{B_i}{r}(v - r) & b_i < b_{3-i}, \quad B_{3-i} < c\, b_i, \quad B_i \leq \left(c - \frac{B_{3-i}}{b_i}\right) r \\ 0 & b_i < b_{3-i}, \quad B_{3-i} \geq cb_i \end{cases} \tag{A1}$$

The first case in (1) describes the situation where advertiser i's bid is higher than its rival, and it has a high enough budget. In this case, it receives all of the c potential clicks and pays b_{3-i} for each click. The last case in (1) captures the reverse situation: when the advertiser i's bid is lower than its rival, and the rival has a high enough budget. In this situation, advertiser i can never get the ad slot during the day, and hence its profit is zero.

All other three middle cases (second, third and fourth) describe the scenarios where advertiser i gets only a part of the traffic. In the second case, A_i is bidding higher than its rival and hence gets the slot first. Advertiser i then should pay b_{3-i} (the rival bid) for every click it receives, and so its budget is depleted after it receives B_i/b_{3-i} number of click. As a results, its profit would be as in the second line of (1). In the third and fourth cases, A_i is bidding lower than its rival and hence will receive the "remaining" clicks in the day, after its rival exhausts its budget. In these two cases, A_i pays the reserve price r for each of these remaining clicks that it receives. If A_i's budget is large enough, it receives all of the renaming clicks (third case). Otherwise, advertiser i only afford for B_i/r number of the remaining clicks (fourth case).

We also need to specify how search engines assign the ad slot to the advertisers when they bid equally for the ad slot (tie-breaking rule). In this situation, platform assigns the ad slot randomly to the advertisers. Therefore, we define the advertiser i's profit when its bid is equal to the rival's bid as follow,

$$\pi_i(b_i|b_i) = \frac{1}{2}\lim_{\epsilon\to 0}\{\pi_i(b_i+\epsilon|b_i) + \pi_i(b_i-\epsilon|b_i)\} \tag{2}$$

The equation (2) implies that there is 50% chance that the ad slot be assigned to the advertiser i when the bids are equal. In general, the profit function (1) is discontinuous at equal bid levels. Therefore, equation (2) is required for our analysis in order to understand an advertiser's incentive to bid at, above or below its rivals' bid.

To determine the equilibrium bid levels, first consider the case where advertiser's budgets are small enough such that they satisfy $B_1 + B_2 < cr$. This condition can also be re-written as $r > (B_1 + B_2)/c$, implying that the platform's reserve price r is relatively high. In this situation, we show that advertisers' equilibrium bids must be equal to the reserve price r. Recall that if $b_1^* = b_2^* = r$, each advertiser has a 50% chance of getting the slot in the beginning of the day. Suppose Advertiser 1 gets the slot first. It then receives B_1/r number of clicks, exhausts its budget, and leaves remaining $c - B_1/r$ number of clicks for Advertiser 2. But Advertiser 2 then also cannot afford to pay for all of the remaining clicks because $c - B_1/r > B_2/r$. As a result, when bids are at the reserve price level, advertiser i receives B_i/r number of clicks, regardless of whether it gets the slot first or second. Therefore, deviating to higher bids does not increase an advertiser's profit. This is because both number of clicks it receives (B_i/r) and the margin on each click ($v - r$) remain unchanged. To summarize, when advertisers' budgets satisfy $B_1 + B_2 \geq cr$, advertisers equilibrium bids are equal to the reserve price and they earn a profit of $(B_i/r\ )(v - r)$. These profits are linear in own budget and do not depend on the rival's budget.

Consider now the case when the reserve price is relatively small, i.e. $r < (B_1 + B_2)/c$. Said differently, the sum of advertisers' budgets is high enough to pay for the *all* potential clicks at the price equal to reserve price, i.e. $B_1 + B_2 > cr$. This implies that bidding at reserve price is not NE anymore since the advertiser who gets the slot second will be left with some extra budget even after receiving and paying for *all* of the remaining clicks ($c - B_1/r < B_2/r$). Therefore, advertisers have incentive to slightly increase their bid from reserve price in order to be the first advertiser who is assigned to the ad slot. Of course, if an advertiser deviates to a bid slightly higher than r, its rival will also respond by increasing its bid. As a result, one could postulate that he equilibrium bids will be higher than the reserve price.

We construct the equilibrium bids, b_1^* and b_2^*, in two steps. First, we prove that it is always a weakly dominant strategy for the advertiser with lower bid to increase its bid to just below its rival's. To see this, suppose $b_1 > b_2$ and hence Advertiser 1 will get the slot in the beginning of the day. From last three cases of the profit function (1), one can observe that $\pi_2(b_1 - \epsilon | b_1) \geq \pi_2(b_2 | b_1)$, for any ϵ that is arbitrary small. Therefore, we conclude that the equilibrium bids should be either $(b^* - \epsilon, b^*)$, where one advertiser jams the other one, or (b^*, b^*) where advertisers bid equally. We now show that the latter is possible only if $B_1 = B_2$. If (b^*, b^*) is Nash equilibrium, each advertiser gets the slot in the beginning of the day with probability 50%. Therefore, advertiser should be indifferent between getting the slot first (and be jammed by the rival), and getting it second (and jam the rival). In other words, in equilibrium bid b^*, they should be indifferent between being a jammee and being a jammer. Given that $B_1 + B_2 > cr$ and $B_1, B_2 < cv$, we should have,

$$\frac{B_i}{b^*}(v - b^*) = \left(c - \frac{B_{3-i}}{b^*}\right)(v - r) \tag{3}$$

The LHS of (3) is advertiser i's profit when it gets the slot first and is jammed by its rival at bid level b^*. The RHS is advertiser i's profit when it jams the rival at bid level b^* and gets the remaining traffic at reserve price. The equation (3) should hold for both advertisers if (b^*, b^*) is Nash equilibrium. This will result in $B_1 = B_2$. We proved that advertisers' equilibrium bids are equal, only if their budgets are equal. In other words, if one advertiser has strictly higher budget than the other, then the equilibrium bids are not equal, and hence bids would be just ϵ different from each other, and one advertiser jams the other one. But who jams whom? To answer this question, suppose in equilibrium advertiser i jams the other advertiser, i.e. $(b_i^*, b_{3-i}^*) = (b^* - \epsilon, b^*)$. This implies that in equilibrium bid level b^*, advertiser i (weakly) prefers to be a jammer, and that advertiser $3 - i$ (weakly) prefers to be jammee. Mathematically,

$$\left(c - \frac{B_{3-i}}{b^*}\right)(v - r) \geq \frac{B_i}{b^*}(v - b^*) \tag{4a}$$

$$\frac{B_{3-i}}{b^*}(v - b^*) \geq \left(c - \frac{B_i}{b^*}\right)(v - r) \tag{4b}$$

Inequality (4a) states that at equilibrium bid of b^*, advertiser i prefers to be jammer, rather than gets the slot first and become a jammee. Conversely, inequality (4b) implies that at equilibrium bid of b^*, advertiser $3 - i$ prefers to get the slot first rather than to bid just below its rival and become a jammer. Adding the two inequalities (4a) and (4b) and simplifying, we obtain $B_i \geq B_{3-i}$. In other words, the jammer should be the advertiser with (weakly) higher budget. The equilibrium bid level b^* is obtained from two inequalities (4a) and (4b), where we replace B_i and $B_{3-i}$with, respectively, B_H and B_L,

$$\underline{b^*} \triangleq \frac{(B_{\mathrm{H}}+B_{\mathrm{L}})v-B_{\mathrm{L}}r}{c(v-r)+B_{\mathrm{H}}} \leq b^* \leq \frac{(B_{\mathrm{H}}+B_{\mathrm{L}})v-B_{\mathrm{H}}r}{c(v-r)+B_{\mathrm{L}}} \triangleq \overline{b^*} \tag{5}$$

We can therefore see that our model has multiple Nash equilibria just as in the basic second price auction. Specifically, any pair $(b^*, b^* - \epsilon), b^* \in [\underline{b^*}, \overline{b^*}]$, is a Nash equilibrium. However, we argue that the only *un-dominated* Nash equilibrium is the highest bid level $\overline{b^*}$. Consider the case $b_L \in [r, \overline{b^*})$, where b_L is the low-budget advertiser's bid. Now there are two possibilities: $b_H \leq b_L$ or $b_H > b_L$. In the first case, we can see that $\pi_L(b_L|b_H) = \pi_L(\overline{b^*}|b_H)$. This is because the bid of advertiser who gets the slot first does not enter its profit function since it is a second price auction. In the second case, $\pi_L(b_L|b_H) < \pi_L(\overline{b^*}|b_H)$. Taken together this means that all bids in $b_L \in [r, \overline{b^*})$ are weakly dominated by $\overline{b^*}$ for the low-budget advertiser. We use this fact to reduce the multiple equilibria to the unique un-dominate one, $(\overline{b^*}, \overline{b^*} - \epsilon)$., which is the expression in Lemma 1. Profits for jammee and jammer are obtained by plugging this bid level into profit functions;

$$\underline{\pi}(B_1, B_2) = \frac{B_L}{\overline{b^*}}\left(v - \overline{b^*}\right) = \frac{B_L(cv-B_H)(v-r)}{(B_H+B_L)v-rB_H} \text{ and } \overline{\pi} = \left(c - \frac{B_L}{\overline{b^*}}\right)(v-r) = \frac{\{cvB_H - B_L^2 - cr(B_H-B_L)\}(v-r)}{(B_H+B_L)v-rB_H}.$$

Proof of Proposition 1

We first write the Advertiser 1's total profit over both platforms, denoted by $\pi_1(\delta_1, \delta_2)$, as a function of δ_1 and δ_2. This function takes on different forms, depending on the values of δ_1, δ_2. For example, if both δ_1, δ_2 are small, then the sum of budgets would be small in platfirm 1 and large in platform 2. Then according to Lemma 1, advertisers profits in platform 1 will be of $\tilde{\pi}(.)$ for while of $\overline{\pi}(.,.)$ or $\underline{\pi}(.,.)$ form in platform 2. We identify five different cases, illustrated in Figure 2 (separated by dashed lines), with $\pi_1(\delta_1, \delta_2)$ differing in each case. We first summarize the Advertiser 1's total profit in the five cases as follows, keeping in mind that $\delta_2 \leq 0.5$,

$$\pi_1(\delta_1, \delta_2) = \begin{cases} \tilde{\pi}(B_1^1) + \underline{\pi}(B_1^2, B_2^2) & cr/T - \delta_2 \geq \delta_1 \geq \delta_2 & case\ 1 \\ \tilde{\pi}(B_1^1) + \overline{\pi}(B_1^2, B_2^2) & cr/T - \delta_1 \geq \delta_2 \geq \delta_1 & case\ 2 \\ \overline{\pi}(B_1^1, B_2^1) + \tilde{\pi}(B_1^1) & \delta_1 \geq \delta_2 \geq 2 - cr/T - \delta_1 & case\ 3 \\ \underline{\pi}(B_1^1, B_2^1) + \overline{\pi}(B_1^2, B_2^2) & 2 - cr/T - \delta_1 \geq \delta_2 \geq \delta_1 \geq cr/T - \delta_2 & case\ 4 \\ \overline{\pi}(B_1^1, B_2^1) + \underline{\pi}(B_1^2, B_2^2) & 2 - cr/T - \delta_2 \geq \delta_1 \geq \delta_2 \geq cr/T - \delta_1 & case\ 5 \end{cases}$$

The profit functions $\overline{\pi}(.,.)$, $\underline{\pi}(.,.)$ and $\tilde{\pi}(.)$ were introduced in Lemma 1. In Case 1, $B_1^1 + B_2^1 \leq cr$, and hence Advertiser 1's profit in platform 1 is $\tilde{\pi}(B_1^1)$. Moreover, $B_1^2 + B_2^2 \geq cr$ and $B_1^2 \leq B_2^2$, resulting in $\underline{\pi}(B_1^2, B_2^2)$ in platform 2. So the Advertiser 1's total profit across platforms is $\tilde{\pi}(B_1^1) + \underline{\pi}(B_1^2, B_2^2)$. To derive the conditions for Case 1, we simplify inequalities $B_1^1 + B_2^1 \leq cr$, $B_1^2 + B_2^2 \geq cr$

and $B_1^2 \leq B_2^2$ to get $cr/T - \delta_2 \geq \delta_1 \geq \delta_2$, which has been illustrated in Figure 2. The explanations for four other cases are similar.

Having specified the Advertiser 1's total profit function in 5 different regions, we next examine how $\pi_1(\delta_1, \delta_2)$ changes with Advertiser 1's allocation strategy δ_1 and construct the best response function. Figure 2 also displays the sign of the first derivative of Advertiser 1's profit with respect to its strategy δ_1. By taking derivative, we show that $\pi_1(\delta_1, \delta_2)$ is increasing in $\delta_1$in Cases 1 and Case 2 (see Technical Appendix Claim OA1 and OA2). Since Case 3 is a mirror image of Case 2 with platforms interchanged again the derivative is positive. In Case 4 and Case 5, unlike the first three cases, the Advertiser 1's profit is not monotonic in δ_1. We rule out an equilibrium corresponding to Case 4 by showing that Advertiser 1's best response to δ_2 cannot fall in the region of Case 4. In fact, we show that any strategy in the Case 4 area is dominated by a Corresponding strategy in Case 5 (see Technical appendix Claim OA3). Thus, we conclude that the best response function $\rho_1(\delta_2)$ cannot cross the interior regions of the first four cases. The profit in Cases 1 to 3 is monotonic and thus the best response cannot fall inside these regions. Moreover, any strategy $\delta_1$inside the region of Case 4 is strictly dominated by the strategy $1 - \delta_1$, which lies in Case 5. Consequently, the best response lies inside the region of Case 5 and its boundary, as illustrated in Figure 2. We can derive $\rho_1(\delta_2)$ by setting the first derivative of $\pi_1(\delta_1, \delta_2)$ in Case 5 to zero and solving for δ_1 as a function of δ_2. In Case 5, the profit function is,

$$\pi_1^{Case\ 5}(\delta_1, \delta_2) = \frac{\left(cv\delta_1 - \delta_2^2 T - cr(\delta_1 - \delta_2)\right)(v-r)}{(\delta_1+\delta_2)v - r\delta_1} + \frac{(1-\delta_1)(cv-(1-\delta_2)T)(v-r)}{(1-\delta_1+1-\delta_2)v - r(1-\delta_2)}$$

Let $\rho_1^{Case\ 5}(\delta_2)$ be the part of the best response function of Advertiser 1 that falls inside the region of Case 5. To find $\rho_1^{Case\ 5}(\delta_2)$, we take the first derivative of $\pi_1^{Case\ 5}(\delta_1, \delta_2)$ and then solve for δ_1 as a function of δ_2,

$$\rho_1^{Case\ 5}(\delta_2) = \frac{\left(2v - r - \delta_2(v-r)\right)\sqrt{\delta_2(cv+\delta_2 T - cr)} - \delta v\sqrt{(1-\delta_2)(c\,v-(1-\delta_2)T)}}{v\sqrt{\delta_2(cv+\delta_2 T - cr)} + (v-r)\sqrt{(1-\delta_2)(c\,v-(1-\delta_2)T)}}$$

This function has been illustrated in Figure 1. Note that $\rho_1^{Case\ 5}(\delta_2)$ intersects the border of Case 2 and Case 5 (line $\delta_1 + \delta_2 = cr/T$) at $0 < \hat{\delta}_2 < 0.5$. To see this, note that, $\lim_{\delta_2 \to 0} \rho_1^{Case\ 5}(\delta_2) = 0$. Moreover, we know that $0 < \rho_1^{Case\ 5}(\delta_2)$. Finally, $\lim_{\delta_2 \to 0} \rho_1'(\delta_2) = +\infty$. Therfore, $\rho_1^{Case\ 5}(\delta_2)$ increases from zero at $\delta_2 = 0$ and intersects the line $\delta_1 + \delta_2 = cr/T$ at $0 < \hat{\delta}_2$. Moreover, $\rho_1^{Case\ 5}(0.5) > 0.5$. This implies that $\rho_1^{Case\ 5}(0.5)$ crosses above the point $(0.5,0.5)$ and hence this point cannot be NE. Finally, the best response function $\rho_1(\delta_2)$ is characterized fully as below,

$$\rho_1(\delta_2) = \begin{cases} cr/T - \delta_2 & 0 < \delta_2 < \hat{\delta}_2 < 0.5 \\ \rho_1^{Case\ 5}(\delta_2) & 0 < \hat{\delta}_2 < \delta_2 < 0.5 \end{cases}$$

The second half of $\rho_1(\delta_2)$ for $\delta_2 > 0.5$ is obtained by reflecting the first half around the point $(0.5,0.5)$. This completes our characterization of best response function $\rho_1(\delta_2)$.

Note that advertisers are symmetric and so their best response functions are similar. By intersecting these best response functions and simplifying we get, $2T\delta^{*2} - (2T + 2cv - cr)\delta^* + T + cv - cr = 0$. This second degree has only one root that satisfies $0 < \delta^* \leq 1$ is,

$$\delta^* = \frac{1}{2} + \frac{1}{4T}\{2cv - cr - \sqrt{4(cv)^2 - 4T^2 + cr(cr + 4T - 4cv)}\}$$

We can easily verify that $0.5 < \delta^* < 1$ when $cr < T < cv$.

Proof of Corollary 1

We have $D = |\delta_1^* - \delta_2^*| = 2\delta^* - 1 = \frac{1}{2T}\{2cv - cr - \sqrt{4(cv)^2 - 4T^2 + cr(cr + 4T - 4cv)}\}$.

Taking the first derivate with respect to T, we show it is always negative,

$$\frac{\partial D}{\partial T} = \frac{\partial \delta^*(T)}{\partial T} = \frac{-c^2r^2 - 2crT + 4c^2rv - 4c^2v^2 + (-cr + 2cv)\sqrt{c^2r^2 + 4crT - 4T^2 - 4c^2rv + 4c^2v^2}}{4T^2\sqrt{c^2r^2 + 4crT - 4T^2 - 4c^2rv + 4c^2v^2}}$$

The numerator is negative because,

$-c^2r^2 - 2crT + 4c^2rv - 4c^2v^2 + (-cr + 2cv)\sqrt{c^2r^2 + 4crT - 4T^2 - 4c^2rv + 4c^2v^2} < 0 \Leftrightarrow$

$(2v - r)\sqrt{c^2r^2 + 4crT - 4T^2 - 4c^2rv + 4c^2v^2} < cr^2 + 2rT - 4crv + 4cv^2 \Leftrightarrow$

$0 < 8T^2(r^2 - 2rv + 2v^2)$. Therefore, $\partial|\delta_1^* - \delta_2^*|/\partial T < 0$.

Proofs of Propositions 2, 3 and 4

Please see Technical Appendix.

Technical Appendix

Claim A1

In fully symmetric model, we have $\frac{\partial \pi_1(\delta_1,\delta_2)}{\partial \delta_1} > 0$ *in Case 1 region.*

Proof. In Case 1, the Advertiser 1's profit is,

$$\pi_1(\delta_1,\delta_2) = \tilde{\pi}(B_1^1) + \overline{\pi}(B_1^2, B_2^2) = \frac{\delta_1 T(v-r)}{r} + \frac{(cv(1-\delta_1)-(1-\delta_2)^2 T - cr(\delta_2-\delta_1))(v-r)}{(2-\delta_1-\delta_2)v - r(1-\delta_1)}$$

Let us define $\overline{\delta}_i \equiv 1-\delta_i$. Then, we take the first derivatives and obtain,

$$\frac{\partial \pi_1(\delta_1,\delta_2)}{\partial \delta_1} = \frac{v-r}{r\left((\overline{\delta}_1+\overline{\delta}_2)v - r\overline{\delta}_1\right)^2} \times f\left(\overline{\delta}_1, \overline{\delta}_2\right),$$

Where, $f\left(\overline{\delta}_1, \overline{\delta}_2\right) = (v^2 - rv + r^2)T\overline{\delta}_2{}^2 + T(v-r)^2\overline{\delta}_1{}^2 + 2v(v-r)T\overline{\delta}_1\overline{\delta}_2 - cr(v-r)^2\overline{\delta}_2$

To prove the claim, it is enough to show $f(\delta_1,\delta_2)$ is always positive in the region of Case 1 ($0 \le \delta_1 \le \delta_2 \le 0.5$ and $\delta_1 + \delta_2 \le cr/T$). First, note that $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$ is increasing in $\overline{\delta}_1$ because coefficients of $\overline{\delta}_1$ and $\overline{\delta}_1{}^2$ are positive. Second, we show that $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$ is also increasing in $\overline{\delta}_2$. To show this, we take the first derivative of $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$ w.r.t $\overline{\delta}_1$ and simplify and obtain,

$$g\left(\overline{\delta}_1, \overline{\delta}_2\right) \triangleq \frac{\partial f(\overline{\delta}_1,\overline{\delta}_2)}{\partial \overline{\delta}_2} = (2v^2 - 2rv + 2r^2)T\,\overline{\delta}_2 + 2v(v-r)T\,\overline{\delta}_1 - cr(v-r)^2$$

We show that $g\left(\overline{\delta}_1, \overline{\delta}_2\right)$is positive in the region of Case 1. The function $g\left(\overline{\delta}_1, \overline{\delta}_2\right)$ is linear in $\overline{\delta}_1$and $\overline{\delta}_2$. Therefore, it suffices to check corners of Case 1's region, which is a quadrilateral (see Figure 2).

-At the corner $(\delta_1,\delta_2) = (0,0)$, we get, $g\left(\overline{\delta}_1, \overline{\delta}_2\right) = (4v^2 - 4rv + 2r^2)T - cr(v-r)^2$, which is positive since $cr \le T$ and $(4v^2 - 4rv + 2r^2) > (v-r)^2$.

-At the corner $(\delta_1,\delta_2) = (0,1/2)$, we have, $g\left(\overline{\delta}_1, \overline{\delta}_2\right) = (3v^2 - 3rv + r^2)T - cr(v-r)^2$,which is positive for the reason similar to previous corner.

-At the corner $(\delta_1,\delta_2) = (cr/2T, cr/2T)$, we get, $g\left(\overline{\delta}_1, \overline{\delta}_2\right) = (4v^2 - 4rv + 2r^2)T - cr(3v^2 - 4rv + 2r^2)$, which is again positive.

-Finally, at the corner $(\delta_1,\delta_2) = (\frac{cr}{T} - 1/2, 1/2)$, $g\left(\overline{\delta}_1, \overline{\delta}_2\right) = (3v^2 - 3rv + r^2)T - cr(3v^2 - 4rv + r^2)$, which is again positive.

Therefore, $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$ in increasing in both $\overline{\delta}_1$ *and* $\overline{\delta}_2$ in the region of Case 1. Thus, to show that $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$ is positive everywhere in Case 1, it suffices to check positivity at minimum values of

$\overline{\delta}_1$ *and* $\overline{\delta}_2$, or equivalently, the maximum values δ_1 *and* δ_2. This happens at the corner of quadrilateral at $(\delta_1, \delta_2) = (cr/2T, cr/2T)$. At this point, we have,

$$f(\delta_1 = cr/2T, \delta_2 = cr/2T) = (1 - \frac{cr}{2T})((4v^2 - 5rv + 2r^2)T - cr(3v^2 - 4.5rv + 2r^2)$$

which is positive since $1 - \frac{cr}{2T} > 0$ and $T > cr$ and $(4v^2 - 5rv + 2r^2) > (3v^2 - 4.5rv + 2r^2) \Leftrightarrow v > 0.5r$. Thus, $f(\overline{\delta}_1, \overline{\delta}_2)$ is positive everywhere in Case 1. Thus, $\frac{\partial \pi_1(\delta_1,\delta_2)}{\partial \delta_1} > 0$ in Case 1 region.

Claim A2

In fully symmetric model, we have $\frac{\partial \pi_1(\delta_1,\delta_2)}{\partial \delta_1} > 0$ *in Case 2 region.*

Proof. In Case 2 region, we have , $\delta_1 \geq \delta_2$ and $\delta_1 + \delta_2 \leq cr/T$.The profit in this region is,

$$\pi_1(\delta_1, \delta_2) = \pi_1^1 + \pi_1^2 = \frac{B_1^1(v-r)}{r} + \frac{B_1^2(c\,v - B_2^2)(v-r)}{(B_1^2 + B_2^2)v - rB_2^2} = \frac{\delta_1 T(v-r)}{r} + \frac{(1-\delta_1)(cv-(1-\delta_2)T)(v-r)}{(2-\delta_1-\delta_2)v - r(1-\delta_2)}$$

We then take the first derivatives and simplify to obtain,

$$\frac{\partial \pi_1(\delta_1,\delta_2)}{\partial \delta_1} = \frac{v-r}{r\left((2-\delta_1-\delta_2)v - r(1-\delta_2)\right)^2} \times f(\overline{\delta}_1, \overline{\delta}_2),$$

Where $f(\overline{\delta}_1, \overline{\delta}_2) = v^2 T \overline{\delta}_1^{\,2} + (v^2 - rv)T\overline{\delta}_2^{\,2} + 2vT(v-r)\overline{\delta}_1\overline{\delta}_2 - crv(v-r)\overline{\delta}_2$ .To prove the claim, it is enough to show $f(\overline{\delta}_1, \overline{\delta}_2)$ is always positive in the region of Case 2. Note that $f(\overline{\delta}_1, \overline{\delta}_2)$ is clearly increasing in $\overline{\delta}_1$. We claim that it is also increasing in $\overline{\delta}_2$. To show this, we take the derivative with respect to $\overline{\delta}_2$ to obtain,

$$g(\overline{\delta}_1, \overline{\delta}_2) \triangleq \frac{\partial f(\overline{\delta}_1, \overline{\delta}_2)}{\partial \overline{\delta}_2} = (2v^2 - 2rv)T\overline{\delta}_2 + 2v(v-r)T\overline{\delta}_1 - cr(v^2 - rv)$$

$g(\overline{\delta}_1, \overline{\delta}_2)$ is a linear expression in terms of $\overline{\delta}_1, \overline{\delta}_2$. We claim it is always positive in the region of case 2. To show this, we check three corners of the triangle of Case 2 region,

-At the corner $(\delta_1, \delta_2) = (0,0)$, we have, $g(\overline{\delta}_1, \overline{\delta}_2) = (4v^2 - 4rv)T - cr(v^2 - rv)$, which is positive

-At the corner $(\delta_1, \delta_2) = (cr/2T, cr/2T)$, we get, $g(\overline{\delta}_1, \overline{\delta}_2) = (v^2 - rv)(4T - 3cr)$, which is positive.

-Finally, at the corner $(\delta_1, \delta_2) = (cr/T, 0)$, $g(\overline{\delta}_1, \overline{\delta}_2) = 4v^2T - cr(3v^2 - 3rv)$, which is positive.

Therefore, $g(\overline{\delta}_1, \overline{\delta}_2)$ is always positive in the region of Case 2, and hence $f(\overline{\delta}_1, \overline{\delta}_2)$ is increasing in both $\overline{\delta}_1$ and $\overline{\delta}_2$. Thus, to show that $f(\overline{\delta}_1, \overline{\delta}_2)$ is positive in region of Case 2, it suffices to

check at the minimum $\overline{\delta}_1$ and $\overline{\delta}_2$ (or the maximum δ_1 and δ_2) in this region, which are the points on the line $\delta_1 + \delta_2 = cr/T$ (see Figure 2). Let us first check two ends of this line,

$$f(\delta_1 = cr/T, \delta_2 = 0) = v^2/\,T(T - cr)^2 + (v^2 - rv)T + 2v(v - r)(T - cr) - cr(v^2 - rv)$$

Which is positive. At $(\delta_1, \delta_2) = (cr/2T, cr/2T)$,

$$f(\delta_1 = cr/2T, \delta_2 = cr/2T) = (1 - \delta_2)\{(4v^2 - 3rv)(T) - cr(3v^2 - 2.5rv)\} > 0$$

Which is again positive. For any other points on $\delta_1 + \delta_2 = cr/T$ where $\delta_1 \leq cr/2T$, we have, $\overline{\delta}_1 + \overline{\delta}_2 = 2 - cr/T \rightarrow \overline{\delta}_1{}^2 + \overline{\delta}_2{}^2 + 2\overline{\delta}_1\overline{\delta}_2 = (2 - cr/T)^2$. By plugging this into $f\left(\overline{\delta}_1, \overline{\delta}_2\right)$,

$$f\left(\overline{\delta}_1, \overline{\delta}_2\right) = (rv)T\overline{\delta}_1{}^2 + crv(v - r)\overline{\delta}_1 + \frac{(v^2 - rv)}{T}(2T - cr)(2T - 2cr) > 0$$

Therefore, $\frac{\partial \pi_1(\delta_1, \delta_2)}{\partial \delta_1} > 0$ in Case 2 region.

Claim A3

In fully symmetric model, suppose $\delta_1 \leq \delta_2 \leq 0.5$. *Then* $\pi_1(\delta_1, \delta_2) \leq \pi_1(1 - \delta_1, \delta_2)$.

Proof. Since $\delta_1 \leq \delta_2 \leq 0.5$, Advertiser 1 is jammee in SE1 and thus,

$$\pi_1(\delta_1, \delta_2) = \frac{\delta_1(cv - \delta_2 T)(v - r)}{(\delta_1 + \delta_2)v - r\delta_2} + \frac{(cv(1 - \delta_1) - (1 - \delta_2)^2 T - cr(\delta_2 - \delta_1))(v - r)}{(2 - \delta_1 - \delta_2)v - r(1 - \delta_1)}$$

On the other hand, since $\delta_2 \leq 0.5 \leq 1 - \delta_2 \leq 1 - \delta_1$, Advertiser 1 is jammer in SE1 and thus,

$$\pi_1(1 - \delta_1, \delta_2) = \frac{\left(cv(1 - \delta_1) - \delta_2^2 T - cr((1 - \delta_1) - \delta_2)\right)(v - r)}{\left((1 - \delta_1) + \delta_2\right)v - r(1 - \delta_1)} + \frac{\delta_1(cv - (1 - \delta_2)T)(v - r)}{(\delta_1 + 1 - \delta_2)v - r(1 - \delta_2)}$$

To show $\Delta\pi \triangleq \pi_1(\delta_1, \delta_2) - \pi_1(1 - \delta_1, \delta_2) \leq 0$ in the region $\delta_1 \leq \delta_2 \leq 0.5$, we first simplify it,

$$\Delta\pi = \frac{(v - r)(1 - 2\delta_2) \times f(\delta_1, \delta_2)}{((\delta_1 + \delta_2)v - r\delta_2)((2 - \delta_1 - \delta_2)v - r(1 - \delta_1))\left(((1 - \delta_1) + \delta_2)v - r(1 - \delta_1)\right)((\delta_1 + 1 - \delta_2)v - r(1 - \delta_2))}$$

Where $f(\delta_1, \delta_2)$ is a degree-4 polynomial in two variables δ_1, δ_2,

$$\begin{aligned} f(\delta_1, \delta_2) = {} & Tv(v - r)\delta_1{}^4 - (v - r)v(cr + 2T - 2cv)\delta_1{}^3 - (v - r)v(cr + 2T - 3cv)\delta_1{}^2 \\ & - (2v - r)(cr^2 - rT - crv + cv^2)\delta_1\delta_2{}^2 + (2v - r)(cr^2 - rT - crv + cv^2)\delta_1\delta_2 \\ & + (v - r)v(cv - T)\delta_1 + T(r - v)v\delta_2{}^4 + 2T(v - r)v\delta_2{}^3 + (v - r)(cr^2 - rT \\ & - 2crv + cv^2)\delta_2{}^2 + (v - r)^2(cr - T - cv)\delta_2 \end{aligned}$$

It is clear that the sign of $\Delta\pi$ is the same as that of $f(\delta_1, \delta_2)$ since the denominator is positive. So it is sufficient to show the polynomial $f(\delta_1, \delta_2)$ is negative when $\delta_1 \leq \delta_2 \leq 0.5$. To do this, we take derivatives of $f(\delta_1, \delta_2)$,

$$f_2(\delta_1, \delta_2) = \frac{\partial f(\delta_1, \delta_2)}{\partial \delta_2} = (1 - 2\delta_2) \times g(\delta_1, \delta_2)$$

$$g(\delta_1, \delta_2) = 2T(v - r)v\left(\delta_2{}^2 - \delta_2\right) + (2v - r)(cr^2 - rT - crv + cv^2)\delta_1 + (r - v)^2(cr - T - cv)$$

The sign of $f_2(\delta_1, \delta_2)$ is the same as that of $g(\delta_1, \delta_2)$. We claim that $g(\delta_1, \delta_2) \leq 0$ when $\delta_1 \leq \delta_2 \leq 0.5$. To show this, first note that,

$$g_2(\delta_1, \delta_2) = \frac{\partial g(\delta_1, \delta_2)}{\partial \delta_2} = 2T(v - r)v(2\delta_2 - 1) < 0$$

This implies that $g(\delta_1, \delta_2)$ is decreasing in δ_2. Therefore, for $g(\delta_1, \delta_2)$ to be always negative in the region, it suffices to show that it is negative for the minimum value of δ_2 given δ_1, that is $g(\delta_1, \delta_1) \leq 0$ for $0 \leq \delta_1 \leq 0.5$,

$$g(\delta_1, \delta_1) = 2T(v - r)v{\delta_1}^2 - (cr^3 - r^2T - 3cr^2v + 3crv^2 + 2Tv^2 - 2cv^3)\delta_1 + (r - v)^2(cr - T - cv)$$

The $g(\delta_1, \delta_1)$ is an upside parabola in δ_1 since the coefficient of δ_1^2 is positive. Moreover, $g(0,0) = (r - v)^2(cr - T - cv) < 0$, since $cr < T$. In addition, $(0.5,0.5) = \frac{1}{2}(cr - T)(r^2 + 3v(v - r)) < 0$, since $cr < T$. Therefore, the upside parabola $g(\delta_1, \delta_1)$ is always negative in $0 \leq \delta_1 \leq 0.5$. We then conclude that □$(\delta_1, \delta_2) < 0$ on the region $\delta_1 \leq \delta_2 \leq 0.5$. As a result, $f_2(\delta_1, \delta_2) < 0$. In other words, $f(\delta_1, \delta_2)$ is decreasing in δ_2 in the region. Therefore, to show that $f(\delta_1, \delta_2) < 0$ in the region, it suffices to show that $f(\delta_1, \delta_1) < 0$ for $0 \leq \delta_1 \leq 0.5$. We have,

$$f(\delta_1, \delta_1) = \delta_1(1 - \delta_1)(cr - T)(2v - r)(v - r + r\delta_1)$$

Which is clearly negative. Therefore, $f(\delta_1, \delta_2) < 0$ in the region $\delta_1 \leq \delta_2 \leq 0.5$. As a result, $\pi_1(\delta_1, \delta_2) - \pi_1(1 - \delta_1, \delta_2) \leq 0$.

Proof of Proposition 2

We first derive the best response function for Advertiser 1. Note that the best response functions would be the same for Advertiser 2 since advertisers are symmetric. We derive $\rho_1(\delta_2)$ for $0 \leq \delta_2 \leq 1$. To do so, we first write the profit of Advertiser 1. For the sake of exposition, assume reserve price is zero. Then, there are only two cases. In Case 1, Advertiser 1 is a jammer in SE_1 and jammee in SE_2. In Case 2, the reverse situation happens.

$$\pi_1(\delta_1, \delta_2) = \begin{cases} \overline{\pi}(B_1^1, B_2^1) + \underline{\pi}(B_1^2, B_2^2) & \delta_1 \geq \delta_2 \quad case\ 1 \\ \underline{\pi}(B_1^1, B_2^1) + \overline{\pi}(B_1^2, B_2^2) & \delta_2 \geq \delta_1 \quad case\ 2 \end{cases}$$

Next, we start to characterize the $\rho_1(\delta_2)$. First, we take first derivative in each cases. In Case 1,

$$\pi_1^{\text{case 1}} = \frac{c_1\, v\, \delta_1 - \delta_2^2\, T}{\delta_1 + \delta_2} + \frac{(1-\delta_1)(c_2\, v - (1-\delta_2)T)}{(2-\delta_1-\delta_2)}$$

$$\frac{\partial \pi_1^{\text{case 1}}}{\partial \delta_1} = \frac{(c_1 v + \delta_2\, T)\delta_2}{(\delta_1+\delta_2)^2} - \frac{(c_2\, v - (1-\delta_2)T)(1-\delta_2)}{(2-\delta_1-\delta_2)^2}$$

Equating the first derivative to zero, we get,

$$\frac{(\delta_1+\delta_2)^2}{(2-\delta_1-\delta_2)^2} = \frac{(c_1 v+\delta_2\, T)\delta_2}{(c_2\, v-(1-\delta_2)T)(1-\delta_2)}$$

Note that the RHS is positive. We take square root to solve for the best response δ_1,

$$\rho_1^{\text{case 1}}(\delta_2) = \frac{2\sqrt{(c_1 v+\delta_2\, T)\delta_2}}{\sqrt{(c_2\, v-(1-\delta_2)T)(1-\delta_2)}+\sqrt{(c_1 v+\delta_2\, T)\delta_2}} - \delta_2$$

We follow a similar procedure for Case 2 to find,

$$\pi_1^{\text{case 2}} = \frac{\delta_1(c_1\, v-\delta_2 T)}{\delta_1+\delta_2} + \frac{c_2\, v\,(1-\delta_1)-(1-\delta_2)^2\, T}{2-\delta_1-\delta_2}$$

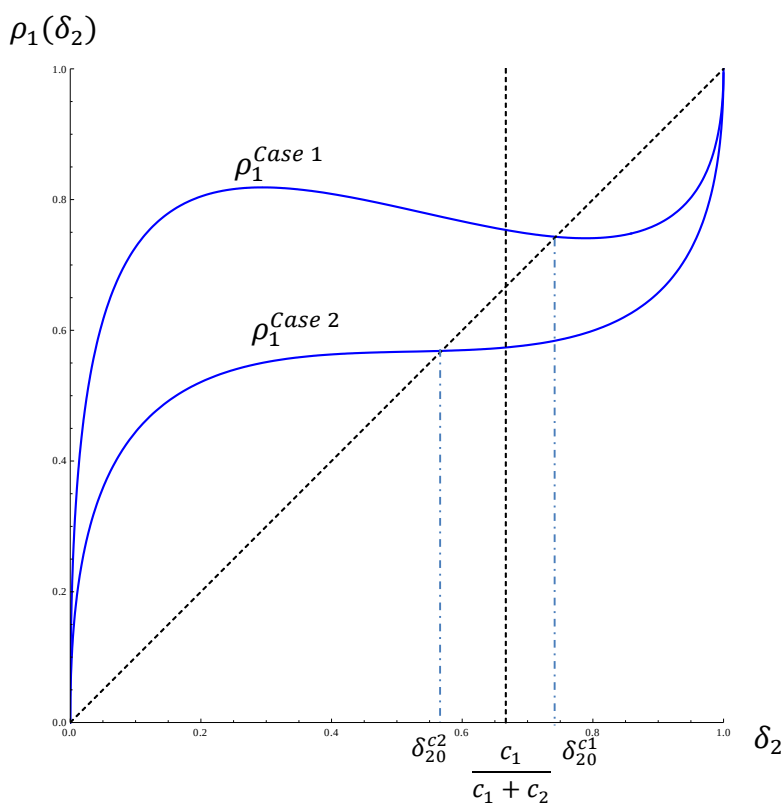
$$\frac{\partial \pi_1^{\text{case 2}}}{\partial \delta_1} = \frac{(c_1 v-\delta_2\, T)\delta_2}{(\delta_1+\delta_2)^2} - \frac{(c_2\, v+(1-\delta_2)\, T)(1-\delta_2)}{(2-\delta_1-\delta_2)^2}$$

$$\frac{(\delta_1+\delta_2)^2}{(2-\delta_1-\delta_2)^2} = \frac{(c_1 v-\delta_2\, T)\delta_2}{(c_2\, v+(1-\delta_2)T)(1-\delta_2)}$$

$$\rho_1^{\text{case 2}}(\delta_2) = \frac{2\sqrt{(c_1 v-\delta_2\, T)\delta_2}}{\sqrt{(c_2\, v+(1-\delta_2)T)(1-\delta_2)}+\sqrt{(c_1 v-\delta_2\, T)\delta_2}} - \delta_2$$

We have found the expression for $\rho_1^{\text{case 1}}(\delta_2)$ and $\rho_1^{\text{case 2}}(\delta_2)$. It can be seen that for any δ_2 we have $\rho_1^{\text{case 1}}(\delta_2) \geq \rho_1^{\text{case 2}}(\delta_2)$, which can be easily verified from their formula. These two functions have been illustrated in Figure OA1. We now find the characteristics of these two functions.

Figure OA1. Best Response Functions
$(T = 6, r = 0, c_1 = 2, c_2 = 1, v = 10)$



First, note that $\rho_1^{\text{case }1}(\delta_2)$ can be best response only if the condition of Case 1, that is $\delta_1 = \rho_1^{\text{case }1}(\delta_2) > \delta_2$ hold. Therefore, only the part of $\rho_1^{\text{case }1}(\delta_2)$ above the 45-degree line is acceptable. Similarly, only the part of $\rho_1^{\text{case }2}(\delta_2)$ below the 45-degree line is acceptable. We first find the intersection of $\rho_1^{\text{case }1}(\delta_2)$ with line $\delta_1 = \delta_2$, and we name it δ_{20}^{c1} (see Figure OA1). It can be calculated from $\rho_1^{\text{case }1}(\delta_{20}^{c1}) = \delta_{20}^{c1}$,

$$\rho_1^{\text{case }1}(\delta_{20}^{c1}) = \delta_{20}^{c1} \rightarrow \delta_{20}^{c1} = \frac{1}{4T}(2T - c_1 v - c_2 v + \sqrt{8c_1 vT + (2T - c_1 v - c_2 v)^2})$$

Similarly,

$$\rho_1^{\text{case }2}(\delta_{20}^{c2}) = \delta_{20}^{c2} \rightarrow \delta_{20}^{c2} = \frac{1}{4T}(2T + c_1 v + c_2 v - \sqrt{-8c_1 vT + (2T + c_1 v + c_2 v)^2})$$

We see that $0 < \delta_{20}^{c2} < \frac{c_1}{c_1+c_2} < \delta_{20}^{c1} < 1$. Hence, there would be an overlap between $\rho_1^{c1}(\delta_2)$ and $\rho_1^{c2}(\delta_2)$ in the range $\delta_{20}^{c2} < \delta_2 < \delta_{20}^{c1}$ where both of them are feasible and we have to determine which one produce higher profit for Advertiser 1 and thus is the best response.

To derive the best response function in the range $\delta_{20}^{c2} < \delta_2 < \delta_{20}^{c1}$, we have to compare profits, it can be verified that the profit is equal at $\hat{\delta}_2 = c_1/(c_1 + c_2)$,

$$\pi_1^{c1}\left(\rho_1^{c1}(\hat{\delta}_2), \hat{\delta}_2\right) = \pi_1^{c2}\left(\rho_1^{c2}(\hat{\delta}_2), \hat{\delta}_2\right)$$

More generally, we have

$$\pi_1^{c2}(\rho_1^{c1}(\delta_2), \delta_2) <=> \pi_1^{c1}(\rho_1^{c1}(\delta_2), \delta_2) \ \ if\ and\ only\ if\ \ \delta_2 <=> c_1/(c_1 + c_2)$$

Therefore, the best response function of Advertiser 1 would be,

$$\rho_1(\delta_2) = \begin{cases} \rho_1^{\text{case }1}(\delta_2) & \delta_2 \leq c_1/(c_1 + c_2) \\ \rho_1^{\text{case }2}(\delta_2) & \delta_2 > c_1/(c_1 + c_2) \end{cases}$$

To find the intersection of advertisers best response function, we should find the intersection of $\rho_1^{c1}(\delta_2)$ with the inverse function of $\rho_1^{c2}(\delta_2)$. This means that the NE is the joint solution of the following system,

$$\rho_1^{\text{case }1}(\delta_2^*) = \delta_1^* = \frac{2\sqrt{(c_1 v + \delta_2^* T)\delta_2^*}}{\sqrt{(c_2 v - (1-\delta_2^*)T)(1-\delta_2^*)} + \sqrt{(c_1 v + \delta_2^* T)\delta_2^*}} - \delta_2^*$$

$$(\rho_1^{\text{case }2})^{-1}(\delta_1^*) = \delta_2^* = \frac{2\sqrt{(c_1 v - \delta_1^* T)\delta_1^*}}{\sqrt{(c_2 v + (1-\delta_1^*)T)(1-\delta_1^*)} + \sqrt{(c_1 v - \delta_1^* T)\delta_1^*}} - \delta_1^*$$

Which is simplified to

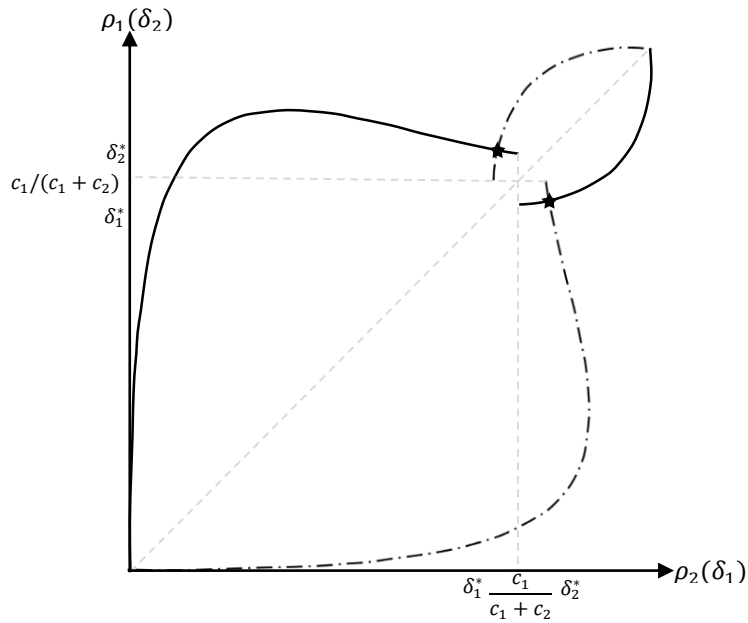
$$\frac{\delta_1^* + \delta_2^*}{2 - \delta_1^* - \delta_2^*} = \frac{\sqrt{(c_1 v + \delta_2 T)\delta_2^*}}{\sqrt{(c_2 v - (1-\delta_2^*)T)(1-\delta_2^*)}} = \frac{\sqrt{(c_1 v - \delta_1^* T)\delta_1^*}}{\sqrt{(c_2 v + (1-\delta_1^*)T)(1-\delta_1^*)}}$$

Or,

$$\left(\frac{\delta_1^* + \delta_2^*}{2 - \delta_1^* - \delta_2^*}\right)^2 = \frac{(c_1 v + \delta_2^* T)\delta_2^*}{(c_2 v - (1-\delta_2^*)T)(1-\delta_2^*)} = \frac{(c_1 v - \delta_1^* T)\delta_1^*}{(c_2 v + (1-\delta_1^*)T)(1-\delta_2^*)}$$

Which are the equations which have been stated in the proposition.

Figure A2. Best Response Functions
$(T = 5, r = 0, c_1 = 3, c_2 = 1, v = 10)$



The final step is to show these two equations have roots satisfying $\delta_2^* < c_1/(c_1 + c_2) < \delta_1^*$. To show this, we first simplify them and get,

$$\frac{(c_1 v+\delta_2^* T)\delta_2^*}{(c_2 v-(1-\delta_2^*)T)(1-\delta_2^*)} = \frac{(c_1 v-\delta_1^* T)\delta_1^*}{(c_2 v+(1-\delta_1^*)T)(1-\delta_2^*)} = \frac{(\delta_1^*+\delta_2^*)(c_1 v-(\delta_1^*-\delta_2^*)T)}{(2-\delta_1^*-\delta_2^*)(c_2 v-(\delta_1^*-\delta_2^*)T)}$$

As a result, we will have, $\frac{\delta_1^*+\delta_2^*}{2-\delta_1^*-\delta_2^*} = \frac{c_1 v-(\delta_1^*-\delta_2^*)T}{c_2 v-(\delta_1^*-\delta_2^*)T}$. This equation immediately implies that, $\delta_1^* + \delta_2^* <=> 1$ if and only if $c_1 <=> c_2$. Moreover, $\delta_1^* = \delta_2^* = \frac{c_1}{c_1+c_2}$ satisfies it. We then calculate δ_1^* as a function of δ_2^*,

$$\delta_1^* = f(\delta_2^*) = \frac{1}{4T}\{2T + (c_1 + c_2)v - \sqrt{(2T + (c_1 + c_2)v)^2 - 8T(2c_1 v + (2T - (c_1 + c_2)v)\delta_2^* - 2T{\delta_2^*}^2)}\}$$

The positive sign is not acceptable since $\delta_1^* < 1$. $f(\delta_2^*)$ is a decreasing function of δ_2^*, because both coefficient of δ_2^* and${\delta_2^*}^2$ are positive and the squared-root is negative.

We would like to show that $\rho_1^{\text{case 1}}(\delta_2^*)$ and the above equation will intersect at some $\delta_2^* < c_1/(c_1 + c_2)$. To show this, we use *Intermediate value theorem* for interval $[0, \frac{c_1}{c_1+c_2}]$. At $\delta_2^* = 0$, $f(0) = \frac{2T+(c_1+c_2)v-\sqrt{(2T+(c_1+c_2)v)^2-8T(2c_1 v)}}{4T} > 0 = \rho_1^{\text{case 1}}(0)$. At $\delta_2^* = \frac{c_1}{c_1+c_2}$, $f\left(\frac{c_1}{c_1+c_2}\right) = \frac{c_1}{c_1+c_2} <$

$\rho_1^{\text{case 1}}\left(\frac{c_1}{c_1+c_2}\right)$. Therefore, $\rho_1^{\text{case 1}}(\delta_2^*)$ and $f(\delta_2^*)$ intersect at some $\delta_2^* \in (0, \frac{c_1}{c_1+c_2})$. It then follows that $\delta_1^* \in (\frac{c_1}{c_1+c_2}, 1)$ since $f(\delta_2^*)$ is a decreasing function of δ_2^* and $f\left(\frac{c_1}{c_1+c_2}\right) = \frac{c_1}{c_1+c_2}$.

Proof of Proposition 3

Let us define $T_H = Max(T_1, T_2)$ and $T_L = Min(T_1, T_2)$ to denote the high-budget and low-budget advertiser. For the sake of exposition, we also assume that reserve prices are zero. We derive the high-budget and Low-budget advertisers' best response functions, $\rho_H(\delta_L)$, $\rho_L(\delta_H)$, and show that their unique intersection is at $\delta_H^* = \delta_L^* = c_1/(c_1 + c_2)$. To do so, we follow a procedure similar to the one in the full symmetric model. We first write the profit in each case

$$\pi_H(\delta_H, \delta_L) = \begin{cases} \overline{\pi}(B_H^1, B_L^1) + \underline{\pi}(B_H^2, B_L^2) & \delta_H T_H \geq \delta_L T_L \ , (1-\delta_H)T_H \leq (1-\delta_L)T_L & case\ 1 \\ \underline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) & \delta_H T_H < \delta_L T_L \ , (1-\delta_H)T_H > (1-\delta_L)T_L & case\ 2 \\ \overline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) & \delta_H T_H \geq \delta_L T_L \ , (1-\delta_H)T_H > (1-\delta_L)T_L & case\ 3 \end{cases}$$

We then write each case as a function of strategies δ_L and δ_H,

$$\pi_H^1(\delta_H, \delta_L) = \overline{\pi}(B_H^1, B_L^1) + \underline{\pi}(B_H^2, B_L^2) = \frac{c_1\, v\, \delta_H T_H - \delta_L^2\, T_L^2}{\delta_H T_H + \delta_L T_L} + \frac{(1-\delta_H)T_H(c_2\, v - (1-\delta_L)T_L)}{(1-\delta_H)T_H + (1-\delta_L)T_L}$$

$$\pi_H^2(\delta_H, \delta_L) = \underline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) = \frac{\delta_H T_H(c_1\, v - \delta_L T_L)}{\delta_H T_H + \delta_L T_L} + \frac{c_2\, v\, (1-\delta_H)T_H - (1-\delta_L)^2\, T_L^2}{(1-\delta_H)T_H + (1-\delta_L)T_L}$$

$$\pi_H^3(\delta_H, \delta_L) = \overline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) = \frac{c_1\, v\, \delta_H T_H - \delta_L^2\, T_L^2}{\delta_H T_H + \delta_L T_L} + \frac{c_2\, v\, (1-\delta_H)T_H - (1-\delta_L)^2\, T_L^2}{(1-\delta_H)T_H + (1-\delta_L)T_L}$$

Next, we take the first derivatives $\partial \pi_H^j / \partial \delta_H$,

$$\frac{\partial \pi_H^1}{\partial \delta_H} = T_L T_H \frac{(c_1\, v\, + \delta_L\, T_L)\delta_L}{(\delta_H T_H + \delta_L T_L)^2} - T_L T_H \frac{(c_2\, v - (1-\delta_L)T_L)(1-\delta_L)}{\left((1-\delta_H)T_H + (1-\delta_L)T_L\right)^2}$$

$$\frac{\partial \pi_H^2}{\partial \delta_H} = T_L T_H \frac{(c_1\, v - \delta_L\, T_L)\delta_L}{(\delta_H T_H + \delta_L T_L)^2} - T_L T_H \frac{(c_2\, v + (1-\delta_L)T_L)(1-\delta_L)}{\left((1-\delta_H)T_H + (1-\delta_L)T_L\right)^2}$$

$$\frac{\partial \pi_H^3}{\partial \delta_H} = T_L T_H \frac{(c_1\, v + \delta_L\, T_L)\delta_L}{(\delta_H T_H + \delta_L T_L)^2} - T_L T_H \frac{(c_2\, v + (1-\delta_L)T_L)(1-\delta_L)}{\left((1-\delta_H)T_H + (1-\delta_L)T_L\right)^2}$$

In Claim A4, we show that if $\frac{T_H}{T_L} > \frac{c_1+c_2}{Min(c_1,c_2)}$, then the derivative in Case 1 and Case 2 are always negative and positive, respectively. Therefore, the best response function is derived by solving $\partial \pi_H^3 / \partial \delta_H = 0$ for δ_H,

$$\rho_H(\delta_L) = \rho_H^{case\ 3}(\delta_L) = \frac{(T_H + T_L)\sqrt{(c_1 v + \delta_L\, T_L)\delta_L}}{T_H\sqrt{(c_2\, v + (1-\delta_L)T_L)(1-\delta_L)} + T_H\sqrt{(c_1 v + \delta_L\, T_L)\delta_L}} - \frac{\delta_L T_L}{T_H}$$

Since the best response of high-budget advertiser to the strategy δ_L is always in Case 3, we only need to calculate the best response of low-budget advertiser in Case 2, in which the low-budget advertiser is jammee in both platforms.

$$\pi_L^2(\delta_H,\delta_L) = \underline{\pi}(B_H^1,B_L^1) + \underline{\pi}(B_H^2,B_L^2) = \frac{\delta_L T_L(c_1\, v - \delta_H T_H)}{\delta_H T_H + \delta_L T_L} + \frac{(1-\delta_L)T_L(c_2\, v - (1-\delta_H)T_H)}{(1-\delta_H)T_H + (1-\delta_L)T_L}$$

$$\rho_L(\delta_H) = \frac{(T_H + T_L)\sqrt{(c_1 v - \delta_H\, T_H)\delta_H}}{T_L\sqrt{(c_2\, v - (1-\delta_H)T_H)(1-\delta_H)} + T_L\sqrt{(c_1 v - \delta_H\, T_H)\delta_H}} - \frac{\delta_H T_H}{T_L}$$

Finally, it is easy to see that the unique intersection of $\rho_L(\delta_H)$ and $\rho_H(\delta_L)$ is the point $\delta_L^* = \delta_H^* = c_1/(c_1 + c_2)$.

Claim A4

In the model with high-budget and low-budget advertisers, $\frac{\partial \pi_H^1}{\partial \delta_H} < 0$ *and* $\frac{\partial \pi_H^2}{\partial \delta_H} > 0$ *in the regions of Case 1 and Case 2, if* $\frac{T_H}{T_L} > \frac{c_1+c_2}{Min(c_1,c_2)}$.

Proof. In Case 1, we have

$$\frac{\partial \pi_H^1}{\partial \delta_H} = T_L T_H \frac{(c_1\, v\ + \delta_L\, T_L)\delta_L}{(\delta_H T_H + \delta_L T_L)^2} - T_L T_H \frac{(c_2\, v - (1-\delta_L)T_L)(1-\delta_L)}{\left((1-\delta_H)T_H + (1-\delta_L)T_L\right)^2}$$

We will show that $\frac{\partial \pi_H^1}{\partial \delta_H} < 0$ in the Case 1 region, $(1-\delta_H)T_H < (1-\delta_L)T_L$, and given $\frac{T_H}{T_L} > \frac{c_1+c_2}{Min(c_1,c_2)}$. Note that this function is continuous and differentiable in the triangular region of Case 1. Moreover, it is decreasing in δ_H. Therefore, it suffices to show it is negative for the lowest value of δ_H in the triangular region of Case 1, that is $(1-\delta_H)T_H = (1-\delta_L)T_L$, the lower boundary line,

$$\frac{\partial \pi_H^1}{\partial \delta_H}(\delta_H = 1 - \frac{(1-\delta_L)T_L}{T_H}\ ) = T_L T_H(\frac{(c_1\, v\ + \delta_L\, T_L)\delta_L}{(T_H - (1-2\delta_L)T_L)^2} - \frac{(c_2\, v - (1-\delta_L)T_L)(1-\delta_L)}{(2(1-\delta_L)T_L)^2})$$

Therefore, we need to should show,

$$4(1-\delta_L)T_L^2(c_1 v\ + \delta_L\, T_L)\delta_L < (T_H - (1-2\delta_L)T_L)^2(c_2\, v - (1-\delta_L)T_L)$$

For $\delta_L \in [0,1]$. Simplifying and collecting terms as a polynomial of δ_L, we need to prove,

$$8T_L^3\delta_L^3 + 4T_L^2(T_H - 3T_L + (c_1+c_2)v)\delta_L^2 + T_L((T_H - T_L)(T_H - 5T_L) + 4v(c_2 T_H - (c_1+c_2)T_L))\delta_L + (T_H - T_L)^2(c_2 v - T_L) > 0$$

This left-hand-side expression is clearly increasing in T_H. Therfore, it suffices to prove it for the minimum heterogeneity, that is $T_H = \frac{c_1+c_2}{Min[c_1,c_2]}T_L$. Without loss of generality, suppose $c_1 > c_2$. Plugging $T_H = \frac{c_1+c_2}{c_2}T_L$, we get

$$8T_L\delta_L^3 + 4(cT_L + c_1v + c_2v - 2T_L)\delta_L^2 + T_L(c^2 - 4c)\delta_L + c^2(c_2v - T_L) > 0$$

Where $c = c_1/c_2 > 1$. All terms are positive except possibly the coefficient of δ_L. Thus, it suffices to prove LHS is positive for $\delta_L = 1$,

$$8T_L + 4(c_1v + c_2v - 2T_L) + T_L(c^2) + c^2(c_2v - T_L) > 0$$

We have completed the proof. Due to symmetry, the proof in case 2 is analogues to that of Case 1.

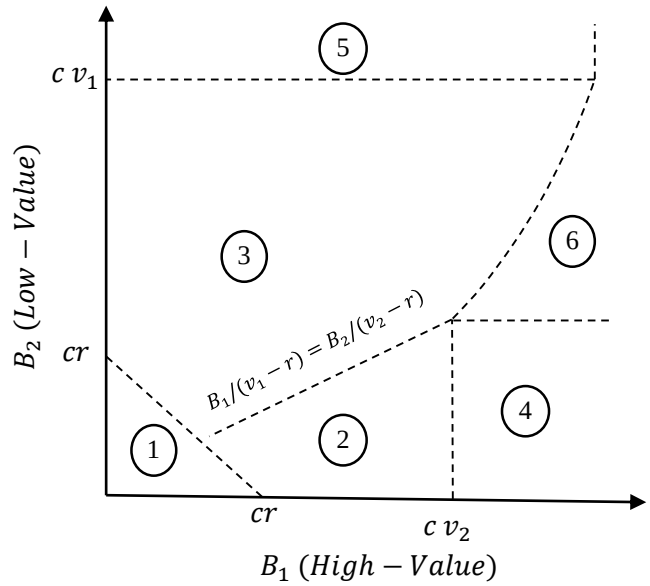
Lemma 2

Suppose $v_1 \geq v_2$,

1) *If* $B_1 + B_2 < cr$, *then* $b^* = r$ *and profits are* $\pi_i = B_i(v_i - r)/r$
2) *If* $B_1 + B_2 \geq cr$, $\frac{B_1}{v_1 - r} \geq \frac{B_2}{v_2 - r}$ *and* $B_1 < cv_2$, *then* A_1 *jams* A_2 *at* $b^* = \frac{v_2B_2 + (v_2 - r)B_1}{c(v_2 - r) + B_2}$. *Profits are* $\pi_1 = \frac{(cv_2B_1 + cr(B_2 - B_1) - B_2^2)(v_1 - r)}{(B_1 + B_2)v_2 - rB_1}$, $\pi_2 = \frac{B_2(cv_2 - B_1)(v_2 - r)}{(B_1 + B_2)v_2 - rB_1}$
3) *If* $B_1 + B_2 \geq cr$, $\frac{B_1}{v_1 - r} < \frac{B_2}{v_2 - r}$ *and* $(B_1^2 - crB_1)/c(v_1 - r) \leq B_2 \leq cv_1$, *then* A_2 *jams* A_1 *at* $b^* = \frac{v_1B_1 + (v_1 - r)B_2}{c(v_1 - r) + B_1}$, *and* $\pi_1 = \frac{B_1(cv_1 - B_2)(v_1 - r)}{(B_1 + B_2)v_1 - rB_2}$, $\pi_2 = \frac{(cv_1B_2 + cr(B_1 - B_2) - B_1^2)(v_2 - r)}{(B_1 + B_2)v_1 - rB_2}$
4) *If* $B_1 > cv_2$ *and* $B_2 \leq \frac{cv_2(v_2 - r)}{v_1 - r}$, *then* A_1 *jams* A_2 *at* $b^* = v_2$. *Profits are,* $\pi_1 = (c - \frac{B_2}{v_2})(v_1 - r)$, $\pi_2 = 0$
5) *If* $B_1 < cv_1 < B_2$, *then* A_2 *jams* A_1 *at* $b^* = v_1$. *Profits are,* $\pi_1 = 0$, $\pi_2 = (c - \frac{B_1}{v_1})(v_2 - r)$
6) *Otherwise, advertisers bid their valuations. Profits are,* $\pi_1 = c(v_1 - v_2)$, $\pi_2 = 0$

Lemma 2 is basically a generalization of Lemma 1 where advertisers have heterogeneous valuations. Figure OA3 illustrates different equilibrium cases.

Figure OA3. Asymmetric Valuations ($v_1 \geq v_2$) – Bidding Stage



In case 1, the reserve price is relatively high. Similar to full symmetric model, equilibrium bid will be at reserve price in case 1. The condition for this case would be, $\frac{B_1}{r} + \frac{B_2}{r} \leq c$. This guaranties that none of advertiser has incentive to deviate to higher bid. This is line with slope -1 in the $B_1 - B_2$ space (see Figure OA3).When budgets increase from case 1 region, advertisers have incentive to increases their bid. In the case 2 and 3, one of the advertiser jams the other one at a bid level which is above reserve price but below valuation. Similar to the model with homogeneous valuations, it is always weakly optimal for the advertiser with lower bid to increase its bid to just below its rival in case 2 and 3 (bid jamming). We again answer two questions, who jams whom and at what bid level. To answer these questions, we write necessary conditions for the NE, assuming advertiser i jams the other one,

$$\left(c - \frac{B_{3-i}}{b^*}\right)(v_i - r) \geq \frac{B_i}{b^*}(v_i - b^*)$$

$$\frac{B_{3-i}}{b^*}(v_{3-i} - b^*) \geq \left(c - \frac{B_i}{b^*}\right)(v_{3-i} - r)$$

Simplifying these two inequalities, we obtain,

$$\frac{v_i B_i + (v_i - r)B_{3-i}}{c(v_i - r) + B_i} \leq b^* \leq \frac{v_{3-i}B_{3-i} + (v_{3-i} - r)B_i}{c(v_{3-i} - r) + B_{3-i}}$$

From first and last expressions, we should have $\frac{B_i}{v_i - r} \geq \frac{B_{3-i}}{v_{3-i} - r}$. Since we had assumed advertiser i jams its rival, this condition states that the advertiser with the higher ratio of $B_i/(v_i - r)$ will jam its rival.

Observe that if valuations are equal, then inequalities implies that the advertiser with higher budget will be the jammer (Lemma 1). On the other hand, if budgets are equal, then the high value advertiser gets the slot first and is jammed by low value advertiser. This results resembles the one in Varian (2007), where in a symmetric Nash equilibrium, the highest value advertiser is assigned to the first slot.

In cases 4, the high-value advertiser's budget is inexhaustible, leading to zero profit for Advertiser 2. Conversely, in Case 5, the low-value advertiser's budget is inexhaustible, leading to zero profit for Advertiser 1. Finally, in case 6, both budget are unlimited. A dominant strategy equilibrium would be to bid the valuation.

To characterize the boundaries, suppose case 2 in which high-value advertiser (Advertiser 1) is the jammer. When B_1 increases, the bid level increases and first reaches to v_L, the Advertiser 2's valuation for each click. At this posit, Advertiser 2's profit becomes zero and it will not be willing to bid higher, $\pi_2 = 0 = \frac{B_2}{v_2}(v_2 - v_2) = (c - \frac{B_1}{v_2})(v_2 - r)$. This implies that the boundary of Case 1 and Case 2 is $B_1 = cv_2$.

In Case 3, low-value advertiser (Advertiser 2) is the jammer. When B_2 increases, the bid level increases and first reaches to v_L. Note that Advertiser 2 might bid *above* its valuation because it is the jammer and it is paying only the reserve price r. Therefore, the boundary of Case 3 and Case 5 is obtained by equating the bid level at higher valuation, $v_1 = v_H$,

$$\pi_1 = 0 = \frac{B_1}{v_H}(v_H - v_H) = (c - \frac{B_2}{v_H})(v_H - r)$$

Which implies that $B_2 = cv_1$ is the boundary of Case 3 and Case 5.

To characterize the boundary of case 3 and case 6, let us start from case 3 and increase B_1. At some point, the budget of Advertiser 1 becomes high enough to receive all the clicks, leaving zero clicks for the jammer, $\pi_1 = c(v_H - b^*) = \left(c - \frac{B_2}{b^*}\right)(v_H - r), b^* = \frac{B_1}{c}$. Simplifying the above equations,

$$cB_2(v_1 - r) = B_1^2 - rcB_1$$

Which is the boundary curve (a parabola) between Case 3 and Case 6. This completes the characterization of boundaries in Figure OA3.

Proof of Proposition 4

Let us define $v_H = Max(v_1, v_2)$ and $v_L = Min(v_1, v_2)$ to denote the high-value and low-value advertiser. For the sake of exposition, we also assume that reserve prices are zero. We show that both advertisers are at their best response when $\delta_H^* = \delta_L^* = c_1/(c_1 + c_2)$.

Consider the High-Value advertiser profit function below,

$$\pi_H(\delta_H, \delta_L) = \begin{cases} \overline{\pi}(B_H^1, B_L^1) + \underline{\pi}(B_H^2, B_L^2) & \delta_H v_L \geq \delta_L v_H \ , (1-\delta_H)v_L \leq (1-\delta_L)v_H \quad case\ 1 \\ \underline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) & \delta_H v_L \leq \delta_L v_H \ , (1-\delta_H)v_L \geq (1-\delta_L)v_H \quad case\ 2 \\ \underline{\pi}(B_H^1, B_L^1) + \underline{\pi}(B_H^2, B_L^2) & \delta_H v_L \leq \delta_L v_H \ , (1-\delta_H)v_L \leq (1-\delta_L)v_H \quad case\ 3 \end{cases}$$

We first take the first derivative from each of the parts of profit function,

$$\pi_H^1(\delta_H, \delta_L) = \overline{\pi}(B_H^1, B_L^1) + \underline{\pi}(B_H^2, B_L^2) = \frac{c_1 v_L\, \delta_H - \delta_L^2\, T}{\delta_H + \delta_L}\frac{v_H}{v_L} + \frac{(1-\delta_H)(c_2\, v_H - (1-\delta_L)T)}{(2-\delta_H-\delta_L)}$$

$$\pi_H^2(\delta_H, \delta_L) = \underline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) = \frac{\delta_H(c_1\, v_H - \delta_L T)}{\delta_H + \delta_L} + \frac{c_2\, v_L\, (1-\delta_H) - (1-\delta_L)^2\, T}{(2-\delta_H-\delta_L)}\frac{v_H}{v_L}$$

$$\pi_H^3(\delta_H, \delta_L) = \overline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) = \frac{c_1 v_L\, \delta_H - \delta_L^2\, T}{\delta_H + \delta_L}\frac{v_H}{v_L} + \frac{c_2\, v_L\, (1-\delta_H) - (1-\delta_L)^2\, T}{(2-\delta_H-\delta_L)}\frac{v_H}{v_L}$$

We should show that $\delta_H^* = c_1/(c_1 + c_2)$ is the best response to $\delta_L^* = c_1/(c_1 + c_2)$. We take the first derivatives $\partial\pi_H^j/\partial\delta_H$,

$$\frac{\partial\pi_H^1}{\partial\delta_H} = \frac{\delta_L(c_1 v_L + \delta_L\, T)}{(\delta_H + \delta_L)^2}\frac{v_H}{v_L} - \frac{(1-\delta_L)(c_2\, v_H - (1-\delta_L)T)}{(2-\delta_H-\delta_L)^2}$$

$$\frac{\partial\pi_H^2}{\partial\delta_H} = \frac{\delta_L(c_1\, v_H - \delta_L T)}{(\delta_H + \delta_L)^2} - \frac{(1-\delta_L)(c_2\, v_L + (1-\delta_L)T)}{(2-\delta_H-\delta_L)^2}\frac{v_H}{v_L}$$

$$\frac{\partial\pi_H^3}{\partial\delta_H} = \frac{\delta_L(c_1 v_L + \delta_L\, T)}{(\delta_H + \delta_L)^2}\frac{v_H}{v_L} - \frac{(1-\delta_L)(c_2\, v_L + (1-\delta_L)T)}{(2-\delta_H-\delta_L)^2}\frac{v_H}{v_L}$$

If $\frac{v_H}{v_L} > \frac{c_1+c_2}{Min(c_1,c_2)}$, then high-value advertiser's best response to $\delta_L^* = c_1/(c_1 + c_2)$ can never be in case 1 or case 2, and it has to be in Case 3, even though it allocates all of its budget to one platform. In Case 3, the best response to $\delta_L^* = c_1/(c_1 + c_2)$ is $\delta_H^* = c_1/(c_1 + c_2)$.

Next consider low-value advertiser. In Claim A5 we show $\frac{\partial\pi_L^1}{\partial\delta_L} < 0$ at $\delta_H^* = c_1/(c_1 + c_2)$, in the region of Case 1. Similarly, we show that in the region of case 2, $\frac{\partial\pi_L^2}{\partial\delta_L} > 0$ at $\delta_H^* = c_1/(c_1 + c_2)$. It is easily seen than $\frac{\partial\pi_L^3}{\partial\delta_L} = 0$, at $\delta_H^* = \delta_L^* = c_1/(c_1 + c_2)$. Therefore, $\delta_L^* = c_1/(c_1 + c_2)$ is the best response to $\delta_H^* = c_1/(c_1 + c_2)$.

Claim A5

If $\delta_H^* = c_1/(c_1 + c_2)$, *then* $\frac{\partial \pi_L^1}{\partial \delta_L} > 0$ *in the region of Case 1*

Proof. In Case 1,

$$\pi_L^1(\delta_H, \delta_L) = \underline{\pi}(B_H^1, B_L^1) + \overline{\pi}(B_H^2, B_L^2) = \frac{\delta_L(c_1\, v_L - \delta_H T_H)}{\delta_H + \delta_L} + \frac{c_2\, v_H\, (1 - \delta_L) - (1 - \delta_H)^2\, T}{(2 - \delta_H - \delta_L)} \frac{v_L}{v_H}$$

$$\frac{\partial \pi_L^1}{\partial \delta_L} = \frac{\delta_H(c_1\, v_L - \delta_H T_H)}{(\delta_H + \delta_L)^2} - \frac{(1 - \delta_H)(c_2\, v_H + (1 - \delta_H)T)}{(2 - \delta_H - \delta_L)^2} \frac{v_L}{v_H}$$

We show it is positive. This is clearly decreasing in δ_L. So it suffices to show it is positive for the highest value of δ_L, that is $\delta_L = v_L \delta_H / v_H$. Plugging $\delta_H^* = c_1/(c_1 + c_2)$ and simplifying, we have,

$$\frac{\partial \pi_L^1}{\partial \delta_L} = \frac{v_H}{\left(2c_2 v_H + c_1(v_H - v_L)\right)^2 (v_H + v_L)^2} \times$$

$$((c_1 + c_2)^2 v_H (v_H - v_L) v_L (c_1 v_H + 3c_2 v_H - c_1 v_L + c_2 v_L) - T(4c_1 c_2 {v_H}^2 (v_H - v_L) + c_1^2 v_H (v_H - v_L)^2 + {c_2}^2 (4{v_H}^3 + {v_H}^2 v_L + 2v_H {v_L}^2 + v_L^3)))$$

The coefficient of $-T$ and the other term is positive. Therefore, to show that it is always positive it suffices to show it for the maximum possible T. First suppose $c_1 > c_2$. Then, the maximum $T = c_2 v_L$. Plugging and simplifying, we have,

$$(c_1^3 + 4c_1^2 c_2 + 3c_1 c_2^2 - c_2^3) v_H^3 v_L + (-2c_1^3 - 4c_1^2 c_2 - 2c_1 c_2^2 - 3c_2^3) v_H^2 v_L^2 + (c_1^3 - c_1 c_2^2 - 3{c_2}^3) v_H v_L^3 - c_2^3 v_L^4$$

This is positive is v_H is sufficiently bigger that v_L. In particular, if $v_H = \left(1 + \frac{c_1}{c_2}\right) v_L$,

$$\frac{v_L^4}{c_2^3} ({c_1}^2 + 2c_1 c_2 + 2{c_2}^2)({c_1}^4 + 3{c_1}^3 c_2 + 3{c_1}^2 {c_2}^2 - 2c_1 {c_2}^3 - 4{c_2}^4) {v_L}^4$$

Which is positive since $c_1 > c_2$. The reverse assumption of $c_1 < c_2$ will be very similar. Therefore, $\frac{\partial \pi_L^1}{\partial \delta_L} > 0$ in the region of Case 1.

Empirical Analysis

In this section I try to provide some empirical evidence for the results in first and last essays of my dissertation. I first describe the data and some related summary statistics. I then proceed by describing the empirical model and estimation results.

Data Description

There are three sources of data which is described in turn: *Google Adwords* data, *Bing Adcenter* data and the scraped data from Bing and Google result pages. I collected these data for three different categories: *Flower*, *Tablets* and *Textbooks*. I have chosen top 100 keywords with highest search volumes. In the most of the analysis, I only analyze the *flower* category. Table 1 shows some sample advertisers and keywords in each category.

Table 1. Categories, Sample Keywords and Advertisers

Category	*Number of Keywords*	*Sample Keywords*	*Top Advertisers*
Flowers	100	Flowers, flower delivery, florist send flowers, flowers online, cheap flowers, wedding flowers	Teleflora ,FlowerDeliveryExpress, ProFlowers, 1800Flowers, BloomsToday, FTD, SendFlower
Tablets	100	Samsung tablet, best tablet, nook tablet, cheap tablet, motion tablet, tablet phones, buy tablet, tablet pc comparison	Google, Microsoft, Dell, Intel, Amazon,Walmart, Bestbuy, Target, Shop411, Pronto
Textbook	100	eBooks, sell textbooks, cheap textbooks, college textbooks	Chegg , Textbooks ,Amazon, Ecampus, DirectTextbook, CampusBookRentals

Source 1: Google Adwords Keyword Planner

Google Adwords keyword planer provides search volume and performance forecast for different keywords. I've collected this data for the set of 100 keywords that I selected (in each category). Keywords planner provides forecasts for number of impressions, clicks and costs for a keyword at a specific bid level. I chose bid levels in range of $0.05 to $10 with increment of 50 cents. It should be noted that this data set is NOT time or advertiser specific. I chose the time period that the

forecast to be based on to be April to July, the same period that I collected advertisers from search engine result page. Google provides three keyword-specific bid-independent data: 1) *Monthly searches*, The average number of times people have searched for this exact keyword, 2) *Competition*, which is a measure of the number of advertisers that showed on each keyword relative to all keywords across Google and shows how competitive ad placement is for a keyword and 3) *suggested bid*, that is calculated by taking into account the costs-per-click (CPCs) that advertisers are paying for this keyword.

Table 2. Summary Statistics of Google Adwords bid-independent Data for Flower Category

Statistics	*Monthly Searches*	*Competition*	*Suggested bid*
Mean	14389	0.85	4.23
Mode	2400	1	5.87
Standard Deviation	68592	0.26	2.13
Minimum	1000	0.08	0.42
Maximum	673000	1	9.67
Count	100	100	100

The bid-dependent data includes four forecasts: estimated clicks, impressions, costs and average position. Figure 1 shows these 4 measures for keyword “flowers”. We have similar graphs for other 99 keywords.

Source 2: Bing Adcenter Keyword Planner

The second source of data is from Bing Adcenter’s Keyword Planner Tool. This environment is very similar to that of Google. I collected the same data that described in previous section for the same set of keywords.

Figure 1. Google Adwords Keyword Planner Tool

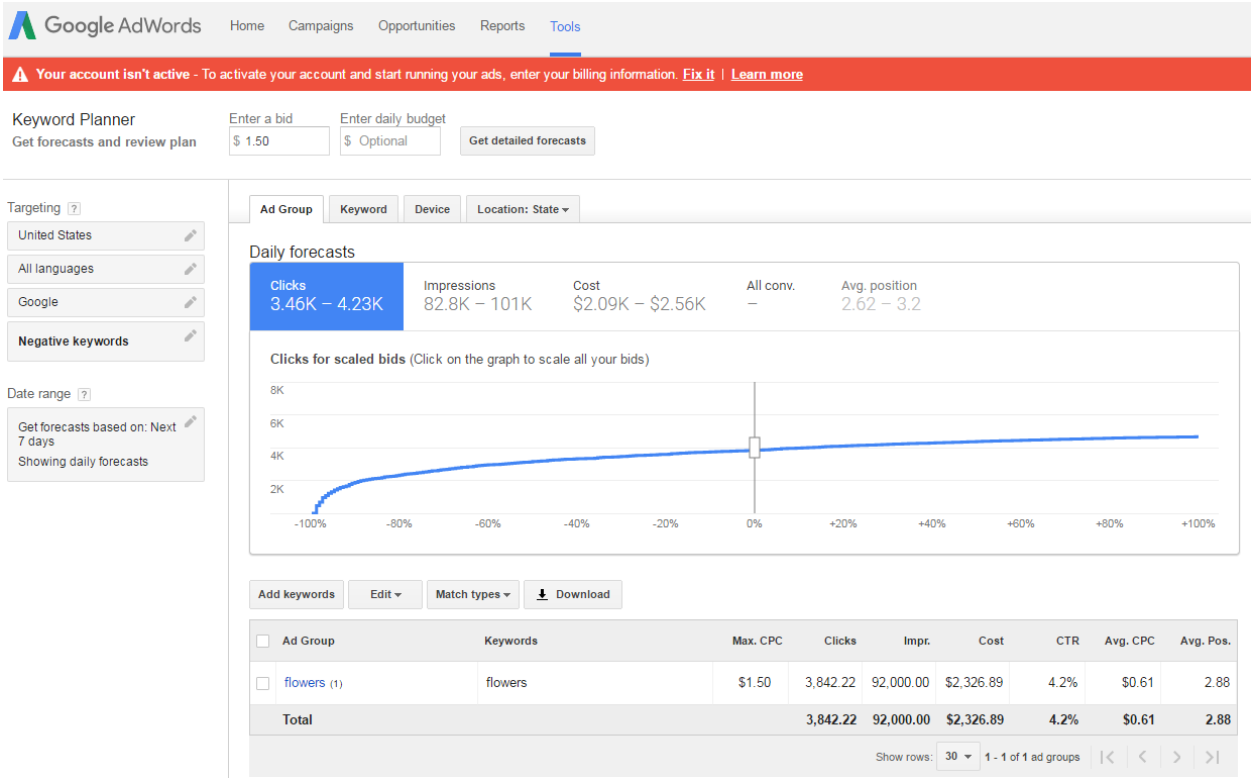


Table 3a. Google Adwords Keyword Planner Data for kw "Flowers"

Bid	Estimated Clicks	Estimated Cost	Estimated CTR	Estimated Average CPC	Estimated Average Position
30	6252.17	30231.97	6.80%	4.84	1.01
20	6224.41	27871.32	6.80%	4.48	1.05
10	6040.69	20683.54	6.60%	3.42	1.34
8	5946.76	17878.69	6.50%	3.01	1.51
6	5801.61	14609.75	6.30%	2.52	1.77
4	5518.47	10394.77	6.00%	1.88	2.09
2	4760.76	4550.65	5.20%	0.96	2.67
1	3631.13	1350.7	3.90%	0.37	3.25
0.5	2989.42	657.78	3.20%	0.22	3.49
0.1	1527.93	94.8	2.40%	0.06	3.9

Table 3b. Bing Adcenter Keyword Planner Data for kw "Flowers"

Bid	Estimated Clicks	Estimated Spend	Estimated CTR	Estimated Avg. CPC	Estimated Avg. Position
30	450.29	3,159.26	6.53%	7.02	2.46
20	382.3	2,007.97	6.24%	5.25	2.84
10	278.82	892.89	5.09%	3.2	3.38
8	246.77	666.97	4.82%	2.7	3.55
6	215.33	493.37	4.64%	2.29	3.74
4	162.76	280.69	4.25%	1.72	4.09
2	70.63	58.19	2.78%	0.82	4.84
1	35.59	18.07	2.20%	0.51	5.48
0.5	13.41	3.89	2.34%	0.29	6.61
0.1	0.61	0.04	2.03%	0.06	8.95

Source 3: Scraped Data

The third source of data is the relevant data extracted from actual result pages in Bing and Google. Figure 2 shows the first result page of Google for keyword Flowers and the data that has been collected. I have extracted list of sponsored links (top, bottom and side), advertisers rating, list of shopping ads and prices and list of organic links. I sent each keyword (a total of 1000 keywords in 10 categories) around 44 times a day to Google and Bing and saved the .html result pages during 4-month period between April to July. The total number of files saved is more than 1 million that is $2\ Platforms\ \times 300\ keywords\ \times 110\ days\ \times 44\ samples\ a\ day$. Below I provide some summary statistics for the collected data in Flower category.

Figure 2. Google Result Page. Stars illustrate the data that has been extracted from page

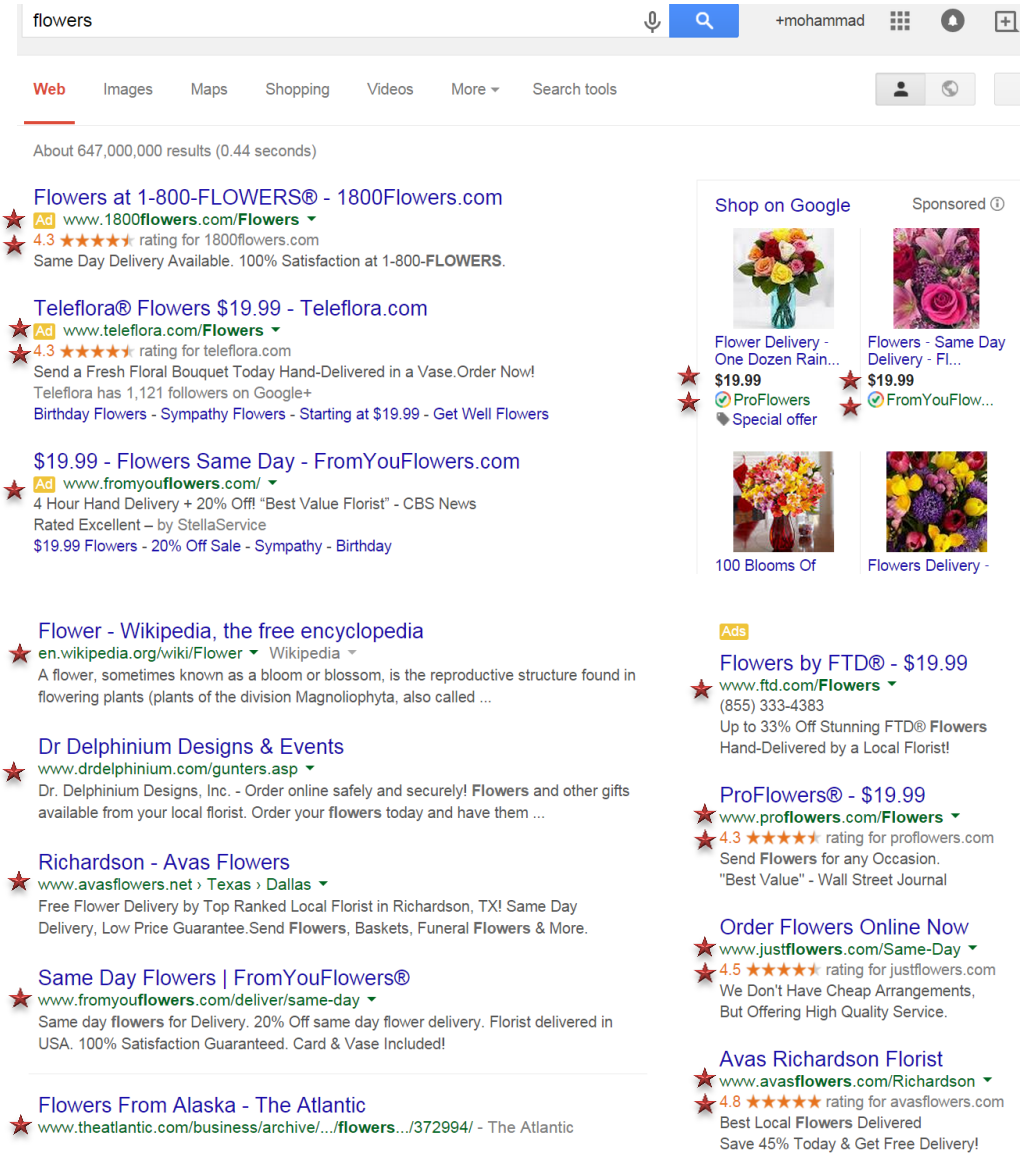


Table 4. Number of Keyword Ads, Shopping Ads and Organic Links

Bing	*No. kw Ads*	*On Top*	*On Bottom*	*On Side*	*No. Shopping Ad*	*No. Organic Links*
Mean	8.26	3.39	1.43	3.44	3.35	8.00
Standard Deviation	2.64	1.19	0.88	1.54	2.88	0.04
Minimum	1*	0	0	0	0	0
Maximum	13	4	2	7	6	9
Count	493582	493582	493582	493582	493582	493582

Google	*No. kw Ads*	*On Top*	*On Bottom*	*On Side*	*No. Shopping Ad*	*No. Organic Links*
Mean	8.86	1.55	0.01	7.31	3.00	9.31
Standard Deviation	2.30	1.38	0.13	1.76	2.11	0.86
Minimum	1*	0	0	0	0	1
Maximum	11	3	3	8	5	10
Count	493582	493582	493582	493582	493582	493582

*Only Pages with keyword Ads retained in Data set

In the entire sample, there are more than 1000 advertisers. Many of these advertisers have been observed only a few time in the entire data set. Table 5 show that the top 10 advertiser account for most of the Ads. Out of 1,392,677 total number of Bing keywords ads in the data set, 1,036,249 (75%) ads belong to top 10 advertiser. Similarly, in Google, 60% of Impression are for Top 10 advertisers. Moreover, most of these top advertisers are active in both platforms. There was one major advertisers (sendflowers.com) that was active in Google only.

Table 5 summarized top advertisers' average organic and paid lists rank on Bing and Google. Teleflora is on average in a higher place than fromyouflowers in Bing but in a lower place on Google.

Table 5. Top 10 Advertisers share of Impressions

Top 10 Advertisers	No. Ads Bing	No. Ads Google
teleflora	221,303	118,615
fromyouflowers	153,767	114,342
1800flowers	143,775	125,050
Ftd	125,521	119,058
proflowers	92,736	111,780
justflowers	80,982	17,092
flowerdeliveryexpress	80,758	74,221
avasflowers	61,940	92,716
bloomstoday	50,929	48,658
thebouqs	24,538	39,681
Sum of Top 10	1,036,249	861,213
All Advertisers	1,392,677	1,493,560

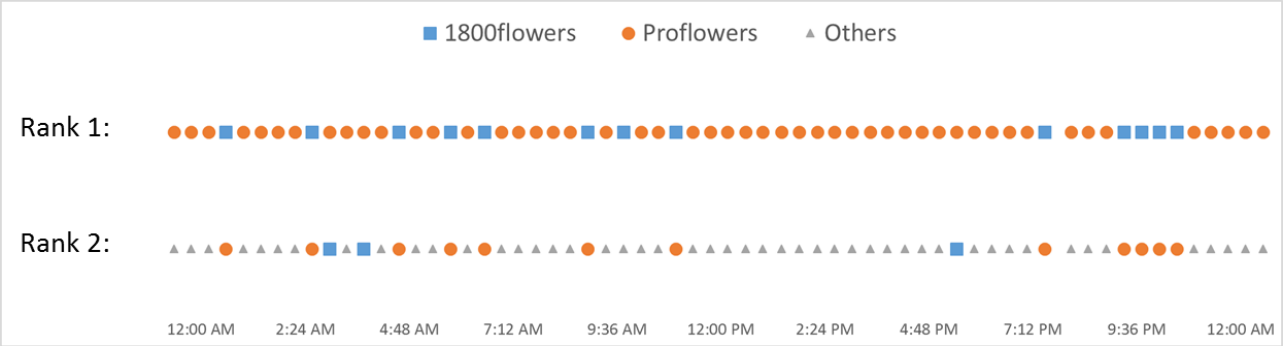
Note. These statistics are for first 6-weeks of data only.

Evidence on Bid Jamming

Bid jamming happens when a high-budget advertiser bids just below its low-budget rival in order to deplete its budget faster and obtain a higher advertising slot at a lower cost-per-click. It is difficult to directly find evidence for this behavior since we are not able to observe advertisers' bids. However, I could indirectly found some anecdotal evidence that the phenomena sometimes happen. Figure 3 shows an example of advertisers' ranking which might be associated with the bid jamming behavior. This figure shows the two top advertisers (Rank 1 and Rank 2) for keyword "*flower delivery*" on a particular day (*Jun 18*) on *Bing* search engine. On that day, this keyword was sent 63 times day and advertiser were scraped. As we can see in figure, *1800flowers* appeared in search result page 16 times (out of 63 observations). This advertiser appeared in first rank 13 times and in the second rank 3 times. However, 1800flowers was not in the search result page for the reset of 63-16=47 observations. *Proflowers*, on the other hand, appeared in all 63 observations in either Rank 1 or Rank 2. These observations could indicative of bi jamming behavior: The reason that 1800flowers is being displayed only in 16 out of 63 observations is because its budget

is restricted. In fact, it seems that 1800 *is being jammed* by its rival Proflowers who gets the firs link whenever 1800flowers is not displayed. Note that in Bing advertises can evenly distribute their (limited) budget during the day. This explains why 1800flowers appearance has been spread during the day.

Figure 3. Two top ads for keyword "flower delivery" on June 18 on Bing.
Total No. Searches made during the day = 63



Testable Hypothesis on Budget Allocation

Based on the theory that was developed in the paper, I found that advertisers' budget allocation strategies depend on the degree of heterogeneity in the market: if advertiser's are similar in terms of their total budgets, then they follow an asymmetric allocation strategy and partially differentiate. On the other hand, if advertisers are asymmetric (one of them has a large budget whereas the other on has a sufficiently small advertising budget), then advertisers follow a symmetric, similar allocation strategy and allocate their budget proportional to each platforms market share. These results imply that the degree of differentiation in allocation strategy must be negatively correlated with the degree of budget heterogeneity in the market. It can be shown the same relationship hold also for spending. Intuitively, when advertisers are similar, they become jammer in one platform and jammee in the other. Therefore, their spending will also be similar. On the other hand, when budgets are heterogeneous, one advertiser dominates both platforms and as a results its spending becomes much lower that its rival. So, the heterogeneity in advertiser budgets are positively

correlated with their total spending across platforms. Therefore, I can develop the following hypothesis to be tested:

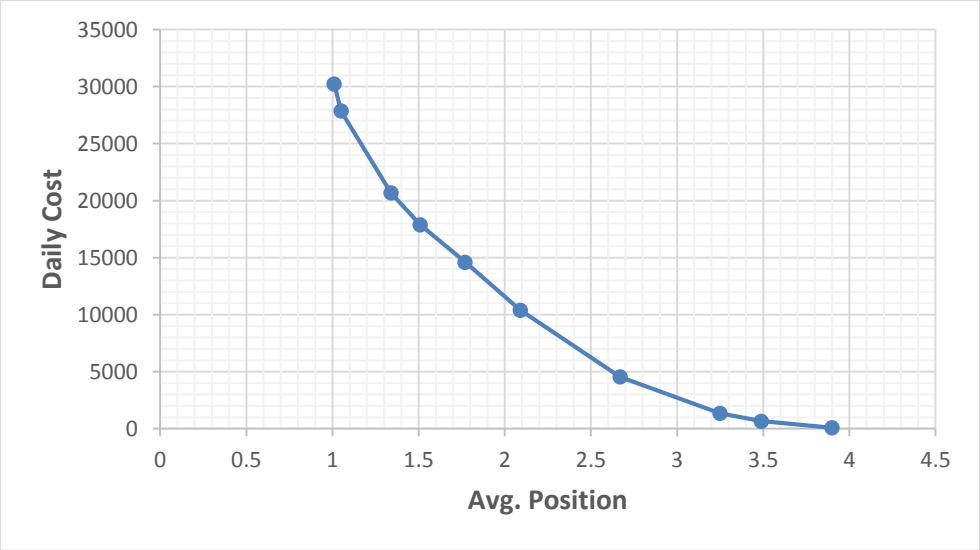
H1: Degree of differentiation in allocation strategies is ***negatively correlated to*** *the Degree of Asymmetry in advertisers' spending across platforms.*

To test this hypothesis, we need to estimate advertisers spending in Google and Bing. I do this by combining three sources of data described above.

Cost Estimation

To estimate advertisers' spending, I combine three sources of the data. Using the data obtained from Google Adwords and Bing Adcenter, I constructed an empirical relationship between average ranking in search engine result page and the cost an advertiser would incur. Fig. 4 show this cost-rank relationship for keyword "Flowers" in google search engine.

Figure 4. Cost-Rank Relationship in Google for keyword "*Flowers*"



Using the data scraped form Google and Bing, I also find the advertisers average daily ranking across these two platform for any keywords. I combine cost-rank data with the advertiser specific ranking data to estimate how much each advertiser is spending in Google and in Bing;

$$S_{ijt}^{k} = m_{ijt}^{k} * \sum_{R=1}^{11} c_{j}^{k}(R) * m_{ijt}^{k}(R)$$

Where S_{ijt} is an estimate of the Advertiser $\boldsymbol{i}$'s total spending for keyword $\boldsymbol{j}$, on search engine $\boldsymbol{k}$ at day $\boldsymbol{t}$. Moreover, m_{ijt} is the percentage of observations (for keyword $\boldsymbol{j}$, search engine $\boldsymbol{k}$ and day $\boldsymbol{t}$) in which Advertiser $\boldsymbol{i}$ was appeared on search engine result page. For example, in Figure 3, $m_{ijt} = \frac{16}{63}$ for 1800flower and $m_{ijt} = \frac{63}{63}$ for Proflowers. Furthermore, $m_{ijt}^{k}(R)$ is the percentage of observations (for keyword $\boldsymbol{j}$, search engine $\boldsymbol{k}$ and day $\boldsymbol{t}$) in which Advertiser $\boldsymbol{i}$ was appeared on Rank $\boldsymbol{R}$ of the result page (conditional on appearance). Finally, $c_{j}^{k}(R)$ is the cost-rank relationship for the keyword $\boldsymbol{j}$ on search engine $\boldsymbol{k}$ (see Figure 4).

Measures

Having estimated the advertisers' spending across Bing and Google, we are able to test relationship between the heterogeneity in total spending and allocation strategies. I first construct total spending and allocation strategy as below:

$$Advertiser\ \boldsymbol{i}'s\ \boldsymbol{Total\ Spent}\ (Keyword\ \boldsymbol{j}, Day\ \boldsymbol{t}) : \boldsymbol{TS_{ijt}} = S_{ijt}^{Bing} + S_{ijt}^{Google}$$

$$Advertiser\ \boldsymbol{i}'s\ \boldsymbol{Allocation\ Strategy}\ (Keyword\ \boldsymbol{j}, Day\ \boldsymbol{t}) : \boldsymbol{\delta_{ijt}} = \frac{S_{ijt}^{Bing}}{TS_{ijt}}$$

Next, I use the following constructs to measure the degree of heterogeneity in total spending and the degree of differentiation in allocation strategies:

$$Degree\ of\ differentiation\ in\ Allocation\ Strategies : X_{jt} = |\delta_{1jt} - \delta_{2jt}|$$

$$Degree\ of\ Asymmetry\ in\ Spending : Y_{jt} = \frac{|TS_{1jt} - TS_{2jt}|}{TS_{1jt} + TS_{2jt}}$$

Where Advertiser 1 and Advertiser 2 are the two largest spenders for keyword $\boldsymbol{j}$ on Day $\boldsymbol{t}$. Note that I scaled Y_{jt} by the summation of the total spending of two advertisers in order to scale it between 0 and 1.

Regression Results

I first ran a regression of Y_{jt} on X_{jt} for only one keyword (which is the most important keyword in the category), j="Flowers". The following table shows the regression results:

Table 6. Degree of Differentiation in Allocation Strategies (kw=" *Flowers*")

	Coefficients	*Standard Error*	*t Stat*
Intercept	0.1616	0.0172	9.3838
Degree of Asymmetry	-0.0960	0.0318	-3.0192
R Square	0.0778		
Observations	110		

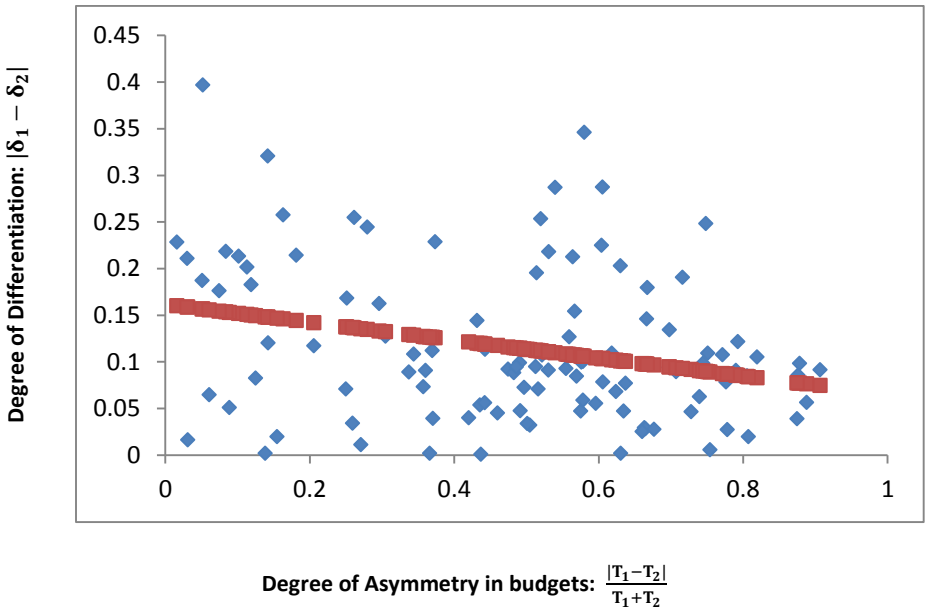
As we can see, the coefficient of Degree of Asymmetry is negative and significant, indicating that similar advertisers are more likely to differentiate across platforms. The Figure 5 show the fitted regression line and data points.

I also ran a regression of Y_{jt} on X_{jt} with fixed effects for all keywords;

$$y_{jt} = \alpha_j + \beta X_{jt} + u_{jt}\ , \quad t = 1,2, \dots, 110\ ,\ j = 1,2, \dots ,33$$

I removed the keyword with missing data points and used the following 33 keywords in flower category:

Figure 5. Fitted Regression Line and Data



33 Keywords: buy flowers, cheap flowers, flower delivery, cheap flowers delivered, discount flowers, floral arrangements, florist delivery, flower arrangements, flower delivery Houston, flowers, flowers delivered today, flowers delivery, flowers for delivery, flowers for funeral, flowers online, flowers same day delivery, flowers to go, funeral flower arrangements, funeral flowers, mothers day flowers, online florist, online flowers, order flowers, order flowers online, ordering flowers, same day delivery flowers, same day flower delivery, same day flowers, send flowers, send flowers cheap, send flowers online, sympathy flowers, cheap flower delivery

The Table 7 summarized the regression results.

Table 7. Degree of Differentiation in Allocation Strategies (33 keywords)
Intercepts not reported

	Coefficients	*Standard Error*	*t Stat*
Degree of Asymmetry	-0.2455	0.1077	-2.2795
R Square	0.0297		
Observations	3630		

As we can see, the coefficient is negative and significant, confirming the findings that the higher the asymmetry in advertisers' budget the lower the degree of differentiation is allocation strategies.

A Model of Bid Coordination Across Platforms

In this section I develop an empirical model to investigate the advertisers ranking across Bing and Google platforms. The main objective is to understand whether and how advertisers coordinate their bidding across platforms. I use different specifications of a 2-dimentiotional ranked Probit model.

The basic idea is that advertisers ranking helps us infer about their bids, which we don't observe and so is a latent construct,

$$Bid_{akt}^{s} > Bid_{bkt}^{s} \quad \leftrightarrow \quad Rank_{akt}^{s} < Rank_{bkt}^{s}$$

Where Bid_{akt}^{s} and $Rank_{akt}^{s}$ are (quality score weighted) bid and rank of advertiser a for keywords k at time t in search engine s. We can't observe bids and it's a latent construct. At any time and for each keyword, we observe two ranked lists of advertisers on Bing and Google. I first specify a very simple model for bid, and estimate it for only three advertisers. Let's

$$Bid_{akt}^{s} = v_{a}^{s} + \varepsilon_{akt}^{s}$$

So I only consider a advertiser-platform fixed effect, v_{a}^{s}. Suppose there is 3 advertisers, A, B and C. Then there are 36 (3! * 3!) different rankings across Search Engines:

$$(A^1 > B^1 > C^1 , A^2 > B^2 > C^2), \qquad (A^1 > B^1 > C^1 , A^2 > C^2 > B^2),$$
$$(A^1 > B^1 > C^1 , B^2 > A^2 > C^2), \ \ldots$$

We should calculate Probability of each. For example:

$Pr(A^1 > B^1 > C^1 , A^2 > B^2 > C^2) = \Pr(\tilde{\varepsilon}_{AB,kt}^{1} < \tilde{v}_{BA}^{1},\ \tilde{\varepsilon}_{BC}^{1} < \tilde{v}_{CB}^{1},\ \ \tilde{\varepsilon}_{AB}^{2} < \tilde{v}_{BA}^{2},\ \tilde{\varepsilon}_{BC}^{2} < \tilde{v}_{CB}^{2}$
where $\tilde{\varepsilon}_{AB}^{s} = \varepsilon_{A}^{s} - \varepsilon_{B}^{s}$ and $\tilde{v}_{AB}^{s} = v_{A}^{s} - v_{B}^{s}$. I assume the following covariance structure for the error terms,

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & r & 0 & 0 \\ 0 & 1 & 0 & 0 & r & 0 \\ 0 & 0 & 1 & 0 & 0 & r \\ r & 0 & 0 & s & 0 & 0 \\ 0 & r & 0 & 0 & s & 0 \\ 0 & 0 & r & 0 & 0 & s \end{bmatrix}$$

Where $r = cov(\varepsilon^1, \varepsilon^2)$, which the correlation of error term across search engines and s is the variance of error term in Google relative to Bing.

I estimate the model using MLE. I use data for three advertisers: A: Proflower.com , B: 1800Flower.com and C: Teleflora.com. I estimated for 3 model with different number of keywords.

Parameter	Model 1 (All Keywords)	Model 2 (4 Keywords only)	Model 3 (1 Keyword=Flowers)
Proflowers Bing	0	0	0
1800Flowers Bing	0.36***	-0.15***	-0.41***
Teleflora Bing	0.83***	0.98***	0.88***
Proflowers Google	0	0	0
1800Flowers Google	-0.14***	-1.02***	-0.48***
Teleflora Google	0.17***	1.08**	0.02***
Variance in Google (s)	1.01***	0.94***	1.07***
Across SE Correlation (r)	0.09***	0.08***	0.25***
No.Obs.	116676	3752	1064
No.Days	91+	91	91

+This is an older scraped data set for 3 months

Now I make the specification more general (for 10 advertisers) and include some covariates: advertiser's organic rank, whether advertiser is in shopping list , the price of advertiser in shopping list and advertiser's rating

$$Bid_{akt}^{s} = \beta_{0a} + \beta_1 org_{akt}^{s} + \beta_2 Ishop_{akt}^{s} + \beta_3 Ishop_{akt}^{s} \times Price_{akt}^{s} + \beta_4\ Rate_{at}^{s} + \varepsilon_{akt}^{s}$$

$Ishop$ is a dummy if the advertiser is present in Shopping list ads, $Price$ is its price at Shopping list, org is the rank in organic list. ε_{akt}^{s} is a normally distributed error term. We assume the only source of correlation is cross-platform one. In other words, we assume that error term is not correlated across keyword or time dimensions.

To estimate this model, I use frequency estimator (or accept/reject estimator). Given a set of parameter, and a fixed set of standard normally distributed error terms, I calculate the bids and maximized the likelihood. This analysis in on progress