We analyze a general model in which, at each echelon of the supply process, an arbitrary number of firms compete, offering one or multiple products to some or all of the firms at the next echelon, with firms at the most downstream echelon selling to the end consumer. At each echelon, the offered products are differentiated and the firms belonging to this echelon engage in price competition. The model assumes a general set of piece-wise linear consumer demand functions for all products (potentially) brought to the consumer market, where each product’s demand volume may depend on the retail prices charged for all products; consumers’ preferences over the various product/retailer combinations are general and asymmetric. Similarly the cost rates incurred by the firms at the most upstream echelon are general as well.

We initially study a two-echelon sequential oligopoly with competing suppliers, each selling multiple products through a pool of multiple competing retailers. We characterize the equilibrium behavior under linear price-only contracts. In the second stage, given wholesale prices selected in the first stage, all retailers simultaneously decide on their retail prices to maximize their total profits among all products of all suppliers they choose to do business with. In the first stage, the suppliers anticipate the retailers’ responses and all suppliers simultaneously maximize their total profits from all channels by selecting the wholesale prices. We show that in this two-stage competition model, a subgame perfect Nash equilibrium always exists. Multiple subgame perfect equilibria may arise but, if so, all equilibria are equivalent in the sense of generating unique demands and profits for all firms. We subsequently generalize our results to supply chain models with an arbitrary set of echelons, and show how all equilibrium performance measures can be computed with an efficient recursive scheme. Moreover, we establish how changes in the model parameters, in particular, exogenous cost rates, or intercept values in the demand functions, impact on the system-wide equilibrium. These comparative statics results allow for the quantification of cost pass-through effects and the measurement and characterization of the brand value of different retailers and suppliers.
1. Introduction

Over the past decades, economists, marketing and operations management researchers have developed a rich literature dealing with models of competition and coordination within supply chains. Most studies have focused on models where competition arises only at one echelon of the supply process. However, oligopolistic competition prevails, typically, at all echelons of the market. We call this general model with multiple echelons of competing upstream firms (e.g., manufacturers, suppliers) selling through multiple competing downstream firms (e.g., distributors, retailers), a sequential multi-product oligopoly.

We analyze a general model in which, at each echelon of the supply process, an arbitrary number of firms compete, offering one or multiple products to some or all of the firms at the next or possibly subsequent echelons, with firms in the most downstream echelon selling to the end consumer. At each echelon, the offered products are differentiated and the firms belonging to this echelon engage in price competition. Recently, a few two-echelon sequential oligopoly models have been addressed, with price competition at both echelons, and among an arbitrary number of competing firms each offering an arbitrary number of products, see Villas-Boas and Hellerstein (2006), Villas-Boas (2007) and Bonnet et al. (2013).

However, to our knowledge, this is the first such model in which the existence of a subgame perfect Nash equilibrium is proven, and a full characterization of the equilibrium behavior is provided. Moreover, we provide such a characterization for a supply network with an arbitrary number of echelons. Finally, our model is one in which the product assortment sold in the market is endogenously determined, along with all associated prices and demand volumes.

Our model assumes a general set of consumer demand functions for all $N$ products (potentially) brought to the consumer market, where each product’s demand volume may depend on the retail prices charged for all products. More specifically, the system of consumer demand functions, for all products potentially offered on the market, is based on a system of general affine functions.

However, the affine structure cannot be assumed to prevail on the complete price space: after all, outside of a special polyhedron $P$, the affine demand functions predict negative demand volumes. Shubik and Levitan (1980) stipulated that the generalization of the affine demand functions (on the complete price space, i.e., beyond $P$) must satisfy a simple and intuitive regularity condition: under any given price vector, when some product is priced out of the market, i.e., has zero customer demand, any increase of its price has no impact on the demand volumes. Soon et al. (2009) showed that, under minimal conditions, there exists one, and only one, such a regular extension. Under this regular extension, the demand volumes associated with an arbitrary price vector are obtained
by applying the affine functions to the projection of the price vector onto $P$. (This projected price vector is determined by solving a simple linear complementarity problem.)

This consumer demand model has many advantages:

(i) The model allows for general combinations of direct and cross-price elasticities and, in particular, asymmetric demand functions.

(ii) The model is parsimonious, nevertheless, as it is fully specified by a single $N \times N$ matrix of price sensitivity coefficients, and a single $N$-dimensional intercept vector.

(iii) Along with variants of the Multinomial Logit (MNL) model (e.g., nested or mixed MNL models), the most frequently used demand model in marketing, operations management and industrial organization studies, employs affine demand functions or extensions thereof, see, e.g., Federgruen and Hu (2013a).

(iv) Depending on the set of prices selected by the competing firms, a different subset of all potential products is offered on the market. Thus, the model specifies a product assortment, along with specific associated demand volumes. This is in sharp contrast to all other commonly used demand models. For example, under the above variants of the MNL model, all products attain some market share, irrespective of their absolute and relative price levels.\footnote{Ailawadi et al. (2010) open their section on “product assortment”, as follows: “In contrast with the vast amount of research on consumer response to product assortment [...] there is scant research on how manufacturers and retailers interact to determine the composition of the assortment.”}

We assume that prices are selected sequentially, starting with the firms in the most upstream echelon, followed by a simultaneous price selection by all firms in the next, more downstream echelon, for all of their products, et cetera until retailers in the most downstream echelon determine their retail prices. In the marketing literature, this type of pricing interaction is referred to as “Manufacturer Stackelberg (MS) models.” Other types of interaction are conceivable, for example, vertical Nash (VN) relationships, in which all firms select their prices simultaneously. Empirical support for (MS) interactions was provided by Sudhir (2001), Che et al. (2007) and Villas-Boas and Zhao (2005), see the recent survey by Ailawadi et al. (2010), describing the MS model as the widely applied “workhorse for modeling manufacturer-retailer interactions.”

We initially study a two-echelon sequential oligopoly with competing suppliers, each selling multiple products through a pool of multiple competing retailers. We characterize the equilibrium behavior under linear price-only contracts. In the second stage, given wholesale prices selected in the first stage, all retailers simultaneously decide on their product assortment and retail prices to maximize their total profits among all products of all suppliers they choose to do business with.
In the first stage, the suppliers anticipate the retailers’ responses and all suppliers simultaneously maximize their total profits by selecting the wholesale prices. We show that in this two-stage competition model, a subgame perfect Nash equilibrium always exists. Multiple, in fact infinitely many, subgame perfect equilibria may arise but, if so, all equilibria are equivalent in the sense of generating unique demands and profits for all firms. This characterization is obtained, as follows: Any choice of wholesale prices by the upstream firms induces a unique set of equilibrium retailer demand quantities, giving rise to an induced set of equilibrium demand functions for the first stage competition model. Moreover, we show that this set of demand functions is structurally similar to the demand functions faced by the retailers, thus allowing for a similar characterization of the equilibrium behavior among the suppliers. Finally, we derive a simple computational scheme for the (unique) equilibrium sales volumes and profit levels of all firms as well as the component-wise lowest equilibrium price vectors at both echelons. (Except for the possible solution of a single Linear Program, the evaluations only require matrix operations with respect to the matrix of marginal price sensitivities and the intercept vector which describe the affine part of the consumer demand functions, as well as matrices constructed from the same data.)

We subsequently generalize our results to supply chain models with an arbitrary set of echelons. The solution scheme is to backwards-inductively show that at every stage of the Stackelberg game, firms face a demand system uniquely specified by the downstream echelon best-response equilibria. This demand system has the same structural properties across all stages.

Our first and foremost contribution is to characterize the equilibrium behavior of a very general sequential oligopoly model with price competition at every echelon, and show how all equilibrium performance measures can be computed via a simple recursive scheme. Moreover, we establish how changes in the model parameters, in particular, exogenous cost rates, or intercept values in the demand functions, impact on the system-wide equilibrium. These comparative statics results allow for the quantification of cost pass-through effects: more specifically, our model can be used to quickly ascertain what impact changes in raw material and component prices have on the equilibrium prices and product assortments of all firms at all echelons of the supply chain network. The channel pass-through problem is of central interest in the marketing literature, see, e.g., Moorthy (2005) and Section 4 of Ailawadi et al. (2010) and the references therein, where it was addressed in a single-echelon setting. Similarly, the comparative statics with respect to the intercept vector, enable the measurement and characterization of the brand value of different retailers and suppliers, following the methodology of Goldfarb et al. (2009).

In addition, we derive the following qualitative insights from our comparative statics results, as applied to a two-echelon supply network:
(i) when the marginal cost rate of any of the suppliers’ products increases, the cost increase is “passed on” to the wholesale prices charged by the suppliers and, subsequently, the retail prices charged by the retailers. Focusing on the component-wise smallest equilibrium price vectors, we prove that the above exogenous cost increase results in all products’ equilibrium wholesale and retail prices to go up.

(ii) While all “direct” and all “cross-brand” pass-through rates are non-negative, these rates decline as a function of the suppliers’ marginal cost rates. In other words, when a supplier experiences an increase in the marginal cost rate for one of his products, the percentage pass-through applied to the equilibrium wholesale price of that product and all substitute products in the market (whether sold by that supplier or any of the competitors) is lower when the absolute level of the marginal cost rate is higher. The same holds for the pass-through rates that are applied to the equilibrium retail prices.

(iii) An upward shock of a supplier’s marginal cost rate for any of his products leaves the equilibrium product assortment in the market unchanged, or it expands the latter: an expansion is explained by the fact that the cost increase may enable other products that failed to be competitive under the original cost structure, to capture a positive market share, after the upward shock.

(iv) As may be expected, when a supplier experiences an increase in the marginal cost rate of one of his products, the equilibrium sales volume of that product declines, but that of all substitute products (whether sold by the same supplier or any of the competitors) increases.

(v) There exists an easily calculable upper bound for any of the supplier products’ marginal cost rates such that increases beyond this bound leave the equilibrium assortment, sales volumes and wholesale and retail prices unchanged.

(vi) The direct pass-through rates with respect to the equilibrium wholesale prices are bounded from below by 50%, assuming the product is part of the market assortment, i.e., when the equilibrium sales volume is positive. In that case, the pass-through rates with respect to the equilibrium retail prices are bounded from below by 25%. No such uniform lower bounds can be obtained for the cross-brand pass-through rates, other than that they are always positive, see above. These threshold results are generally, although not uniformly, consistent with empirical findings, see Besanko et al. (2005) and Dubé and Gupta (2008), inter alia. For example, Dubé and Gupta (2008, Table 1) report that less than 7% of all estimated cross-brand pass-through rates have a negative value that is significantly different from zero. Besanko et al. (2005, Figure 2) estimate that one-third of all direct pass-through rates between wholesalers and retailers, are less than 50%.
(vii) An increase in the value of any of the demand functions’ intercepts elicits an increase in the equilibrium wholesale and retail prices, demand volumes, and the retailers’ and suppliers’ profit margins for all products. It also increases each firm’s profit level and expands the product assortment (or leaves the latter unchanged).

(viii) One implication of the result in (vii) is that the methodology in Goldfarb et al. (2009) to measure brand values, assigns a positive value to any of these brand measures. (The latter is not apparent in the model used by Goldfarb et al. 2009 themselves.)

All vectors in this paper are column vectors and are represented by lowercase symbols. All matrices are denoted by capital letters. $\mathbb{R}_+ \equiv \{ r \in \mathbb{R} \mid r \geq 0 \}$. $\overline{S} \equiv \mathcal{N} \setminus S$ denotes the complement of a set $S$, where $\mathcal{N}$ is the set of all products. The cardinality of a set $S$ is denoted by $|S|$. For a vector $a$ and an index set $S$, $a_S$ denotes the subvector with entries specified by $S$. Similarly, for a matrix $M$ and index sets $S, T \in \mathcal{N}$, $M_{S,T}$ denotes the submatrix of $M$ with rows specified by the set $S$ and columns by the set $T$. The transpose of a matrix $M$ (vector $a$) is denoted by $M^T$ ($a^T$).

For notational simplicity, 0 may denote a scalar, a vector or a matrix of any dimensions with all entries being zeros, and $I$ is an identity matrix of appropriate dimensions. The matrix inequality $X = (x_{i,j}) \geq 0$ means that $x_{i,j} \geq 0$ for all $i, j$. A matrix is a $P$-matrix if all of its principal minors are positive. It is well known that a positive definite matrix is a $P$-matrix.

The remainder of the paper is organized as follows: Section 2 provides a review of the related literature. Starting with a supply chain network of two echelons, Section 3 presents the model, and reviews and extends the equilibrium behavior among the retailers under a given vector of wholesale prices, as developed in Federgruen and Hu (2013a). In Section 4, we characterize the equilibrium behavior of the suppliers and retailers in the sequential two-stage competition model. Section 5 studies the comparative statics. Section 6 extends, under minor conditions, all of our results in the two-echelon sequential oligopoly model to asymmetric price-sensitivity matrices. Section 7 generalizes the model and results to general sequential oligopolies involving any number of echelons. Section 8 concludes the paper.

2. Literature Review

As mentioned in the Introduction, most oligopoly or supply chain competition models, assume horizontal competition within a single echelon, supplied by a single supplier or selling to a single buyer, see, e.g., Vives (1999).

Four papers initiate the study of sequential oligopolies, all with two duopoly echelons. McGuire and Staelin (1983), later reprinted as McGuire and Staelin (2008), one of the 8 most cited classical
Marketing Science papers, consider the special case of our model where there are two suppliers each selling a single product exclusively to a dedicated retailer. The model is used to investigate the impact of vertical integration in one or both supply chains, see Section 3 for a more detailed discussion. Choi (1996) generalizes this model to allow each of the suppliers to sell to both retailers, giving rise to 4 products. The four demand functions are assumed to be affine on the complete price space. The model is used to compare various channel structures that arise when only some of the possible supplier/retailer combinations are able to trade.

Salinger (1988) assumes that at both echelons, two identical firms engage in Cournot competition for a homogenous good. If all four firms act independently, the equilibrium is found by simple backwards induction: the equilibrium in the downstream market induces a demand function relating the price for the intermediate good to the total quantity sold in the market. Ordover et al. (1990) model an upstream duopoly of two identical firms that produce a homogenous good and engage in price competition, combined with a downstream duopoly of two firms each selling a differentiated product and engaging in Bertrand competition as well. If all four firms are independent, in equilibrium, the upstream suppliers sell the product at their (common) marginal cost, so that the model reduces to a standard Bertrand duopoly. Hart and Tirole (1990) consider a variant of the Ordover et al. (1990) model in which the two downstream firms sell a homogenous good and engage in Cournot competition, allowing for the two upstream firms to incur different cost rates. See Chen (2001) and Chen and Riordan (2007) for recent variants of the Ordover et al. (1990) model with sequential price competition among two duopolies.

Very few sequential oligopoly papers consider supply chain networks with an arbitrary number of firms at some or all of the echelons: Corbett and Karmarkar (2001) consider a market consisting of any number of echelons, however one in which a single homogenous final good is sold to the end consumer. At the most downstream echelon, firms engage in Cournot competition for the single homogenous good, with an affine demand function. At each echelon, all competing firms are assumed to have identical characteristics and to engage in Cournot competition as well. The authors characterize the chain-wide equilibrium, showing that, at each echelon, the induced indirect demand function has the product price as an affine function of the aggregate quantity offered to the market. Cho (2014) investigates the impact of horizontal mergers in the Corbett and Karmarkar model. Saggi and Vettas (2002) consider a two-echelon market with two suppliers each selling a single product via its own dedicated network of retailers. Since all retailers within the same supplier’s network sell the same undifferentiated product, they all charge the same price for this
product. The prices for the two products are affine functions of the aggregate quantities sold in the market. Both the two suppliers, and the retailers engage, sequentially, in quantity competition.

Adida and DeMiguel (2011) recently analyzed the following generalization of Saggi and Vettas (2002): their model assumes $M$ suppliers, each selling the same collection of $P$ products to a set of $N$ retailers. The consumers perceive each of the $P$ products to be identical irrespective of which of the suppliers it is procured from. The demand model is specified by a set of affine inverse demand functions for all retailer/product combinations with multiplicative random noise factors. The retailers engage in quantity competition responding to announced wholesale prices, one for each of the $P$ products, and optimizing a linear combination of the mean and standard deviation of their profits. The suppliers engage in homogenous Cournot competition for each of the $P$ products, separately, based on the equilibrium aggregate function obtained from the retailer competition game. The equilibrium wholesale prices are those where aggregate retailer demand matches aggregate supplier supply. DeMiguel and Xu (2009) analyze a sequential competition model involving two groups of suppliers ultimately delivering the same homogenous good to the same consumer market, under a stochastic demand function relating the common product price to the aggregate quantity sold.

To our knowledge, Villas-Boas and Hellerstein (2006) and Villas-Boas (2007) are the first sequential price competition models, with an arbitrary number of suppliers, products and retailers. In the latter, the demand functions are generated from a mixed MNL model. A system-wide sequential equilibrium is computed, assuming that the price competition game, at each stage, has a unique price Nash equilibrium and that this equilibrium is obtained as the unique solution of the system of First Order Conditions. However, even the equilibrium behavior in the retailer game, under exogenously given wholesale prices, is unknown, as of yet. Allon et al. (2013) recently developed a set of sufficient conditions for the special case where each retailer sells a single product, but the equilibrium behavior in the multi-product case is still an open question. This applies, a fortiori, to the suppliers’ competition game in which the induced demand functions need to be derived from the equilibrium conditions in the retailer game. Villas-Boas and Hellerstein (2006) outline the same approach for a general set of differentiable demand functions; they proceed to illustrate the approach for the case of a two-supplier, two-retailer model with 3 products (discussed in Section 3, Figure 2), assuming affine demand functions.

The sequential oligopoly model in Villas-Boas (2007) was recently used by Bonnet et al. (2013) to characterize the German coffee market and to estimate cost pass-throughs, one of the applications of sequential oligopoly models, discussed in Section 5.
As mentioned, in Section 5, we show how our model can be used to provide general insights and easily computable estimates, in two important managerial areas: cost pass-throughs and the measurement of brand values in an equilibrium framework. We defer the review of the relevant literature in these two areas, to that section. Suffice it to state here, that the seminal structural approach to measure brand values is due to Goldfarb et al. (2009). These authors develop their approach in a sequential price competition model, however assuming a single retailer\(^2\), the same network structure as that in Villas-Boas and Zhao (2005).

As mentioned in the Introduction, traditional demand models for oligopolies with differentiated products, invoke a set of demand functions under which all potential products capture part of the market, irrespective of what prices are selected by the competing firms. A few papers have focused on the fact that retailers compete not only in terms of their price choices for a given assortment of products, but in terms of the selected assortment, itself, see Rusmevichientong et al. (2010a,b), Sauré and Zeevi (2013) and Besbes and Sauré (2010). To our knowledge, there are no existing papers addressing competitive assortment choices in multi-echelon settings.

### 3. The Two-Echelon Model

Our base model considers a market where a set \( J \equiv \{1, 2, \ldots, J\} \) of suppliers compete by selling any number of grossly substitutable products, via the same pool \( I \equiv \{1, 2, \ldots, I\} \) of competing retailers. In Section 7, we generalize this to settings with an arbitrary number of echelons. As a concrete example, consider the market for television sets. Each of the mega brands (Samsung, RCA, Magnavox, Mitsubishi etc.) sells a line of television types, differentiated by type (LCD or plasma), screen size (19”, 27”, 32” etc.) and screen resolution (720 vs. 1080 pixels), among other features. Different brands offer different subsets of the collection of all possible combinations; each sells these to some or all of the consumer electronics chains and general department stores.

We denote by \( \mathcal{N} \) the set of all products offered in the market and let \( N \equiv |\mathcal{N}| \). To differentiate among different products, we employ a triple of indices \((i, j, k)\): \( i \) denotes the retailer via which the product is sold, \( j \) the supplier procuring the product. We allow a supplier to sell multiple products through a retailer, and use the index \( k \) to differentiate among the various products sold by supplier \( j \) to retailer \( i \). We sometimes replace the triple index \((i, j, k)\) by a single index \( l \), where \( l \) may range

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\(^2\) In their “Limitations and Future Research” section, the authors state “Our model contains several assumptions about retailer behavior that may influence the results. We assume that there is a single monopolistic retailer and that retailers stock all brands. Modeling retail competition [...] and retail stocking decisions may provide a deeper understanding of the influence of the retail environment on brand values.” Our paper relaxes both restrictions.
from 1 to \(N\). Let \(K(i, j)\) denote the set of products offered by supplier \(j\) to retailer \(i\). A supplier may offer different sets of products to different retailers. For \(i \in \mathcal{I}, j \in \mathcal{J}\) and \(k \in K(i, j)\), let

\[
\begin{align*}
c_{ijk} &= \text{the constant marginal supply cost of supplier } j \text{ for product } k \text{ sold at retailer } i, \\
p_{ijk} &= \text{the retail price charged by retailer } i \text{ for product } k \text{ provided by supplier } j, \\
w_{ijk} &= \text{the wholesale price charged by supplier } j \text{ for product } k \text{ sold at retailer } i, \\
d_{ijk} &= \text{the consumer demand for product } k \text{ provided by supplier } j \text{ at retailer } i.
\end{align*}
\]

Let \(p, w, c\) and \(d\) be the corresponding vectors.

**Figure 1** A Two-Product Channel Structure (see McGuire and Staelin 1983, 2008)

![Diagram of a two-product channel structure with supplier 1 offering products 1 and 2 to retailer 1 and supplier 2 offering product 1 to retailer 2.]

*Note.* We have \(N = 2\) items: \(N = \{(1, 1, 1), (2, 2, 1)\}\).

Figure 1 depicts a simple structure with only \(N = 2\) products. This structure was considered by McGuire and Staelin (1983), later reprinted as McGuire and Staelin (2008). The authors compute the sequential equilibrium and compare it with those arising when (i) each supplier merges with his retailer; (ii) only one of the suppliers merges with his retailer. The equilibrium under (i) may be determined by fixing \(w_{111} = c_{111}\) and \(w_{221} = c_{221}\), with that under (ii) by fixing only \(w_{111} = c_{111}\).

**Figure 2** Another Channel Structure

![Diagram of a three-product channel structure with supplier 1 offering products A and B, supplier 2 offering product C to retailer 1 and retailer 2.]

*Note.* We have \(N = 5\) items: \(N = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (2, 2, 1)\}\). For instance, the triple \((2, 1, 1)\) refers to the sole product (product B) that is sold through retailer 2 and provided by supplier 1.

See Figure 2 for an example of a possible channel structure in which three products are offered by two suppliers. Since items are differentiated on the basis of the distributing retailer, the channel structure gives rise to \(N = 5\) distinct items. When eliminating product A, one retrieves the channel structure in Choi (1996) and Moorthy (2005), so that \(N = 4\). (The latter and Villas-Boas and Hellerstein 2006 also consider settings where either product B or C is offered exclusively to one of the retailers, reducing \(N\) to \(N = 3\).)
For any $i \in I$, let $\mathcal{N}(i)$ denote the set of products offered to retailer $i$ by the various suppliers, i.e.,

$$\mathcal{N}(i) = \{(i, j, k) \mid j \in J, k \in K(i, j)\}.$$ 

This set is determined by the channel structure. Depending upon the prices selected by the suppliers and retailers, it is possible that only a subset of the products offered to any given retailer are actually sold there. Indeed, part of the retailers’ choices is to determine which products offered by the various suppliers are worth carrying and at which set of prices.

We assume that the suppliers may select arbitrary combinations of wholesale prices. In some settings, these price choices may need to be constrained: for example, in some countries, firms are restricted in terms of their ability to differentiate their prices for an “identical” product sold to different retailers. In the US, such restrictions arise potentially because of the Robinson-Patman Act, a Federal law enacted at the start of the 20th century. To appreciate the importance and prevalence of such price restrictions, it is important to note that, for example, in the European Community, there is no direct legislative equivalent to the US Robinson-Patman Act, see, e.g., Spinks (2000) and Whelan and Marsden (2006).

Even in the US, Kirkland and Ellis (2005) write, when reviewing the “realities of the Robinson-Patman Act” that “everyone price discriminates. [...] Manufacturers of all kinds, selling to national accounts and local distributors, do it.” The same authors point out that over the past several decades, many economists and federal judges, as well as the antitrust enforcement agencies of the Department of Justice and the Federal Trade Commission, have come “to view the Robinson-Patman Act as itself – potentially – ‘anticompetitive,’ leading to higher rather than lower prices, hurting rather than benefiting consumers.” Based on the same consideration, the Antitrust Modernization Commission (AMC), established by the 2002 Antitrust Modernization Commission Act, recommended in its final report AMC (2007) that Congress finally repeal the Robinson-Patman Act.3 As a result, there has been no government challenge, in the ten years preceding the Kirkland and Ellis (2005) report, to any company’s price discrimination under the Act, with just one exception, described as being “anomalous”. Moreover, “even the number of those [privately originated] challenges has diminished in recent years, reflecting the poor record of success that Robinson-Patman Act claims have experienced in recent years.”

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3 The Commission wrote: “The Commission recommends that Congress finally repeal the Robinson-Patman Act (RPA). This law, enacted in 1936, appears antithetical to core antitrust principles. Its repeal or substantial overhaul has been recommended in three prior reports, in 1955, 1969, and 1977. That is because the RPA protects competitors over competition and punishes the very price discounting and innovation in distribution methods that the antitrust laws otherwise encourage. At the same time, it is not clear that the RPA actually effectively protects the small business constituents that it was meant to benefit. Continued existence of the RPA also makes it difficult for the United States to advocate against the adoption and use of similar laws against U.S. companies operating in other jurisdictions. Small business is adequately protected from truly anticompetitive behavior by application of the Sherman Act.”
Indeed, there are many defenses a “price differentiating” firm may invoke, see, e.g., Kirkland and Ellis (2005), as well as the discussion in Moorthy (2005). In addition, in many industries, there has been a steady increase in the use of retailer specific “private” labels and brand variants, with manufacturers offering different variants of the same product for different retail chains; the differentiation in packaging/labeling/after-sales support is sufficient to consider the products essentially different and protected from the implications of the Robinson-Patman Act. Bergen et al. (1996) discuss this advantage as one of many benefits associated with “branded variants”.

While our base model, therefore, assumes that distributors and suppliers may differentiate their prices in arbitrary ways, we describe in Online Appendix C how the equilibrium analysis and behavior is to be amended when every supplier has to charge an identical price to all retailers, for each of her products.

3.1. The Demand Model

The demand value of each product may depend on the prices of all products offered in the market. As in the majority of supply chain competition models, we assume that this dependence is in principle described by general affine functions. In matrix notation, this gives rise to a system of demand equations:

\[ q(p) = a - Rp, \]  

where \( a \in \mathbb{R}^N_+ \) and \( R \in \mathbb{R}^{N \times N} \).

The matrix \( R \) is assumed to satisfy two properties: First, we assume that the various products are substitutes; this means that any product’s demand volume does not decrease when the price of an alternative product is increased: See, however, Federgruen and Hu (2013b) for a generalized model, allowing for certain types of complementarities.

**Assumption (Z).** The matrix \( R \) is a Z-matrix, i.e., has non-positive off-diagonal entries.

In addition, we assume:

**Assumption (P).** The matrix \( R \) is positive definite.

(A matrix is called as \( ZP \)-matrix if it is both a \( Z \)-matrix and a \( P \)-matrix.)

There are many standard numerical procedures to verify the positive definiteness property. The following dominant diagonality conditions are often assumed in the literature, as they are very intuitive and likely to hold in most applications:

\[(D) \text{ (strict row dominant diagonality)} \quad R_{ijk,ijk} > \sum_{(i',j',k') \neq (i,j,k)} |R_{i'j'k',ijk}|, \quad \forall (i,j,k), \]

\[(D') \text{ (strict column dominant diagonality)} \quad R_{ijk,ijk} > \sum_{(i',j',k') \neq (i,j,k)} |R_{ij'k',ijk}|, \quad \forall (i,j,k). \]
(D) is equivalent to the assumption that no product’s demand value increases due to a uniform price increase of all products by the same amount. (D’) is equivalent to the assumption that aggregate sales do not increase due to a price increase of any one of the products. The two dominant diagonality conditions are indeed sufficient to ensure that $R$ is positive definite, see Lemma 1 in Federgruen and Hu (2013a).

However, the affine structure (1) can only apply on the polyhedron

$$P = \{ p \geq 0 \mid q(p) = a - Rp \geq 0 \},$$

since, for a price vector $p \notin P$, the raw demand functions $q(\cdot)$ predict negative demand volumes, for some of the products. Shubik and Levitan (1980) suggest that the extension of the demand functions, beyond $P$, satisfy the following intuitive regularity condition:

**Definition 1 (Regularity).** A demand function $D(p) : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is said to be regular, if for any price vector $p$ and product $l$, $D_l(p) = 0 \implies D(p + \Delta e_l) = D(p)$ for any $\Delta > 0$, where $e_l$ denotes the $l$-th unit vector.

In other words, if a product, is priced out of the market, under a given price vector, an increase in its price does not affect any of the demand volumes. Soon et al. (2009) showed that there is one, and only one, regular extension $d(p)$ of the affine demand functions $q(p)$ in (1). Under this extension, the demand volumes generated under an arbitrary price vector $p$, are obtained by applying the affine function $q(\cdot)$ to the projection $\Omega(p)$ of $p$ onto the polyhedron $P$, i.e.,

$$d(p) = q(\Omega(p)), \quad (2)$$

where:

**Definition 2 (Projection).** For any $p \in \mathbb{R}^N_+$, the projection $\Omega(p)$ of $p$ onto $P$ is defined as the vector $p' = p - t$, with $t$ the unique solution to the following Linear Complementarity Problem (LCP):

$$d(p) = a - R(p - t) \geq 0, \quad t \geq 0 \quad \text{and}$$

$$t^T[a - R(p - t)] = 0. \quad (4)$$

The existence of a unique solution $t$ follows from the general theory of LCPs (see Cottle et al. 1992, Theorem 3.3.7).

We need the following properties of the projection operator.

**Lemma 1 (Projection).** (a) $\Omega(p) \in P$; if $p \in P$, $\Omega(p) = p$. 
(b) If $p \notin P$, $\Omega(p)$ is on the boundary of $P$.
(c) $\Omega(p)$ may be computed by minimizing any linear objective $\phi^T t$ with $\phi > 0$ over the polyhedron, described by (3).
(d) The projection operator $\Omega(\cdot)$ is monotonically increasing, and each component of $\Omega(\cdot)$ is a jointly concave function.

Proof of Lemma 1. See Online Appendix B. □

To simplify the exposition, we initially characterize the equilibrium behavior of the sequential oligopoly under the additional assumption that the matrix $R$ is symmetric:

Assumption (S). The matrix $R$ is symmetric.

Indeed, the matrix $R$ is necessarily symmetric, if we assume that the demand functions $d(p)$ are based on a “representative consumer” maximizing a quadratic utility function, see, e.g., Federgruen and Hu (2013a, Problem (QP)). However, the symmetry assumption is restrictive, as shown in empirical studies, see, e.g., Manchanda et al. (1999), Vilcassim et al. (1999), Dubé and Manchanda (2005) and Li et al. (2013). Fortunately, all of our results can be shown for asymmetric price sensitivity matrices $R$, under weak restrictions. This important extension is covered in Section 6.

3.2. The Retailer Competition Model

In order to characterize the equilibrium behavior in the sequential oligopoly model, we, first, need to understand, how the retailers respond to a given vector of wholesale prices $w$, selected by the suppliers. However, Federgruen and Hu (2013a) characterize the equilibrium behavior in this retailer competition game. To summarize the main results, we need to define the following vectors and matrices:

$$T(R) = \begin{pmatrix} R_{N(1),N(1)}^T & 0 & \cdots & 0 \\ 0 & R_{N(2),N(2)}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{N(I),N(I)}^T \end{pmatrix},$$

(5)

$$\Psi(R) = T(R)[R + T(R)]^{-1},$$

(6)

$$S \equiv \Psi(R)R = T(R)[R + T(R)]^{-1}R \text{ and } b \equiv \Psi(R)a.$$  

(7)

Federgruen and Hu (2013a) show that a pure Nash equilibrium always exists. Often, multiple, sometimes infinitely many, equilibria exist; however, all equilibria are equivalent in the sense of generating identical equilibrium sales volumes for all products. The dependence of these equilibrium sales volumes on $w$ is described by a new set of affine functions:

$$Q(w) \equiv b - Sw,$$  

(8)
when \( w \in W \equiv \{ w \geq 0 \mid Q(w) = b - Sw \geq 0 \} \), or more generally, by the unique regular extension \( D(w) \) of this set of affine functions (for arbitrary \( w \)):

\[
D(w) \equiv Q(w') = b - Sw',
\]

where \( w' = \Theta(w) \) is the projection of \( w \) onto the effective wholesale price polyhedron \( W \), defined as in Definition 2: \( w' \equiv w - t \), with \( t \) the unique vector such that \( t \geq 0, b - S(w - t) \geq 0 \) and \( t^T[b - S(w - t)] = 0 \). (When \( R \) is symmetric, one can verify that \( b \geq 0 \) and \( S \) is a \( Z \)-matrix, see Lemma 4 below; the projection operator thus has all the properties mentioned in Lemma 1, replacing \( P \) by \( W \).)

**Proposition 1 (Retailer Competition Model).** Fix \( w \in W \).

(a) The retailer competition game has a pure Nash equilibrium.

(b) Multiple, pure, Nash equilibria may exist; however, there exists a component-wise smallest equilibrium \( p^* \), and \( p^* \in P \).

(c) All equilibria \( \tilde{p} \) of this game have \( p^* \) as their projection, i.e., \( \Omega(\tilde{p}) = p^* \). This implies that all equilibria \( \tilde{p} \) share the same sales volumes \( d(\tilde{p}) = q(\Omega(\tilde{p})) = q(p^*) = a - Rp^* \) and the same profit levels for all retailers.

(d) If \( w \in W \),

\[
p^* = w + [R + T(R)]^{-1}q(w) = [R + T(R)]^{-1}a + [R + T(R)]^{-1}T(R)w
\]

is an affine function of \( w \).

(e) If \( w \notin W \), \( p^* = w' + [R + T(R)]^{-1}q(w') = [R + T(R)]^{-1}a + [R + T(R)]^{-1}T(R)w' \), where \( w' = \Theta(w) \) is the projection of \( w \) onto \( W \).

(f) \( p^* \) is an increasing function of \( w \).

**Proof of Proposition 1.** See Online Appendix B. \( \square \)

4. **The Two-Stage Competition Model**

In the previous section, we have shown that any wholesale price vector \( w \) induces a retailer competition game with an essentially unique equilibrium: there always exists an equilibrium price vector, and while there may be many, perhaps infinitely many, equilibria, they are all equivalent in the sense of generating identical sales volumes and profit levels for all retailers. If \( w \in W \), the equilibrium sales volumes are given by

\[
Q(w) = d(p^*(w)) = q(p^*(w)) = a - Rw - R[R + T(R)]^{-1}q(w)
\]
\begin{align}
\{I - R[R + T(R)]^{-1}\}q(w) &= \{[R + T(R)][R + T(R)]^{-1} - R[R + T(R)]^{-1}\}q(w) \\
T(R)[R + T(R)]^{-1}q(w) &= \Psi(R)q(w) = b - Sw,
\end{align}

where the second identity follows from \(p^* \in P\) and the third identity from Proposition 1(d). This confirms (8). Similarly, if \(w \notin W\), we have by Proposition 1(e) that (9) is confirmed since

\begin{equation}
D(w) = Q(w') = \Psi(R)q(w') = b - Sw'.
\end{equation}

Thus, the induced demand functions encountered by the suppliers, are the (unique) regular extension of the affine functions (8), as long as we can show that the matrix \(S\) is positive definite and has non-positive off-diagonal elements, i.e., it satisfies properties (P) and (Z), in the same way the original matrix of price sensitivity coefficients \(R\) does. Fortunately, both properties can be shown to apply. This implies that the supplier competition model is structurally analogous to the retailer competition model. We, first, obtain the following characterization of the equilibrium behavior in this first stage competition model: We, again, define equilibria to be equivalent if they result in the same sales volumes for all supplier/product combinations and the same profit values for all suppliers.

**Theorem 1.** (a) \(D(\cdot)\) is the unique regular extension of the affine induced demand function, see (8); the matrix \(S = \Psi(R)R\) is a positive definite, symmetric Z-matrix while the intercept vector \(b \geq 0\).

(b) If a pure equilibrium exists, there exists one and only one equilibrium in \(W\).

(c) Any equilibrium \(w^0 \notin W\), has \(\Theta(w^0) = w^*\); moreover, all equilibria are equivalent.

**Proof of Theorem 1.** (a) We need to show that the regular extension \(D(w)\) of the affine functions \(Q(w) = b - Sw\), is obtained by applying the affine functions to the projection \(\Theta(w)\). The latter result was shown in Soon et al. (2009), when the matrix \(S\) is a positive definite Z-matrix. By Lemma 4 below, \(S\) is a Z-matrix. Moreover,

\begin{equation}
S = \Psi(R)R = T(R)[R + T(R)]^{-1}R = T(R)[I + R^{-1}T(R)]^{-1} = [T(R)^{-1} + R^{-1}]^{-1}
\end{equation}

is positive definite, since \(R\), and hence \(T(R)\), are positive definite and the inverse of a positive definite matrix is positive definite.

If \(R\) is symmetric, so is \(S\): By (13), \(S = [R^{-1} + T(R)^{-1}]^{-1}\). Since \(R\) is symmetric, so is \(T(R)\) and so are their inverses \(R^{-1}\) and \(T(R)^{-1}\); the same applies to \([R^{-1} + T(R)^{-1}]\) and its inverse.

Finally, \(b = \Psi(R)a \geq 0\) follows from \(\Psi(R) \geq 0\), a result shown in Federgruen and Hu (2013a) Proposition 4(e).
Parts (b) and (c). The proof is analogous to that of Theorem 1 in Federgruen and Hu (2013a). The proof of part (b) only requires that $S$ is positive definite, a property just verified. The proof of part (c) merely requires that $b \geq 0$ and $S$ is a positive definite $Z$-matrix, properties verified in part (a). □

Thus, Theorem 1 establishes, for the supplier competition model, a major part of the full equilibrium characterization in Proposition 1 (the latter pertaining to the retailer competition model): if a pure Nash equilibrium exists, there exists a component-wise smallest equilibrium $w^*$ and that equilibrium belongs to $W$; all other equilibria have $w^*$ as its projection on $W$, and are equivalent to $w^*$.

To complete the full equilibrium characterization of Proposition 1, we need to show that a pure Nash equilibrium is guaranteed to exist and to provide an explicit formula for the component-wise smallest equilibrium $w^*$. Reorder the products so that their supplier index in the triple of indices $(i,j,k)$ comes first. Let $M(1),\ldots,M(J)$ denote the sets of products supplied by supplier 1,...,J, respectively. Define the matrix

$$T(S) \equiv \begin{pmatrix}
S_{M(1),M(1)}^T & 0 & \cdots & 0 \\
0 & S_{M(2),M(2)}^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & S_{M(J),M(J)}^T
\end{pmatrix}$$

and $\Psi(S) \equiv T(S)[S + T(S)]^{-1}$. Note that the operator which transforms $R$ into $T(R)$ is different from that mapping $S$ into $T(S)$. We nevertheless, use the same mapping $T(\cdot)$ for both operators to simplify the notation. Finally, analogous to the definition of the effective wholesale price polyhedron $W$, define the effective polyhedron $C$ of supplier cost vectors as follows: $C = \{c \geq 0 \mid \Psi(S)Q(c) \geq 0\}$.

Let $\Gamma(\cdot)$ denote the projection operator onto $C$.

**Theorem 2. (Characterization of Equilibria in the Supplier Competition Game).**

(a) $C \neq \emptyset$, since $0 \leq c^0 = S^{-1}b = R^{-1}a \in C$.

(b) If $c \in C$, there exists a unique wholesale price equilibrium $w^*(c)$ in $W$. Any equilibrium $w^0$ outside of $W$ has $\Theta(w^0) = w^*(c)$ and is equivalent to $w^*(c)$.

(c) If $c \notin C$, let $c' = \Gamma(c)$ denote the projection of $c$ onto $C$. Then $w^*(c')$ is the unique wholesale price equilibrium in $W$. Any equilibrium $w^0$ outside of $W$ has $\Theta(w^0) = w^*(c')$ and is equivalent to $w^*(c')$.

**Proof of Theorem 2.** (a) Clearly $Q(c^0) = b - S(S^{-1}b) = 0$. Moreover, since by Theorem 1(a), $S$ is a positive definite $Z$-matrix, $S^{-1} \geq 0$, see, e.g., Horn and Johnson (1991, Theorem 2.5.3). By Theorem 1(a), $b \geq 0$, so that $0 \leq c^0 = S^{-1}b = R^{-1}\Psi(R)^{-1}b = R^{-1}a$. (To verify the second
equality, note that both $R$ and $\Psi(R)$ are invertible: $R$ is invertible because it is positive definite, by assumption (P); $\Psi(R) = T(R)[T(R) + R]^{-1}$ is invertible as the product of two invertible matrices, with $T(R)$ invertible because it is positive definite, as well.)

(b) Analogous to the proof of Theorem 2 in Federgruen and Hu (2013a).

(c) Analogous to the proof of Theorem 3 in Federgruen and Hu (2013a), after establishing that

\[ \Psi(S)b \geq 0 \quad \text{and} \quad \Psi(S)S \text{ is a positive definite } Z\text{-matrix} \]  \hspace{1cm} (14)

to ensure that the projection onto the polyhedron $C$ in the space of cost rate vectors, is well defined, in the sense that any vector $c \in \mathbb{R}^N_+$ is projected onto a non-negative vector $c'$. By Theorem 1(a), $S$ is a symmetric positive definite $Z$-matrix and $b \geq 0$. Thus, the induced demand functions $D(w)$ are the unique regular extension of the system of affine functions $Q(w) = b - Sw$ with $(b, S)$ sharing the same properties as $(a, R)$. Applying the above arguments to the functions $Q(\cdot)$, (14) follows. □

Thus, for any cost rate vector $c \in \mathbb{R}^N_+$, there exists a component-wise smallest equilibrium $w^*(c)$ in the supplier competition game. Moreover, analogous to Proposition 1(d), one can show

\[ w^*(c) = c + [S + T(S)]^{-1}Q(c) = [S + T(S)]^{-1}b + [S + T(S)]^{-1}T(S)\Gamma(c). \]  \hspace{1cm} (15)

Note that on the polyhedron $C$, $\Gamma(c) = c$ so that $w^*(\cdot)$ is an affine function on this polyhedron.

Finally, we can show that the effective price polyhedra $P$, $W$ and $C$ are nested.

**Proposition 2.** $P \subseteq W \subseteq C$.

**Proof of Proposition 2.** $P \subseteq W$: Since $W = \{w \geq 0 : \Psi(R)q(w) \geq 0\}$, it suffices to show that $\Psi(R) \geq 0$: if $0 \leq x \in P$, $q(x) \geq 0$ and $\Psi(R)q(x) \geq 0$, i.e., $x \in W$. But, $\Psi(R) \geq 0$ follows from Proposition 4(e) in Federgruen and Hu (2013a). The proof of $W \subseteq C$ is analogous. □

**Example 1.** Consider the distribution structure analyzed in McGuire and Staelin (2008), as depicted in Figure 1 where supplier $i$, $i = 1, 2$, sells product $i$ exclusively through retailer $i$.

Assume that

\[ a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix}, \quad \text{with } 0 \leq \gamma_1, \gamma_2 \leq 1. \]

It is easily verified that $R$ is a positive definite $Z$-matrix. When $\gamma_1 \neq \gamma_2$, $R$ fails to be symmetric but all of the results in Propositions 1 and 2, and Theorems 1 and 2 continue to hold, see §6. Clearly, $T(R) = 1$ and

\[ \Psi(R) = [I + R]^{-1} = \frac{1}{4 - \gamma_1 \gamma_2} \begin{pmatrix} 2 & \gamma_1 \\ \gamma_2 & 2 \end{pmatrix}. \]
Then we have

\[ S = \Psi(R)R = \frac{1}{4 - \gamma_1 \gamma_2} \begin{pmatrix} 2 - \gamma_1 \gamma_2 & -\gamma_1 \\ -\gamma_2 & 2 - \gamma_1 \gamma_2 \end{pmatrix}, \]

\[ \Psi(S) = T(S)[S + T(S)]^{-1} = \frac{2 - \gamma_1 \gamma_2}{4(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2} \begin{pmatrix} 2(2 - \gamma_1 \gamma_2) & \gamma_1 \\ \gamma_2 & 2(2 - \gamma_1 \gamma_2) \end{pmatrix}. \]

Hence,

\[ C = \left\{ c \geq 0 \left| \begin{array}{c} (8 + 6\gamma_1 - 3\gamma_1 \gamma_2 - 2\gamma_1^2 \gamma_2) - (8 - 9\gamma_1 \gamma_2 + 2\gamma_1^2 \gamma_2^2) c_1 + \gamma_1 (2 - \gamma_1 \gamma_2) c_2 \\ (6 + 6\gamma_1 - 3\gamma_1 \gamma_2 - 2\gamma_1^2 \gamma_2) + \gamma_2 (2 - \gamma_1 \gamma_2) c_1 - (8 - 9\gamma_1 \gamma_2 + 2\gamma_1^2 \gamma_2^2) c_2 \end{array} \right. \geq 0 \right\}. \]

In Figure 3, we exhibit the effective retail price polyhedron \( P \), the effective wholesale price polyhedron \( W \) and the effective marginal cost polyhedron \( C \). As stated in Proposition 2, \( P \subseteq W \subseteq C \).

We also provide an example where \( c \in C \) and \( w^*(c) \in (W \setminus P) \). Let \( \gamma_1 = 0.7, \gamma_2 = 0.3 \). Then, with

\[ a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & -0.7 \\ -0.3 & 1 \end{pmatrix}, \]

it is easily verified that

\[ b = \Psi(R)a = \begin{pmatrix} 0.7124 \\ 0.6069 \end{pmatrix} \quad \text{and} \quad S = \Psi(R)R = \begin{pmatrix} 0.4723 & -0.1847 \\ -0.0792 & 0.4723 \end{pmatrix}, \]

and moreover,

\[ \Psi(S) = T(S)[S + T(S)]^{-1} = \begin{pmatrix} 0.5083 & 0.0994 \\ 0.0426 & 0.5083 \end{pmatrix}. \]

Consider \( c = (1, 1.5)^T \). It is easily verified that

\[ \Psi(S)Q(c) = \Psi(S)(b - Sc) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0, \]
By Theorem 2,

\[ w^*(c) = c + [S + T(S)]^{-1}Q(c) = \begin{pmatrix} 1.5519 \\ 1.5225 \end{pmatrix} \in W^o. \]

By Proposition 1(d),

\[ p^*(w^*(c)) = w^*(c) + [R + T(R)]^{-1}q(w^*(c)) = \begin{pmatrix} 1.8125 \\ 1.5331 \end{pmatrix} \in P^o \]

and

\[ d(p^*(w^*(c))) = a - Rp^*(w^*(c)) = \begin{pmatrix} 0.2607 \\ 0.0106 \end{pmatrix} > 0. \]

However, note that

\[ a - Rw^*(c) = \begin{pmatrix} 0.5139 \\ -0.0569 \end{pmatrix}, \]

i.e., \( w^*(c) \notin P \).

**Example 2 (The Implications of Disintermediation).** Consider a market with \( J \) manufacturers each selling a group of products to a manufacturer-associated chain of independently owned retailers. Initially, each manufacturer \( j \) sells its products via a dedicated wholesaler at a given price vector \( c^j \). Recall from Proposition 2 that \( P \subseteq W \subseteq C \). Assume \( c = (c^1, \ldots, c^J)^T \) is in the interior of \( (C \setminus W) \). What is the impact of disintermediation, i.e., what happens when the retailers can buy the products directly from their manufacturers? On the one hand, it is possible to show that retail prices will come down. More surprisingly, however, product variety will decrease, all cost efficiencies not withstanding: in the presence of the intermediary wholesalers, we get \( w^*(c) \in W^o \), hence \( p^*(w^*(c)) \in P^o \), i.e., all products are sold in the market. Without the intermediaries, the vector \( c \) becomes the new vector of “wholesale” purchase prices for the retailers. Since \( c \notin W \), \( p^*(c) \) is on the boundary of \( P \) implying that some products are no longer part of the retailer assortment.

### 5. Comparative Statics

In this section, we characterize the impact of various of the model parameters on equilibrium wholesale and retail prices, equilibrium sales volumes and product assortments. Among the model primitives, we focus, in particular, on the suppliers’ cost rate vector \( c \) and the intercept vector \( a \) of the demand functions \( q(\cdot) \), as both relate to important managerial questions.

As mentioned in the Introduction, much attention has been given to understanding the “pass-through” rates of exogenous cost changes: when a supplier changes the wholesale price for a given product, how will the different retailers respond to this price change for the same product (“direct pass-through rate”) as well as for other products in the same product category (“cross-brand
pass-through rates"). The literature has adopted two approaches, (i) structural/theoretical models and (ii) reduced form econometric models. The former derive the pass-through rates from a formal market model, by characterizing how equilibrium prices depend on exogenous cost rates. The reduced form approach stipulates a specific functional relationship between cost rates and equilibrium prices, unsupported by any underlying competition model, and uses empirical data to estimate the parameters in the resulting regression model.

Few papers have chosen to follow the first approach, presumably because of the difficulty to characterize the equilibria in multi-product multi-retailer models. Besanko et al. (2005, Section 2.2) provided a review of five such papers; all, but one, assume a retail market with a single retailer. Moorthy (2005) addressed the question in a special model with two manufacturers and two retailers, which arises as a special case of the network structure in Figure 2, without product A. (Moorthy also considers non-linear demand functions, with several concavity, supermodularity and dominant diagonal properties, to ensure the existence of a unique equilibrium.) Goldberg (1995) characterized the pass-through rates of exogenous wholesale prices in a retailer competition model with nested Logic demand functions.

As to the reduced form approach, the seminal paper is Besanko et al. (2005), estimating the pass-through behavior at Dominick’s Finer Foods, a major U.S. supermarket chain. The study involved 78 products over 11 categories. (Earlier contributions, going back to Chevalier and Curhan (1976) used accounting measures rather than a rigorous econometric study.) Besanko et al. (2005) stipulate either an affine dependency of the equilibrium retail prices with respect to wholesale prices, or an affine relationship among the logarithms of these prices. We will prove that the former (affine) structure prevails in our model, but only as long as the wholesale prices are selected within $W$. When $w \notin W$, the same affine functions need to be applied to $\Theta(w)$, its projection onto $W$. The statistical validity of the estimation results in Besanko et al. (2005), in particular the significance of cross-brand pass-through rates, was challenged by McAlister (2007). This resulted in a refined study by Dubé and Gupta (2008), confirming that most cross-brand pass-through rates are significant, indeed. To our knowledge, ours represents the first paper in which the impact of exogenous cost changes is characterized in a multi-echelon supply network of competing firms, i.e., in a sequential oligopoly.

Goldfarb et al. (2009) have argued that a firm’s brand value should be measured in an equilibrium framework. More specifically, consumer demand functions should be modeled as a function of the suppliers’ and/or retailers’ brands, represented by brand indicator variables.
The brand value of a firm is then defined as the difference between its profit value when the brand indicator variable equals one (i.e., in the presence of the brand effect), versus a counterfactual equilibrium value, when it is set equal to zero (i.e., in the absence of the brand effect). The authors apply this framework to a sequential two-echelon price competition model, with a single retailer, i.e., with $I = 1$, but $J$ and $K$ arbitrary. (Even so, the authors must assume that the first stage competition model among the suppliers is well defined and has a unique price Nash equilibrium, arising as the unique solution of the system of First Order Conditions.) The model was then applied to the ready-to-eat breakfast cereal market, with $J = 5$ national suppliers, each offering one or more products or subbrands. Each of 65 U.S. cities was represented as a single retailer, effectively ignoring competition among different supermarkets within a city.

More specifically, Goldfarb et al. (2009) assume that demands for the various products are specified by a mixed MNL model, in which the intercept of the utility measure of each product is specified as an affine function of suppliers’ brand indicator variables. Following the same approach in our demand model, we specify the intercepts as follows:

$$a_{ijk} = \alpha^T x_{ijk} + \sum_{j'=1}^{J} \beta_{j'z_{jj'}}$$

where $z_{jj'} = 1$ if $j = j'$ and $z_{jj'} = 0$ if $j \neq j'$; and $x_{ijk}$ represents a vector of observable attribute values for product $(i,j,k)$. The same methodology may be used to measure brand values associated with the different retailers, or with different subbrands, i.e., $(j,k)$-combinations. All of these brand value estimations amount to conducting comparative statics analyses with respect to the intercept vector $a$; this is the subject of subsection 5.2.

5.1. Comparative Statics with Respect to the Cost Rates $c$

In this subsection, we characterize the impact of changes in the suppliers’ cost rates, with respect to equilibrium prices, sales volumes and the product assortment. All effects are computable with little effort, requiring at most a few matrix multiplications and inversions and the solution of a single Linear Program with $N$ variables and constraints. Moreover, we derive various general first and second order monotonicity properties for the relationship between equilibrium retail and wholesale prices, on the one hand, and the cost rates on the other.

**Theorem 3 (Comparative Statics for the Cost Rates $c$).** Fix a cost rate vector $c^0$ and a product $l = (i,j,k)$, and consider the impact of an increase of $c_{ijk}$ from $c^0_{ijk}$ to $c'_{ijk} = c^0_{ijk} + \delta$.

(a) (Equilibrium Demand Volumes) There exists a minimal threshold $\Delta^+ \geq 0$ such that an increase of $\delta$ beyond $\Delta^+$ has no impact on any of the equilibrium demand volumes; when $\delta \leq \Delta^+$, product $l$’s demand volume decreases and the demand volume of all other products increases.
(b) (Equilibrium Assortment) An increase of $\delta$ beyond $\Delta^+$ has no impact on the equilibrium assortment; when $\delta \leq \Delta^+$, the equilibrium assortment remains the same or expands. There exists a second threshold $\Delta \leq \Delta^+$ such that, for $\delta \leq \Delta$, the equilibrium assortment does not change while product $l$’s demand volume decreases and that of all other products $l' \neq l$ increases proportionally with $\delta$.

(c) (Equilibrium Prices) The component-wise smallest equilibrium retail and wholesale price vectors $p^*$ and $w^*$ increase concavely with $\delta$.

Proof of Theorem 3. (a) Let $\gamma \equiv \Psi(S)b$ and $U \equiv \Psi(S)S$. When the suppliers choose the vector of cost rates $[c + \delta e_i]$, the resulting equilibrium sales volumes are obtained as the unique regular extension $D^S(c + \delta e_i)$ of the affine functions $Q^S(c + \delta e_i) = \gamma - U[c + \delta e_i]$. It follows from (14) that $\gamma \geq 0$ while $U$ is a positive definite Z-matrix, i.e., $U$ has positive diagonal elements and non-positive off-diagonal elements. Thus, as $c_i$ increases by $\delta$, $Q^S_l(c + \delta e_i)$ decreases linearly. Let $\Delta^+$ denote the root of the equation $Q^S_l(c + \delta e_i) = 0$. Thus $D^S(c + \Delta^+ e_i) = Q^S(c + \Delta^+ e_i) = 0$. By the definition of regularity, any increase of $\delta$ beyond $\Delta^+$ has no impact on any of the demand volumes, see Definition 1. When $\delta \leq \Delta^+$, the remaining monotonicity properties follow from Proposition 2 in Federgruen and Hu (2013a), applied to the system of demand functions $D^S(\cdot)$.

(b) It follows from part (a) that when $\delta > \Delta^+$, an increase in $\delta$ has no impact on any of the products’ demand volumes, and, a fortiori, on the equilibrium assortment. When $\delta \leq \Delta^+$, it follows from part (a), that the demand volume of all other products $l' \neq l$ increases (weakly), while that of product $l$ remains positive, by the definition of the root $\Delta^+$. This implies that, for $\delta \leq \Delta^+$, the product assortment remains the same or expands.

Finally, let $\Delta \equiv \max \{0 \leq \delta \leq \Delta^+ : \text{assortment } A^0 \}$ is the assortment under the cost rate vector $c = c^0 + \delta e_1$. Thus, when $\delta \in [0, \Delta]$, the product assortment remains given by the set $A^0$. This implies that the demand volume of any product $l \notin A^0$ remains equal to 0, while that of the products in $A^0$ varies linearly with $\delta$, see Proposition 2 in Federgruen and Hu (2013a) applied to the affine functions $Q^S(\cdot)$ and the set $A^0$.

(c) It follows from Theorem 2(b) that $w^*$, the component-wise smallest equilibrium in the supplier competition game, has $w^* \in W$. Recall from (10) and (15) that

$$p^* = [R + T(R)]^{-1}a + [R + T(R)]^{-1}T(R)w^*, \quad (16)$$
$$w^* = [S + T(S)]^{-1}b + [S + T(S)]^{-1}T(S)\Gamma(c). \quad (17)$$

By Lemma 1(d) applied to the projection operator, $\Gamma(\cdot)$ is a monotonically increasing operator. The fact that both $p^*$ and $w^*$ are increasing vector-functions of $c$, thus follows by using the fact that $[R + T(R)]^{-1}T(R) \geq 0$ and $[S + T(S)]^{-1}T(S) \geq 0$, see Proposition 1 parts (d) and (f).
Finally, we have shown that every product \( l \)'s equilibrium retail and wholesale price \( p^*_l \) and \( w^*_l \) is an increasing affine function of \( \Gamma(c) \), while Lemma 1(d) shows that \( \Gamma(\cdot) \) is a vector of jointly concave functions. This establishes that \( p^*_l \) and \( w^*_l \) are jointly concave functions of \( c \).

Beyond the various monotonicity properties in Theorem 3, our model allows for simple expressions of the marginal pass-through rates of cost changes. The simplest expressions are available when \( c \in C^o \), the interior of the effective cost polyhedron \( C \).

**Corollary 1.** (a) Consider the retailer competition model under an arbitrary wholesale price vector \( w \in W^o \).

\[
\left( \frac{\partial p^*}{\partial w} \right) = [R + T(R)]^{-1}T(R) \geq 0.
\]

(b) Assume \( c^0 \in C^o \).

\[
\left( \frac{\partial p^*}{\partial c} \right) = [R + T(R)]^{-1}T(R)[S + T(S)]^{-1}T(S) \geq 0.
\]

**Proof of Corollary 1.** (a) If \( w \in W^o \), there exists a ball around the vector \( w \) which is contained within \( W \). The result is immediate from (16) and Theorem 2(c).

(b) If \( c^0 \in C^o \), there exists a ball around the vector \( c^0 \) which is contained within \( C \) so that, in this ball, \( \Gamma(c) = c \). The result then follows by substituting (17) into (16), while the sign of the elements of the pass-through matrix follows from Theorem 3(c).

When \( c \notin C^o \), the marginal pass-through rates of cost changes are no longer immediate from (16) and (17). To address this case, we first need the following lemma.

**Lemma 2.** Fix \( c^0 \in \mathbb{R}^N_+ \). Let \( A \) denote the (unique) assortment associated with the equilibria, under \( c^0 \). Let \( P^o_A \) denote the subspace of the retail price space \( \mathbb{R}^N_+ \) on which the same assortment \( A \) arises.

(a) \( P^o_A = \{ p \in \mathbb{R}^N_+ \mid d_A(p) = a^A - R^A p_A > 0 \text{ and } p_A \geq R^{-1}_A a^A R^A_a \} \), where \( a^A \equiv a_A - R^A_a R^{-1}_A a^A R^A_a \geq 0 \), and \( R^A \equiv R_{A,A} R^{-1}_A a^A \).

(b) \( p^*_A = [R^A + T(R^A)]^{-1}[a^A + T(R^A)w^*_A] \), \( w^*_A = [S^A + T(S^A)]^{-1}[b^A + T(S^A)c_A] \), \( b^A = \Psi(R^A)a^A \) and \( S^A = \Psi(R^A)R^A \).

**Proof of Lemma 2.** See Online Appendix B.
Theorem 3(c) shows that for every product \( l \), the (component-wise smallest) retail and wholesale price equilibrium, \( p^*_l \) and \( w^*_l \) are concave functions of the cost-rate vector \( c \). This implies that they are differentiable, almost everywhere. Moreover, their left- and right-hand (partial) derivatives with respect to any of the cost rates exist, everywhere. These can be obtained, straightforwardly, from the vector equations (18) and (19) in Lemma 2. Define \( (\frac{\partial p^*_l}{\partial c})^- \) as the matrix of left-hand derivatives.

**Corollary 2.** Fix \( c \in \mathbb{R}^N_+ \). Let \( A \) denote the associated equilibrium assortment. Then

\[
\left(\frac{\partial p^*_A}{\partial c}\right)^- = [R^A + T(R^A)]^{-1}T(R^A)[S^A + T(S^A)]^{-1}T(S^A).
\]  

**(20)**

**Proof of Corollary 2.** Fix a product \( l = (i, j, k) \) and consider the marginal impact of a decrease of its cost rate \( c_l \) to \( c_l - \delta \). As in the proof of Theorem 3(b), one verifies that a positive threshold \( \Delta > 0 \) exists, such that the assortment \( A \) remains unchanged, as long as \( \delta \leq \Delta \). (Unlike for cost increases, the threshold for cost reductions must be positive.) Then, equations (18) and (19) apply for all \( c_l \in (c^0_l - \Delta, c^0_l] \). The left-hand derivative expressions follow immediately. \( \square \)

**Remark 1.** An expression, similar to (19), provides the matrix of right-hand derives \( (\frac{\partial p^*_A}{\partial c})^+ \). In fact, as explained, almost everywhere, \( (\frac{\partial p^*_A}{\partial c})^+ = (\frac{\partial p^*_A}{\partial c})^- : (\frac{\partial p^*_A}{\partial c})^- \). However, as shown in the proof of Theorem 3(b), it is possible that, for a given product \( l \), any increase of its cost rate \( c_l \) results in a new product \( l' \), to be added to the equilibrium assortment \( A \), resulting in a new assortment \( A^+ \). (This corresponds with the case where in Theorem 3(b), the threshold \( \Delta = 0 \).) In that case, the matrix \( (\frac{\partial p^*_A}{\partial c})^+ \) is given by (20), with \( A \) replaced by \( A^+ \).

While the **exact** expressions of the cost pass-through rates in (20) are easily computed with a few matrix multiplications and inversions, we derive simpler lower and upper bounds that provide insights into the pass-through rates. For example, the lower bound shows that at least 50% of a reduction in the wholesale price of a product and at least 25% of a reduction in the supply cost rate are passed on to the consumer.

**Proposition 3 (Bounds for the Cost Pass-Through Rates).** (a) Consider the retailer competition model under a given wholesale price vector \( w \). Let \( A \) denote the equilibrium assortment. Then

\[
\frac{1}{2} \leq \left(\frac{\partial p_A}{\partial w_A}\right)^- = [R^A + T(R^A)]^{-1}T(R^A) \leq \frac{(R^A)^{-1}T(R^A)}{2}.
\]

**(21)**

(b) Fix \( c \in \mathbb{R}^N_+ \). Let \( A \) denote the equilibrium assortment. Then

\[
\frac{1}{4} \leq \left(\frac{\partial p_A}{\partial c_A}\right)^- = [R^A + T(R^A)]^{-1}T(R^A)[S^A + T(S^A)]^{-1}T(S^A) \leq \frac{(R^A)^{-1}T(R^A)(S^A)^{-1}T(S^A)}{4}.
\]

**(22)**
Proof of Proposition 3. (a) We write

\[ [R^4 + T(R^4)]^{-1}T(R^4) = [(T(R^4))^{-1}R^4 + I]^{-1}. \]

Since \( R \) is symmetric, \( T(R^4) \) is symmetric and then \( T(R^4) \geq R^4 \). By Lemma A.1(c), since \( T(R^4) \) is a ZP-matrix and \( R^4 \) is a Z-matrix, \( [T(R^4)]^{-1}R^4 \) is a ZP-matrix. By Lemma A.1(d), \( [T(R^4)]^{-1}R^4 + I \) is a ZP-matrix, as well. By Lemma A.1(a),

\[ [[T(R^4)]^{-1}R^4 + I]^{-1} \geq 0. \] (23)

Since \( T(R^4) \) is a ZP matrix, hence \( [T(R^4)]^{-1} \geq 0 \) by Lemma A.1(a). Then because \( R^4 \leq T(R^4) \), \( [T(R^4)]^{-1}R^4 \leq I \) and hence \( [T(R^4)]^{-1}R^4 + I \leq 2I. \) Since \( [T(R^4)]^{-1}R^4 + I \) is a ZP-matrix, we have by Lemma A.1(b), \( [R^4 + T(R^4)]^{-1}T(R^4) = [[T(R^4)]^{-1}R^4 + I]^{-1} \geq \frac{1}{2}. \)

To prove the upper bound in (21), let \( \Delta \equiv T(R^4) - R^4. \) Then \( R^4 = T(R^4) - \Delta. \) By the symmetry of \( T(R^4) \), \( \Delta \geq 0. \) Then

\[ R^4[R^4 + T(R^4)]^{-1}T(R^4) = [T(R^4) - \Delta][R^4 + T(R^4)]^{-1}T(R^4) \]
\[ \leq \frac{1}{2}[2T(R^4) - \Delta][R^4 + T(R^4)]^{-1}T(R^4) \]
\[ = \frac{1}{2}[T(R^4) + R^4][R^4 + T(R^4)]^{-1}T(R^4) = \frac{1}{2}T(R^4), \]

where the inequality is due to \( \Delta \geq 0 \) and \( [R^4 + T(R^4)]^{-1}T(R^4) \geq 0, \) by (23). Since \( (R^4)^{-1} \geq 0, \) we have the desired upper bound.

(b) Analogously, since \( S \) is symmetric (see Theorem 1(a)), \( T(S^4) \) is symmetric. Hence,

\[ \frac{1}{2} \leq [S^4 + T(S^4)]^{-1}T(S^4) \leq \frac{(S^4)^{-1}T(S^4)}{2}. \]

The bounds on the supplier cost pass through matrix follow immediately because all the bounds are non-negative matrices. ◊

Example 3 (Example 1 Continued). We revisit Example 1 and calculate the matrix of cost pass-through rates. Let \( \mathcal{A} \) denote the equilibrium assortment. We distinguish between two cases.

Case 1: \( \mathcal{A} = \mathcal{N} \). It follows from Corollary 1 and Proposition 3 that

\[ \frac{1}{2} \leq \left( \frac{\partial p^*}{\partial w} \right) = [R + T(R)]^{-1}T(R) = [I + R]^{-1} = \frac{1}{4 - \gamma_1 \gamma_2} \begin{pmatrix} 2 & \gamma_1 \\ \gamma_1 & 2 \end{pmatrix} \leq \frac{1}{2(1 - \gamma_1 \gamma_2)} \begin{pmatrix} 1 & \gamma_1 \\ \gamma_2 & 1 \end{pmatrix} = \frac{R^{-1}T(R)}{2}, \]

\[ \frac{1}{2} \leq \left( \frac{\partial w^*}{\partial c} \right) = [S + T(S)]^{-1}T(S) = \frac{2 - \gamma_1 \gamma_2}{4(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2} \begin{pmatrix} 2(2 - \gamma_1 \gamma_2) & \gamma_1 \\ \gamma_2 & 2(2 - \gamma_1 \gamma_2) \end{pmatrix} \leq \frac{2 - \gamma_1 \gamma_2}{2[(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2]} \begin{pmatrix} 2 - \gamma_1 \gamma_2 & \gamma_1 \\ \gamma_2 & 2 - \gamma_1 \gamma_2 \end{pmatrix} = \frac{S^{-1}T(S)}{2}, \]
\begin{align*}
\frac{1}{4} \leq \left( \frac{\partial p^*}{\partial c} \right) &= \left( \frac{\partial p^*}{\partial w} \right) \left( \frac{\partial w^*}{\partial c} \right) = \frac{2 - \gamma_1 \gamma_2}{4(1 - \gamma_1 \gamma_2)(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2} \left( \begin{array}{c}
8 - 3 \gamma_1 \gamma_2 \\
2 \gamma_1(3 - \gamma_1 \gamma_2)
\end{array} \right) \left( \begin{array}{c}
2 - \gamma_1 \gamma_2 \\
2 \gamma_1(3 - \gamma_1 \gamma_2)
\end{array} \right) \\
&\leq \frac{2 - \gamma_1 \gamma_2}{4(1 - \gamma_1 \gamma_2)(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2} \left( \frac{8 - 3 \gamma_1 \gamma_2}{2 \gamma_1(3 - \gamma_1 \gamma_2)} \right) = \frac{R^{-1}T(R)S^{-1}T(S)}{4}.
\end{align*}

Thus, the own-brand pass-through rate for the retailers (in response to an increase of a wholesale price) grows as either \( \gamma_1 \) or \( \gamma_2 \) increases from 0 to 1, from a minimum value of 50\% to a maximum value of \( \frac{2}{4-I} = 66\frac{2}{3}\% \). The cross-brand pass-through rates grow from 0\% to 33\frac{1}{3}\% as \( \gamma_1 \) and \( \gamma_2 \) increases from 0 to 1 (their maximum value).

Similarly, the own-brand pass-through rates for the suppliers in response to an increase of their input costs, is given by \( \frac{2 - \gamma_1 \gamma_2}{4(1 - \gamma_1 \gamma_2)(2 - \gamma_1 \gamma_2)^2 - \gamma_1 \gamma_2} \), an increasing function of \( \gamma_1 \gamma_2 \), which again increases from 50\% to 66\frac{2}{3}\%. Note that the cross-brand pass-through rate of product \( i \) due to a cost increase of product \( j \) is given by \( \gamma_i \left[ \frac{\left( \frac{2 - \gamma_1 \gamma_2}{4-I} \right)^{-1}}{\left( \frac{2 - \gamma_1 \gamma_2}{4-I} \right)^2} \right] \); both the numerator and the denominator of the expression within square brackets are increasing in \( \gamma_1 \gamma_2 \). Thus, for a given value of \( \gamma_i \), the cross-brand pass-through rate increases from \( \frac{2}{4-I} \) to \( \gamma_i \left[ \frac{\left( \frac{2 - \gamma_1 \gamma_2}{4-I} \right)^{-1}}{\left( \frac{2 - \gamma_1 \gamma_2}{4-I} \right)^2} \right] \), as \( \gamma_i \) increases from 0 to 1. Once again, the cross-brand pass-through rate varies between 0 and 33\frac{1}{3}\%. Finally, the marginal change rate in a product’s retail price, due to an increase of its supplier’s cost rate, increases from a minimum of 25\% to a maximum value of 55.6\%, as \( \gamma_1 \gamma_2 \) increases from 0 to 1.

The \( \gamma \)-parameters are a measure for the competitive intensity. The above results show that all cost pass-through rates increase as competition becomes more intense.

Case 2: \( A = \mathcal{N}(1) = \{1\} \), i.e., only one of the products, without loss of generality, product 1, is sold in the market. In this case, \( a^A = a_A - R_A a_A R_A^{-1} a_A = 1 + \gamma_1 \) and \( R^A = R_A a_A - R_A a_A R_A^{-1} a_A R_A a_A = 1 - \gamma_1 \gamma_2 \). Thus, \( \left( \frac{\partial p_A}{\partial c_A} \right)^{-1} = [S^A + T(S^A)]^{-1} T(S^A) = \frac{1}{2} \) and \( \left( \frac{\partial p_A}{\partial w_A} \right)^{-1} = \left( \frac{\partial p_A}{\partial w_A} \right)^{-1} = \frac{1}{4} \). In other words, in this monopoly case, the cost pass-through rates are at their minimum levels of 50\% and 25\%, see Proposition 3.

5.2. Comparative Statics with Respect to the Intercept Vector \( a \)

In this subsection, we derive comparative statics results for the intercept vector \( a \). We show, in particular, that all equilibrium retail and wholesale prices increase and that the equilibrium product assortment expands when the intercept vector \( a \) increases. In addition, an increase of one of or more of the intercept values, causes all of the suppliers’ and retailers’ profit margins to grow, as well as their aggregate profit values. One implication is that all brand values, discussed at the beginning of the section, are positive. We start with the following lemma. (In the remainder of this section, we write \( p^* \) and \( w^* \), as well as the projection operator \( \Gamma \), as \( p^*(w,a) \), \( w^*(c,a) \) and \( \Gamma(c,a) \), i.e., as functions of the prevailing intercept vector \( a \) and the vector of projected wholesale prices \( w \) and supplier cost rates \( c \), respectively.)
**Lemma 3.** Fix $a^1 \leq a^2$.

(a) $P(a^1) \subseteq P(a^2)$.
(b) $W(a^1) \subseteq W(a^2)$.
(c) $C(a^1) \subseteq C(a^2)$.
(d) $\Gamma(c, a^1) \leq \Gamma(c, a^2)$.
(e) $p^*(w, a^1) \leq p^*(w, a^2)$.
(f) $w^*(c, a^1) \leq w^*(c, a^2)$.

**Proof of Lemma 3.** See Online Appendix B. □

We are now ready for our main results.

**Theorem 4 (Comparative Statics on $a$).** Fix $c \geq 0$ and $0 \leq a^1 \leq a^2$. An increase in $a$ elicits an increase in the equilibrium wholesale and retail prices, demand volumes, and the retailers’ and suppliers’ profit margins for all products. It also increases each firm’s profit level and expands the product assortment. In other words:

(a) (Wholesale Prices) $w^*(\Gamma(c, a^1), a^1) \leq w^*(\Gamma(c, a^2), a^2)$.
(b) (Retail Prices) $p^*(w^*(\Gamma(c, a^1), a^1), a^1) \leq p^*(w^*(\Gamma(c, a^2), a^2), a^2)$.
(c) (Demand Volumes) $d(p^*(w^*(\Gamma(c, a^1), a^1), a^1)) \leq d(p^*(w^*(\Gamma(c, a^2), a^2), a^2))$.
(d) (Assortment) $A(a^1) \subseteq A(a^2)$.
(e) (Retail Profit Margins) $p^*(w^*(\Gamma(c, a^1), a^1), a^1) - w^*(\Gamma(c, a^1), a^1) \leq p^*(w^*(\Gamma(c, a^2), a^2), a^2) - w^*(\Gamma(c, a^2), a^2)$.
(f) (Wholesale Profit Margins) $w^*(\Gamma(c, a^1), a^1) - \Gamma(c, a^1) \leq w^*(\Gamma(c, a^2), a^2) - \Gamma(c, a^2)$.
(g) (Profit Levels) The profit earned by each firm increases with the intercept vector $a$.

**Proof of Theorem 4.**

(a) $w^*(\Gamma(c, a^1), a^1) \leq w^*(\Gamma(c, a^2), a^2) \leq w^*(\Gamma(c, a^2), a^2)$,

where the first inequality is due to Lemma 3(f), and the second inequality is due to Lemma 3(d) and Theorem 3(c), i.e., $w^*(c)$ is increasing in $c$.

(b) $p^*(w^*(\Gamma(c, a^1), a^1), a^1) \leq p^*(w^*(\Gamma(c, a^1), a^1), a^2) \leq p^*(w^*(\Gamma(c, a^2), a^2), a^2)$,

where the first inequality is due to Lemma 3(e) and the second inequality is due to part (a) and Theorem 1(f), i.e., $p^*(w)$ is increasing in $w$.

(c) Fix an arbitrary product $l$. We consider two cases.

Case 1. Suppose $d_l(p^*(w^*(\Gamma(c, a^1), a^1), a^1)) = 0$. Then the claim trivially holds.
Case 2. Suppose \( d_i(p^*(w^*(\Gamma(c,a^1),a^1)),a^1) = D_i^S(\Gamma(c,a^1),a^1) > 0 \), where \( D_i^S(\cdot) \) is defined in the proof of Theorem 3(a). Note that \( c_l \geq [\Gamma(c,a^2)]_l \geq [\Gamma(c,a^1)]_l = c_l \), where the first inequality follows from (3), the second one from Lemma 3(d), and the equality from \( D_i^S(\Gamma(c,a^1),a^1) > 0 \) and the complementary slackness conditions (3) and (4), applied to the demand system \( D \). Thus,

\[
0 < D_i^S(\Gamma(c,a^1),a^1) = [\Psi(S)b^1]_l - [\Psi(S)S]_{l,N}\Gamma(c,a^1) \\
= [\Psi(S)\Psi(R)a^1]_l - [\Psi(S)S]_{l,N}\Gamma(c,a^1) \\
\leq [\Psi(S)\Psi(R)a^2]_l - [\Psi(S)S]_{l,N}\Gamma(c,a^2) = D_i^S(\Gamma(c,a^2),a^2),
\]

where the inequality is due to (i) \( \Psi(S) \geq 0 \) since \( T(S) \) is symmetric and \( \Psi(R) \geq 0 \) since \( T(R) \) is symmetric, see Proposition 4(e) in Federgruen and Hu (2013a), which leads to \( \Psi(S)\Psi(R)a^1 \leq \Psi(S)\Psi(R)a^2 \); (ii) \( \Psi(S)S \) is a Z-matrix since \( T(S) \) is symmetric by Proposition 5(b) in Federgruen and Hu (2013a), and (iii) \( [\Gamma(c,a^1)]_l \geq [\Gamma(c,a^2)]_l \) for all \( l' \neq l \), and \( [\Gamma(c,a^1)]_l = [\Gamma(c,a^2)]_l = c_l \).

(d) Immediate from part (c).

(e) For any \( w \in W \), \( p^*(w) - w = [R + T(R)]^{-1}q(w) \), see Proposition 1(d). By Proposition 4(a) in Federgruen and Hu (2013a), the demand volumes under \( p^*(w) \) satisfy \( q(p^*(w)) = \Psi(R)q(w) \), with \( \Psi(R) = T(R)[R + T(R)]^{-1} \) invertible. Thus,

\[
p^*(w) - w = [R + T(R)]^{-1}[\Psi(R)]^{-1}q(p^*(w)) = [T(R)]^{-1}q(p^*(w)) = [T(R)]^{-1}d(p^*(w)),
\]

since \( p^*(w) \in P \). Moreover, \( [T(R)]^{-1} \geq 0 \), since \( T(R) \) is a ZP-matrix, see Lemma A.1(a). In other words, the vector of the retailers’ profit margins for all \( N \) products is an increasing function of the vector of equilibrium sales volumes and the latter increases in \( a \) by part (c).

(f) Let \( c^1 = \Gamma(c,a^1) \subseteq C(a^1) \subseteq C(a^2) \) and \( c^2 = \Gamma(c,a^2) \subseteq C(a^2) \). It follows from Lemma 3(f) that

\[
w^*(c^1,a^1) - c^1 \leq w^*(c^1,a^2) - c^1 = w^*(\Gamma(c,a^1),a^2) - \Gamma(c,a^1) \\
= [S + T(S)]^{-1}\Psi(R)a^2 + [S + T(S)]^{-1}T(S)\Gamma(c,a^1) \\
\leq [S + T(S)]^{-1}\Psi(R)a^2 + [S + T(S)]^{-1}T(S)\Gamma(c,a^2) = w^*(\Gamma(c,a^2),a^2) - \Gamma(c,a^2),
\]

where the second and last equality follow from Proposition 1(d) applied to the supplier competition game under \( a = a^2 \), respectively with \( c = c^1 \in C(a^1) \subseteq C(a^2) \) and \( c = c^2 \in C(a^2) \). The second inequality follows from \( [S + T(S)]^{-1}T(S) \geq 0 \), as shown in the proof of Proposition 1(f), and \( \Gamma(c,a^1) \leq \Gamma(c,a^2) \) by Lemma 3(d).

(g) Immediate from parts (c), (e) and (f). \( \square \)

Remark 2. One implication of part (g) of the above theorem is that brand values, as defined in Goldfarb et al. (2009), are always non-negative.
6. Asymmetric Price-Sensitivity Matrices

To simplify the exposition, we have, thus far, assumed that the matrix $R$ is symmetric. As explained in §3.1, this is a somewhat restrictive assumption, as many price-sensitivity coefficients often fail to be symmetric. In this section, we show how all of our results can be extended to asymmetric $R$ matrices, under far less restrictive conditions.

Starting with the retailer competition model, under a given wholesale price vector $w$, all of the characterizations in Proposition 1 continue to apply, under an asymmetric $R$-matrix, as long as the following property holds.

**Assumption (A).** $b = \Psi(R)a \geq 0$ and $S$ is a $Z$-matrix.

(Indeed, the proof of Proposition 1, in Appendix B, is obtained under this assumption (A), as opposed to the far stronger assumption (S) of a symmetric $R$ matrix.)

Assumption (A) is easily verified from the primitives of the model (i.e., the vector $a$ and the matrix $R$), with a single matrix inversion and a few matrix multiplications, see (5)-(7). The following lemma provides a strong but broad sufficient condition, see Proposition 5(b) in Federgruen and Hu (2013a).

**Lemma 4.** Assumption (A) applies, if the matrix $T(R)$ is symmetric.

Symmetry of the matrix $T(R)$ means that the cross-price sensitivity coefficients are identical for any pair of products sold by the same retailer. This symmetry assumption is considerably weaker than the global symmetry assumption (S) for the full matrix $R$. (As mentioned, when demand functions $d(p)$ are derived from a representative consumer maximizing a quadratic utility function, the resulting matrix $R$ of price sensitivity coefficients is always symmetric, implying that Assumption (A) is automatically satisfied.)

Even, the weak assumption (A) is only required when $w \not\in W$. Moreover, even when $w \not\in W$, many of the results in Proposition 1 can be guaranteed, simply on the basis of properties (P) and (Z) alone; in particular, there exists at most one equilibrium $p^*$ in $P$ and if $\bar{p} \not\in P$ is an equilibrium, then its projection $\Omega(\bar{p})$ is an equilibrium as well. Thus, if an equilibrium exists, there is a component-wise smallest equilibrium. Assumption (A) is required to ensure that the projection $\Theta(\cdot)$ is well defined, in particular that it maps any vector $w \in \mathbb{R}^N_+$ into a non-negative vector.

**Remark 3.** Assumption (A) may be replaced by a more complex but even weaker condition, referred to as Assumption (NPW) in Federgruen and Hu (2013a), see Proposition 5(a) and Theorem 3 there.
As far as the two-stage competition model is concerned, the results in Theorem 1 and Theorem 2 parts (a) and (b) all continue to apply under assumption (A). Theorem 2(c), i.e., the characterization of the suppliers’ equilibrium choices, when \( c \notin C \), requires a similar condition to assumption (A), now to ensure that the projection \( \Gamma(\cdot) \) onto \( C \) is well-defined, i.e., any suppliers’ cost rate vector \( c \in \mathbb{R}^N_+ \) is projected onto a non-negative vector \( \Gamma(c) \geq 0 \):

**Assumption (A’).** \( \Psi(S)b \geq 0 \) and \( \Psi(S)S \) is a Z-matrix.

Condition (A’) is, again, easily verified, numerically. As with assumption (A), it may be replaced by an even weaker although more complex condition, see Remark 3, above.

On the other hand, similar to condition (A), a sufficient condition for (A’) is that \( T(S) \) be symmetric:

**Theorem 5.** Assume \( T(R) \) and \( T(S) \) are symmetric. All of the results in Theorems 1 (except for the symmetry of matrix \( S \)) and 2, and Proposition 2, continue to apply.

**Proof of Theorem 5.** See Online Appendix B. □

Finally, symmetry of \( T(R) \) and \( T(S) \) is also sufficient to maintain all of our comparative statics results in section 5:

**Theorem 6.** Assume \( T(R) \) and \( T(S) \) are symmetric. All of the results in Theorems 3 and 4, Corollaries 1 and 2, and Proposition 3 continue to apply.

**Proof of Theorem 6.** See Online Appendix B. □

### 7. A Chain of Oligopolies: More Than Two Echelons

In this section, we discuss the generalization of our two-echelon model to one in which products (potentially) travel through an arbitrary number of distribution/production stages before reaching the end consumer. In the chain of oligopolies, there are \( m \) echelons, \( E_1, \ldots, E_m \), each with an arbitrary number of competing distributors. We number the echelons sequentially, starting with the most downstream echelon of retailers until reaching the most upstream echelon \( m \). We assume that firms in a given echelon only sell to firms in the next more downstream echelon, i.e., firms in echelon \( e \) only sell to those in echelon \( e - 1 \), while the retailers in echelon 1 sell to the consumer.

Products are partially differentiated by the route \( r \) traveled in the above multi-partite network. For any such path \( r \in \mathcal{R} \), the set of all possible paths, there may be up to \( K \) distinct products. We thus label each distinct product with a pair of indices \( (r,k) \): product \( (r,k) \) is the \( k^{th} \) product distributed along the route \( r \), \( r \in \mathcal{R} \) and \( k = 1, \ldots, K \).
Our starting point is, again, a set of retailer demand functions $d^{(1)}(p^{(1)})$, with $p^{(1)}$ the vector of retail prices, specified as follows on $\mathbb{R}^N$:

$$d^{(1)}(p^{(1)}) = \begin{cases} q^{(1)}(p^{(1)}) = a^{(1)} - R^{(1)}p^{(1)} & \text{if } p^{(1)} \in P^{(1)} \\ q^{(1)}(\Omega^{(1)}(p^{(1)})) & \text{if } p^{(1)} \notin P^{(1)}, \end{cases}$$

(24)

Here $a^{(1)}$ and $R^{(1)}$ are exogenously given, and $\Omega^{(1)}(p^{(1)})$ is the projection of the vector $p^{(1)}$ onto $P^{(1)}$, defined by Definition 2, with $a$ and $R$ replaced by $a^{(1)}$ and $R^{(1)}$. Following the analysis of Section 3, one verifies that each of the remaining echelons $e = 2, \ldots, m$ experiences equilibrium demand functions of a similar structure. Define, recursively,

$$a^{(e)} = \Psi^{(e)}(R^{(e-1)})a^{(e-1)},$$

$$R^{(e)} = \Psi^{(e)}(R^{(e-1)})R^{(e-1)},$$

(25)

where $\Psi^{(e)}(R^{(e-1)}) \equiv T^{(e-1)}(R^{(e-1)})[R^{(e-1)} + T^{(e-1)}(R^{(e-1)})]^{-1}$, $e = 2, \ldots, m$. The matrix $T^{(e)}(R^{(e)})$ is obtained from the matrix $R^{(e)}$ by replacing by zero, any entry which corresponds with a pair of products that is distributed via different distributors in echelon $e$; for any pair of products $(l, l')$ that is distributed via the same distributor in echelon $e$, $T^{(e)}(R^{(e)})_{l,l'} = R^{(e)}$. As shown in Section 4, after appropriate sequencing of the products, the matrix $T^{(e)}(R^{(e)})$ is block diagonal, with each block corresponding with a specific distributor in echelon $e$. The indirect demand functions for the firms in echelon $e$ are, again, of the form given by (24):

$$d^{(e)}(p^{(e)}) = \begin{cases} q^{(e)}(p^{(e)}) = a^{(e)} - R^{(e)}p^{(e)} & \text{if } p^{(e)} \in P^{(e)} \\ q^{(e)}(\Omega^{(e)}(p^{(e)})) & \text{if } p^{(e)} \notin P^{(e)}, \end{cases}$$

where $\Omega^{(e)}(p^{(e)})$ is the projection of the price vector $p^{(e)}$ onto $P^{(e)}$, $e = 2, \ldots, m$. As before $P^{(e)} \neq \emptyset$, since $0 \leq R^{(e)-1}a^{(e)} = R^{(e-1)-1}a^{(e-1)} = \cdots = R^{-1}a \in P^{(e)}$, $e = 2, \ldots, m$.

Applying Theorem 1(a) recursively one verifies that, the matrix $R^{(e)}$, $e = 2, \ldots, m$, is positive definite. By Proposition 1(a) (or Theorem 1(b)), this, by itself, guarantees the existence of equilibria at any stage of the sequential competition game.

However, to ensure that the indirect equilibrium demand functions for echelon $e$ are well defined, we, of course, need to establish that a unique sales volume equilibrium exists at any downstream echelon $l = 1, 2, \ldots, e - 1$, i.e., the equivalency of equilibria in the sense of generating the same sales volumes and profit levels throughout all downstream echelons (if multiple equilibria exist, they all project onto the same price vector in $P^{(l)}$ for any downstream echelon $l = 1, 2, \ldots, e - 1$). By Proposition 1 (or Theorem 2), this is guaranteed as long as each of the matrices $R^{(1)}, R^{(2)}, \ldots, R^{(e)}$ is a Z-matrix, i.e., has non-positive off-diagonal elements, and $a^{(1)}, a^{(2)}, \ldots, a^{(e)} \geq 0$. This condition can easily be checked numerically. In addition, by the proof of Lemma 4, the condition can be guaranteed, inductively, when $R^{(1)} = R$ is a symmetric matrix.
(If $R^{(e)}$ is symmetric, then $R^{(e+1)} = \Psi^{(e+1)}(R^{(e)})R^{(e)} = [R^{(e)}^{-1} + T^{(e)}(R^{(e)})^{-1}]^{-1}$; thus, $R^{(e+1)^T} = [(R^{(e)^T})^{-1} + (T^{(e)}(R^{(e)})^T)^{-1}]^{-1} = [R^{(e)}^{-1} + T^{(e)}(R^{(e)})^{-1}]^{-1} = R^{(e+1)}$ is symmetric as well.)

We conclude that as long as we can guarantee that each of the matrices $R^{(1)}, R^{(2)}, \ldots, R^{(m+1)}$ is a Z-matrix and $a^{(1)}, a^{(2)}, \ldots, a^{(m+1)} \geq 0$ (where $R^{(m+1)}$ and $a^{(m+1)}$ follow the same type of definition as $R^{(e)}$ and $a^{(e)}$, $e = 2, \ldots, m$), one essentially unique equilibrium exists at each stage of the sequential competition game; the resulting chain-wide equilibrium is a subgame perfect Nash equilibrium. Moreover, for any echelon $e$, the values of the unique price equilibrium in $P^{(e)}$, and the component-wise smallest price equilibrium among all equilibria, are computed as follows: Let $c$ denote the vector of marginal cost rates incurred by the firms in the most upstream echelon:

$$p^{*(m)} = \begin{cases} \{ c + [R^{(m)} + T^{(m)}(R^{(m)})]^{-1}q^{(m)}(c) \} & \text{if } c \in C \equiv P^{(m+1)}, \\ \{ \Omega^{(m+1)}(c) + [R^{(m)} + T^{(m)}(R^{(m)})]^{-1}q^{(m)}(\Omega^{(m+1)}(c)) \} & \text{if } c \notin C, \end{cases}$$

(26)

where $P^{(m+1)}(\supseteq R^{-1}a)$ follows the same type of definition as $P^{(e)}$, $e = 2, \ldots, m$, and

$$p^{*(e)} = p^{*(e+1)} + [R^{(e)} + T^{(e)}(R^{(e)})]^{-1}q^{(e)}(p^{*(e+1)}), \quad e = m - 1, \ldots, 1.$$  \hspace{1cm} (27)

To verify (26) and (27), invoke Proposition 1. Moreover, since $p^{*(m)} \in P^{(m)}$, it follows from Proposition 1 that $p^{*(m-1)}$ is given by (27) and $p^{*(m-1)} \in P^{(m-1)}$. One thus verifies, by induction that $p^{*(e)} \in P^{(e)}$ for all echelons $e = 1, \ldots, m$, so that (27) applies to all echelons $e = 1, \ldots, m - 1$.

The computation of all echelons’ equilibrium price vector $\{p^{*(e)} \mid e = 1, \ldots, m\}$ is thus confined to the following: first one recursively computes the matrices $R^{(e)}$ and intercept vectors $a^{(e)}$ for $e = 1, \ldots, m$, via (25). Determination of $p^{*(m)}$ may involve the computation of the unique solution of an LCP – but only if $c \notin C$ –, which can be achieved by solving a single Linear Program, see Lemma 1(c). The remaining computations involve only multiplications and inversions of matrices related to the price sensitivity matrix $R$.

Finally, the effective price polyhedra expand, at each echelon, as we move upstream in the supply chain network, generalizing Proposition 2. This is shown in Proposition 4, below, along with the fact that as $m$, the number of echelons, grows, the sequence of effective price polyhedra $\{P^{(e)}\}$ converges to a limiting polyhedron $P^*$.

**Proposition 4.** Assume $R = R^{(1)}$ is symmetric.

(a) For any $m \in \mathbb{N}$, $P^{(e)} = \{ p \geq 0 \mid a^{(e)} - R^{(e)}p \geq 0 \} \subseteq P^{(e+1)}$ for all $e = 1, 2, \ldots, m$.

(b) For any $m \in \mathbb{N}$, $P^{(e)} \subseteq H \equiv \{ p \mid 0 \leq p \leq R^{-1}a \}$ for all $e = 1, 2, \ldots, m + 1$.

(c) The sequence $\{P^{(e)}, e = 1, 2, \ldots, m + 1\}$ converges to a limiting polyhedron $P^*$.

**Proof of Proposition 4.** See Online Appendix B. □
8. Conclusion
We have analyzed a general sequential oligopoly model, in which, at each echelon of the supply process, an arbitrary number of firms compete by offering a single or multiple products to some or all of the firms in the next echelon. The model assumes sequential non-cooperative pricing in the sense that at the first stage of the multi-stage competition model, the firms of the most upstream echelon select their prices for all products. At the second stage competition model, the firms of the next more downstream echelon select their price menu. This process continues until at the last stage, the retailers select all of their retail prices.

Our consumer demand model is parsimonious. It is fully specified by a single $N \times N$-matrix of price sensitivity coefficients and a single $N$-dimensional intercept vector, with $N$ the number of potential products on the market. Unlike most traditional oligopoly models, under this demand model, the very assortment of products carried by the retailers, say, and the set of suppliers from which they procure are determined endogenously, rather than being pre-specified.

We provide a full characterization of the equilibrium behavior in this model: Consider, for example, a model with two echelons. We show that in this two-stage competition model, a subgame perfect Nash equilibrium always exists. Multiple subgame perfect equilibria may arise but, if so, all equilibria are equivalent in the sense of generating unique demands and profits for all firms. Indeed, even for a given vector of wholesale prices, the second stage retailer competition game always has an equilibrium but may possess multiple, possibly infinitely many, equilibria. Nevertheless, these various equilibria are equivalent in the above sense.

Moreover, we have shown that all equilibrium performance measures can be computed via a simple recursive scheme requiring only a few multiplications and inversions of matrices obtained from the primitive data of the model, plus the solution of at most one Linear Program with $N$ variables and constraints.

We have shown general comparative statics results with respect to the exogenous cost parameters in the model, as well as the intercept vector in the (affine part of) the demand functions: these comparative statics results have important implications for the assessment of cost pass-through rates and the measurement of brand values.

We have shown that all firms in a particular echelon sell all their products only to firms in the next downstream echelon. Sometimes upstream firms skip intermediate echelons for some of their products, possibly even selling directly to the end consumers. We refer the reader to Federgruen and Hu (2013b) for an analysis of this direct sales option. All of our results continue to apply, under a partial symmetry condition for the matrix $R$. We have also assumed that all products are
substitutes: see, however, Federgruen and Hu (2013b), for a generalized model, allowing for certain
types of complementarities.

Finally, the current model assumes that firms compete in terms of price only: it would be
desirable to extend our demand model to allow for dependence on other strategic instruments,
for example customer waiting times or quality measures, and firms competing with all available
strategic instruments. For recent examples of such models, in single-echelon settings, see Allon and
Federgruen (2007) and Yang et al. (2014) and the references therein.

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Online Appendix to
“Sequential Multi-Product Price Competition in
Supply Chain Networks”

A. Preliminaries.

We use the following properties of $ZP$-matrices.

**Lemma A.1 (Properties of $ZP$-matrices).** Let $X$ be a $ZP$-matrix and $Y$ be a $Z$-matrix such that $X \leq Y$, i.e., $Y - X \geq 0$. Then

(a) $X^{-1}$ exists and $X^{-1} \geq 0$;

(b) $Y$ is a $ZP$-matrix and $Y^{-1} \leq X^{-1}$;

(c) $XY^{-1}$ and $Y^{-1}X$ are $ZP$-matrices; and

(d) If $D$ is a positive diagonal matrix, then $DX$, $XD$ and $X + D$ are $ZP$-matrices.

**Proof of Lemma A.1.** (a)-(d). By Horn and Johnson (1991, Theorem 2.5.3), a $ZP$-matrix is a nonsingular, so-called, $M$-matrix. Properties (a)-(d) of $ZP$-matrices can be found in Horn and Johnson (1991, Section 2.5) as properties of $M$-matrices. □

B. Proofs.

**Proof of Lemma 1.** (a) See Lemma 2 in Federgruen and Hu (2013).

(b) Since $p \notin P$, the correction vector $t \neq 0$; thus, there exists a product $l$ with $t_l > 0$ and by (4), $[a - R(p - t)]_l = 0$, implying that $\Omega(p)$ is on the boundary of $P$.

(c) Follows from Theorem 2 in Mangasarian (1976), since $R$ is a $Z$-matrix.

(d) Let $p^1 \leq p^2$. Fix a product $l$. To show $\Omega(p^1)_l \leq \Omega(p^2)_l$, choose $\phi \in \mathbb{R}^N$ as follows: let $\phi_l = 1$ and $\phi_{l'} = \epsilon$ for all $l' \neq l$ and $\epsilon > 0$ arbitrarily small. Note that with the change of variables $u \equiv p - t$, the Linear Program described in part (c) is equivalent to

$$z_\epsilon(p) \equiv \max \phi^T u$$

s.t. $u \leq p$,

$$a - Ru \geq 0.$$  

Clearly, $z_\epsilon(p^1) \leq z_\epsilon(p^2)$, since the feasible region under $p = p^2$ contains that under $p = p^1$. Thus, $\Omega(p^1)_l = \lim_{\epsilon \downarrow 0} z_\epsilon(p^1) \leq \lim_{\epsilon \downarrow 0} z_\epsilon(p^2) = \Omega(p^2)_l$. Finally, by a standard argument, $z_\epsilon(p)$ is a jointly concave function, for any $\epsilon > 0$, and the same applies to $\Omega(p)_l = \lim_{\epsilon \downarrow 0} z_\epsilon(p)$. □

**Proof of Proposition 1.** Parts (a) and (b) follow from Theorems 2 and 3 in Federgruen and Hu (2013a). The same pair of theorems also show part (c), i.e., $\Omega(\tilde{p}) = p^*$ for any equilibrium $\tilde{p}$. This
implies that all equilibria share the same retailer sales volumes \(d(p^*)\); moreover, since \(p^* = \tilde{p} - t\) with \(t \geq 0\) and \(d(\tilde{p} - t) = q(p^*)\), we have
\[
\pi_i(\tilde{p}) = \sum_{(j,k) \in N(i)} (p^*_i + t_{ij} - w_{ijk})[q(\tilde{p} - t)]_{ijk}
\]
\[
= \sum_{(j,k) \in N(i)} (p^*_i - w_{ijk})[q(p^*)]_{ijk} + \sum_{(j,k) \in N(i)} t_{ij} [q(\tilde{p} - t)]_{ijk}
\]
\[
= \sum_{(j,k) \in N(i)} (p^*_i - w_{ijk})[q(p^*)]_{ijk} = \pi_i(p^*),
\]
where the one next to last identity holds because of (4).

Parts (d) and (e) follow from Proposition 4(a) and Theorem 3 in Federgruen and Hu (2013a), respectively.

(f) It suffices to show that \([R + T(R)]^{-1}T(R) \geq 0\). \([R + T(R)]^{-1}T(R) = [T(R)^{-1}(R + T(R))]^{-1} = [I + T(R)^{-1}R]^{-1}\). Since \(R\) is positive definite, it is a so-called \(P\)-matrix. Since \(R\) is a \(Z\)-matrix and \(T(R)\) is symmetric, we have \(R \leq T(R)\). Since \(T(R)\) is a \(Z\)-matrix and \(R\) is both a \(Z\)- and \(P\)-matrix, it follows from Lemma A.1(c) that \(T(R)^{-1}R\) is a \(Z\)-matrix and a \(P\)-matrix, and the same two properties apply when adding the diagonal matrix \(I\) to the matrix \([I + T(R)^{-1}R]\), see Lemma A.1(d). This implies that its inverse \([I + T(R)^{-1}R]^{-1} \geq 0\), see Lemma A.1(a).

Proof of Lemma 2. (a) Fix \(p \in P^*_A\). Let \(t\) denote the unique price correction vector such that \(d(p) = q(p - t)\). It follows from (4) that \(d_A = 0\), since \(d_A(p) > 0\). It follows from 0 = \(d_A(p) = a_{\tilde{A}} - R_{\tilde{A},Ap} - R_{\tilde{A},A}(p_{\tilde{A}} - t_{\tilde{A}})\) and \(t_{\tilde{A}} \geq 0\), that
\[
p_{\tilde{A}} - t_{\tilde{A}} = R_{\tilde{A},A}^{-1}[a_{\tilde{A}} - R_{\tilde{A},Ap}A],
\]
and \(p_{\tilde{A}} \geq R_{\tilde{A},A}^{-1}[a_{\tilde{A}} - R_{\tilde{A},Ap}A]\). (Since \(R\) is positive definite, so is \(R_{\tilde{A},A}\), so that \(R_{\tilde{A},A}\) is invertible.)
This verifies the second set of inequalities in the characterization of \(P^*_A\). Substituting (B.1) into \(d_A(p) = a_{A} - R_{A,Ap} - R_{A,A}(p_{A} - t_{A})\), we get
\[
\]
thus verifying the first set of inequalities in the description of \(P^*_A\). By Assumptions (Z) and (P), \(R_{A,\tilde{A}} \leq 0, R_{\tilde{A},A} \leq 0 \) and \(R_{A,\tilde{A}}\) is a positive definite \(Z\)-matrix. By Lemma A.1(a), \(R_{A,\tilde{A}}^{-1} \geq 0\) and hence \(a^A = a_{A} - R_{A,A}R_{A,Ap}A \geq a_{A} \geq 0\). Since \(R^A\) is the Schur complement of a principal submatrix \(R_{A,A}\) of matrix \(R\), \(R^A\) is positive definite by Lemma A.2(a) in Federgruen and Hu (2013a). Moreover, since \(R_{A,\tilde{A}}R_{A,\tilde{A}}R_{A,\tilde{A}} \geq 0, R^A \leq R_{A,A}\). Therefore, \(R^A\) is a \(Z\)-matrix.

Conversely, fix \(p \in R^N_+\). Assume
\[
a^A - R^A p_A > 0,
\]

Rearranging terms, we get
\[ p_{\bar{A}} \geq R_{\bar{A},\bar{A}}^{-1} [a_{\bar{A}} - R_{\bar{A},A} p_A]. \]  \hspace{1cm} (B.4)
Let \( t_{\bar{A}} \) denote the surplus variables in (B.4) and \( t_A = 0 \). Thus, \( t = (t_{\bar{A}}, t_A) \geq 0 \). It suffices to show that \([a - R(p - t)]_A \geq 0\) and \([a - R(p - t)]_{\bar{A}} = 0\). The latter follows immediately from (B.1), while \([a - R(p - t)]_A = a_A - R_{A,A} p_A - R_{A,\bar{A}} (p_\bar{A} - t_{\bar{A}}) = a_A - R_{A,A} p_A - R_{A,\bar{A}} R_{\bar{A},A}^{-1} [a_{\bar{A}} - R_{\bar{A},A} p_A] = a_A - R^A p_A > 0\), by (B.3).

(b) The vector equations (18) and (19) follow immediately by proving the following result: Consider the retailer competition model under a given wholesale price vector \( w \). Let \( p^* \) denote the component-wise smallest equilibrium in this game, and let \( A \) denote the associated assortment of products. In other words, \( p_A^* \in P_A^0 \). Then
\[ p_A^* = [R^A + T(R^A)]^{-1} [a^A + T(R^A) w_A]. \]  \hspace{1cm} (B.5)
Applying (B.5) to \( w = w^* \), we get (18). The proof of (19) is analogous.

To prove (B.5), for \( p \in P^0_A \), \( d_A(p) = a^A - R^A p_A > 0 \) and \( d_\bar{A}(p) = 0 \), where \( a^A \geq 0 \) and \( R^A \) is a positive definite \( Z \)-matrix. For \( p \in P^0_A \),
\[ \pi(p) = (p_{N(i) \cap A} - w_{N(i) \cap A}) [a^A - R^A p_A]_{N(i) \cap A}. \]
\( p^* \) must satisfy the First Order Conditions (FOC): for \((i, j, k) \in A, \)
\[ 0 = \frac{\partial \pi_i(p)}{\partial p_{ijk}} = [a^A - R^A p_A]_{ijk} - R^A_{ij,k,ijk} (p_{ijk} - w_{ijk}) - \sum_{(i',j',k') \in A, (i',j',k') \neq (j,k)} R^A_{ij,k',ijk} (p_{ijk'} - w_{ijk'}). \]
Rearranging terms, we get
\begin{align*}
2 R^A_{ij,k,ijk} p_{ijk} + \sum_{(i',j',k') \in A, (i',j',k') \neq (j,k)} (R^A_{ij,k',ijk} + R^A_{ij,k,ijk'}) p_{ijk'} + \sum_{(i',j',k') \in A, (i',j',k') \neq (i,k)} R^A_{ij,k,i'j'k'} p_{i'j'k'} & \\
& = a^A_{ijk} + R^A_{ij,k,ijk} w_{ijk} + \sum_{(i',j',k') \in A, (i',j',k') \neq (j,k)} R^A_{ij,k',ijk} w_{ijk'}.
\end{align*}
It is convenient to write this system of \(|A|\) linear equations in \(|A|\) unknowns in matrix form as:
\[ [R^A + T(R^A)] p_A = a^A + T(R^A) w_A, \]
or equivalently,
\[ [R^A + T(R^A)] (p_A - w_A) = a^A - R^A w_A. \]
Hence, we can write
\begin{align*}
p^*_A & = w_A + [R^A + T(R^A)]^{-1} (a^A - R^A w_A) \\
& = [R^A + T(R^A)]^{-1} a^A + [I - (R^A + T(R^A))^{-1} R^A] w_A \\
& = [R^A + T(R^A)]^{-1} a^A + [(R^A + T(R^A))^{-1} (R^A + T(R^A)) - (R^A + T(R^A))^{-1} R^A] w_A \\
& = [R^A + T(R^A)]^{-1} a^A + [R^A + T(R^A)]^{-1} T(R^A) w_A \\
& = [R^A + T(R^A)]^{-1} [a^A + T(R^A) w_A]. \hspace{1cm} \square
\end{align*}
Proof of Lemma 3. Rather than assuming that the matrix \( R \) is symmetric, we prove the lemma under the far weaker assumptions that \( T(R) \) and \( T(S) \) are symmetric. (Recall that under \( R \) is symmetric, \( S \) is symmetric as well, and so are \( T(R) \) and \( T(S) \).) Part (a) does not require any symmetry assumption. Parts (b) and (e) of the lemma only require that \( T(R) \) be symmetric.

(a) Recall that
\[
P(a) \equiv \{ p \geq 0 \mid a - Rp \geq 0 \}.
\]
Suppose \( p \in P(a^1) \), i.e., \( p \geq 0 \) and \( a^1 - Rp \geq 0 \). Since \( a^1 \leq a^2 \), we have \( p \geq 0 \) and \( a^2 - Rp \geq a^1 - Rp \geq 0 \), i.e., \( p \in P(a^2) \). Hence, \( P(a^1) \subseteq P(a^2) \).

(b) Recall that
\[
W(a) \equiv \{ w \geq 0 \mid \Psi(R)a - \Psi(R)Rw \geq 0 \}.
\]
Since \( T(R) \) is symmetric, by Proposition 4(e) in Federgruen and Hu (2013a), \( \Psi(R) \geq 0 \). If \( a^1 \leq a^2 \), then \( \Psi(R)a^1 \leq \Psi(R)a^2 \). Hence if \( w \in W(a^1) \), \( w \in W(a^2) \), i.e., \( W(a^1) \subseteq W(a^2) \).

(c) Recall that
\[
C(a) \equiv \{ c \geq 0 \mid \Psi(S)\Psi(R)a - \Psi(S)\Psi(R)Rc \geq 0 \}.
\]
Since \( T(R) \) and \( T(S) \) are symmetric, \( \Psi(R) \geq 0 \) and \( \Psi(S) \geq 0 \), so that \( \Psi(S)\Psi(R) \geq 0 \). The result follows, as in part (b).

(d) The proof is analogous to that of Lemma 1(d): one easily verifies that the Linear Program to be solved to compute the projection, under \( a = a^1 \), has a feasible region that is contained within the feasible region of the LP associated with \( a = a^2 \).

(e) By Proposition 1 parts (d) and (c),
\[
p^*(w; a) = [R + T(R)]^{-1}a + [R + T(R)]^{-1}T(R)\Theta(w).
\]
The monotonicity in \( a \) follows from \( [R + T(R)]^{-1} \geq 0 \), an immediate consequence of \( R + T(R) \) being a \( ZP \)-matrix, see Lemma A.1(a).

(f) In view of Theorem 2 and (15), it suffices to show that \( \Psi(R) \geq 0 \) and \( [S + T(S)]^{-1} \geq 0 \); the former follows from \( T(R) \) being symmetric, and the latter from the symmetry of \( T(S) \). □

Proof of Theorem 5. Theorem 1(a): The part that \( D(w) \) arises as the unique regular extension of the affine functions \( Q(w) = b - Sw \) follows by Soon et al. (2009), if the matrix \( S \) is a positive definite \( Z \)-matrix and \( b \geq 0 \) (see Soon et al. 2009, Theorem 4 and Lemma 6). By Lemma 4, \( b \geq 0 \) and \( S \) is a \( Z \)-matrix, since \( T(R) \) is symmetric. Moreover, by the proof of Theorem 1(a), \( S \) is positive definite. Hence, all results in Theorem 1(a), except for the symmetry of \( S \), continue to apply.
Theorem 1 parts (b) and (c): The proof is analogous to that of Theorem 1 in Federgruen and Hu (2013a). The proof of part (b) only requires that $S$ is positive definite. The proof of part (c) merely requires that $b \geq 0$ and $S$ is a positive definite Z-matrix, properties verified in part (a).

Theorem 2(a): The proof of Theorem 2(a) only requires that $b \geq 0$ and $S$ is a positive definite Z-matrix, properties verified in Theorem 1(a).

Theorem 2(b): The proof of Theorem 2(b) is analogous to the proof of Theorem 2 in Federgruen and Hu (2013a), requiring only that $S$ is a positive definite Z-matrix, a property verified in Theorem 1(a).

Theorem 2(c): Since $b \geq 0$ and $S$ is a positive definite Z-matrix, properties verified in Theorem 1(a), the proof of Theorem 2(c) is analogous to the proof of Theorem 3 in Federgruen and Hu (2013a), after establishing that

\[ \Psi(S)b \geq 0 \text{ and } \Psi(S)S \text{ is a positive definite Z-matrix,} \]

which can be shown analogously to Lemma 4 under the symmetry assumption of matrix $T(S)$. Since $S$ is positive definite, the positive definiteness of matrix $\Psi(S)S$ can be shown analogously to (13).

Proposition 2: The proof of $P \subseteq W$ requires $\Psi(R) \geq 0$, which follows from Proposition 4(e) in Federgruen and Hu (2013a) under the symmetry assumption of matrix $T(R)$. Analogously, the proof of $W \subseteq C$ requires $\Psi(S) \geq 0$, which is guaranteed by the symmetry assumption of matrix $T(S)$. □

Proof of Theorem 6. Theorem 3: The proof requires that $\gamma = \Psi(S)b \geq 0$ and $U = \Psi(S)S$ is a positive definite Z-matrix, which can be shown analogously to Lemma 4 under the symmetry assumption of matrix $T(S)$. In addition, for the proof of Theorem 3(c), one needs to show that $[R + T(R)]^{-1}T(R) \geq 0$ and $[S + T(S)]^{-1}T(S) \geq 0$, which can be guaranteed by the symmetry assumptions of matrices $T(R)$ and $T(S)$, see the proof of Proposition 1(f).

Corollary 1: The proof resorts to Theorems 2 and 3, which have been shown to hold under the symmetry assumptions of matrices $T(R)$ and $T(S)$.

Lemma 2: The proof requires that $a \geq 0$ and $R$ is a positive definite Z-matrix, and that $b \geq 0$ and $S$ is a positive definite Z-matrix. Since $T(R)$ is symmetric, the latter set of properties follow from Lemma 4.

Corollary 2: The proof resorts to Theorem 3, which has been shown to hold under the symmetry assumptions of matrices $T(R)$ and $T(S)$.
Proposition 3: The proposition was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix $R$ is symmetric. (Note that if $T(R)$ and $T(S)$ are symmetric, so are $T(R^A)$ and $T(S^A)$ for any assortment set $A$.)

Lemma 3: The lemma was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix $R$ is symmetric. Part (a) does not require any symmetry assumption. Parts (b) and (e) of the lemma only require that $T(R)$ be symmetric.

Theorem 4: The theorem was proved under the far weaker assumptions that $T(R)$ and $T(S)$ are symmetric, rather than assuming that the matrix $R$ is symmetric. □

Proof of Proposition 4. (a) We show that $\{P^e, e = 1, 2, \ldots, m + 1\}$ is nested. Analogously to (11), for any $e$,

$$a^{(e+1)} - R^{(e+1)}p = \Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p),$$

see (25). Since $R = R^{(1)}$ is symmetric, $\Psi(R^{(1)}) \geq 0$, a result shown in Federgruen and Hu (2013a) Proposition 4(e). Recursively, for any $e$, $R^{(e)}$ is symmetric and hence, $\Psi^{(e+1)}(R^{(e)}) \geq 0$. Therefore, for any $e$,

$$P^e = \{p \geq 0 \mid a^{(e)} - R^{(e)}p \geq 0\} \subseteq \{p \geq 0 \mid \Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p) \geq 0\} = P^{e+1}.$$ (b) We show that $\{P^e, e = 1, 2, \ldots, m + 1\}$ is contained in hypercube $H$. For any $p \in P^e$, $p \geq 0$ and $a^{(e)} - R^{(e)}p \geq 0$. Since $R^{(e)}$ is a ZP-matrix, $R^{(e)-1} \geq 0$, see Lemma A.1(a). Then for any $p \in P^e$, $p \geq 0$ and $p \leq R^{(e)-1}a^{(e)} = \cdots = R^{-1}a$, i.e., $p \in H$. (c) An alternative characterization of polyhedron $P^e$ is by its extreme points. Note that $P^e$ is an $N$-dimensional polyhedron with $2N$ linear constraints: $p \geq 0$ and $a^{(e)} - R^{(e)}p \geq 0$. An extreme point is the intersection of $N$ hyperplanes corresponding with $N$ constraints chosen from the total of these $2N$ constraints. The set of constraints may be referred to by a pair of index sets $(A^1, A^2)$, where $A^1 \subseteq N$ is the index set for the set of constraints $p \geq 0$ and $A^2 \subseteq N$ is the index set for the set of constraints $a^{(e)} - R^{(e)}p \geq 0$, which are binding at the extreme point:

$$p_{A^1} = 0 \text{ and } [q^{(e)}(p)]_{A^2} = [a^{(e)} - R^{(e)}p]_{A^2} = 0.$$ Note that $A^1$ and $A^2$ must be mutually exclusive, if $a > 0$: When a product $l$ has its price equal to 0, since $a_l > 0$, its demand cannot be equal to zero; Thus, since $|A^1 \cup A^2| = N$, $A^2 = N \setminus A^1$. If for some product $l$, $a_l = 0$, the extreme points may be degenerate, the set of products that have zero prices may be strictly larger than $A^1$. Nevertheless, it is still sufficient to use one index set
\( \mathcal{A} \subseteq \mathcal{N} \) to characterize an extreme point. That is, an extreme point, denoted by \( z^{(e)}(\mathcal{A}) \), is the unique solution of the system of linear equations:

\[
p_{\mathcal{A}} = 0 \quad \text{and} \quad [q^{(e)}(p)]_{\mathcal{A}} = [a^{(e)} - R^{(e)}p]_{\mathcal{A}} = 0.
\]

(Note that for degenerate extreme points, there exists an index set \( \mathcal{S} \supseteq \mathcal{A} \) such that \( p_\mathcal{S} = 0 \).) Since \( p_{\mathcal{A}} = 0 \),

\[
[q^{(e)}(p)]_{\mathcal{A}} = [a^{(e)} - R^{(e)}p]_{\mathcal{A}} = a^{(e)}_{\mathcal{A}} - R^{(e)}a_{\mathcal{A}}p_{\mathcal{A}} = 0.
\]

Hence, for any \( e \) and \( \mathcal{A} \),

\[
[z^{(e)}(\mathcal{A})]_{\mathcal{A}} = [R^{(e)}a_{\mathcal{A}}]^{-1}a^{(e)}_{\mathcal{A}} \geq 0,
\]

where, because of Lemma A.1(a), the inequality is due to the fact that \( R^{(e)} \) is a ZP-matrix and \( a^{(e)} \geq 0 \), as shown in part (b) (since \( R^{(e)} \) is a ZP-matrix, so is \( R^{(e)}_{\mathcal{A}}, \mathcal{A} \)). This also verifies that the extreme points are indeed non-negative.

The extreme point, \( z^{(e+1)}(\mathcal{A}) \), for polyhedron \( P^{(e+1)} \) satisfies: \( [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} = [z^{(e)}(\mathcal{A})]_{\mathcal{A}} = 0 \) and

\[
[q^{(e+1)}(p)]_{\mathcal{A}} = [\Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}z^{(e+1)}(\mathcal{A}))]_{\mathcal{A}} = 0.
\]

For notational simplicity, we temporarily denote \( \Psi^{(e+1)}(R^{(e)}) \) by \( \Psi \). Moreover, since \( [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} = 0 \), \( z^{(e+1)}(\mathcal{A}) \) satisfies:

\[
0 = [\Psi^{(e+1)}(R^{(e)})(a^{(e)} - R^{(e)}p)]_{\mathcal{A}} = \Psi_{\mathcal{A},\mathcal{A}}(a^{(e)}_{\mathcal{A}} - R^{(e)}a_{\mathcal{A}}p_{\mathcal{A}}) + \Psi_{\mathcal{A},\mathcal{A}}(a^{(e)}_{\mathcal{A}} - R^{(e)}a_{\mathcal{A}}p_{\mathcal{A}}),
\]

i.e.,

\[
\Psi_{\mathcal{A},\mathcal{A}}a^{(e)}_{\mathcal{A}} + \Psi_{\mathcal{A},\mathcal{A}}a^{(e)}_{\mathcal{A}} = [\Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}} + \Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}}] [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}}.
\]

We write:

\[
\Psi_{\mathcal{A},\mathcal{A}}a^{(e)}_{\mathcal{A}} \leq \Psi_{\mathcal{A},\mathcal{A}}a^{(e)}_{\mathcal{A}} + \Psi_{\mathcal{A},\mathcal{A}}a^{(e)}_{\mathcal{A}} = [\Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}} + \Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}}] [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \leq \Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}} [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}},
\]

where the first inequality is due to \( \Psi \geq 0 \), hence \( \Psi_{\mathcal{A},\mathcal{A}} \geq 0 \) and \( a^{(e)} \geq 0 \), and the second inequality is due to the fact that \( R^{(e)} \) is a Z-matrix, hence \( R^{(e)}_{\mathcal{A},\mathcal{A}} \leq 0 \), while \( \Psi_{\mathcal{A},\mathcal{A}} \geq 0 \) and \( [z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \geq 0 \) (see (B.6)).

Analogous to the fact that \( \Psi^{(e+1)}(R^{(e)})R^{(e)} \) is a ZP-matrix, we can show that \( \Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}} \) is a ZP-matrix since \( R^{(e)}_{\mathcal{A},\mathcal{A}} \) is a symmetric ZP-matrix. Hence, \( [\Psi_{\mathcal{A},\mathcal{A}}R^{(e)}_{\mathcal{A},\mathcal{A}}]^{-1} \geq 0 \). By (B.6) and (B.7),

\[
[z^{(e+1)}(\mathcal{A})]_{\mathcal{A}} \geq [R^{(e)}_{\mathcal{A},\mathcal{A}}]^{-1}a^{(e)}_{\mathcal{A}} = [z^{(e)}(\mathcal{A})]_{\mathcal{A}}.
\]
Hence, the series \( \{z^{(e)}(A)\} \) is monotone.

By part (b), this series of extreme points \( \{z^{(e)}(A)\} \) is bounded above by the corresponding extreme point, \( h(A) \), of the hypercube \( H \), characterized by \([h(A)]_A = [R^{-1}a]_A\) and \([h(A)]_{\bar{A}} = 0\). By the Monotone Convergence Theorem, this series of extreme points \( \{z^{(e)}(A)\} \) converges. This argument holds for an arbitrary index set \( A \), and the corresponding extreme point. Since a polyhedron can be characterized by its extreme points, the sequence of polyhedra \( \{P^{(e)}\} \) converges. □

C. Uniform Retailer Prices.

In this appendix, we outline how the equilibrium behavior in the base model of Section 3 needs to be adapted in case all suppliers are required to charge uniform prices across all retailers, for each of the products they offer to the market. See Section 3 for a discussion of the limited settings where such price restrictions may prevail.

The above price restrictions require that for each \( j \in J \) and \( k \in K(i, j) \) for some \( i \):

\[
\begin{align*}
w_{ijk} &= \bar{w}_{jk} & \text{for all retailers } i = 1, \ldots, I \text{ such that } (i, j, k) \in N. \tag{C.1}
\end{align*}
\]

The restricted choice of wholesale price vectors has, of course, no bearing, whatsoever, on the equilibrium behavior of the second stage retailer competition game. This implies that Proposition 1 continues to apply. More specifically, any vector of wholesale prices \( w \) induces a unique set of equilibrium demand volumes. If \( w \in W \),

\[
D(w) = Q(w) = \Psi(R)a - [\Psi(R)R]w, \tag{C.2}
\]

see (8), where the matrix \( S \equiv \Psi(R)R \) is positive definite, as shown in Theorem 1(a). Similarly, if \( w \notin W \),

\[
D(w) = Q(\Theta(w)). \tag{C.3}
\]

Turning, next, to the first stage competition game among the upstream suppliers, it should be noted that the induced demand functions are given in closed form, by (C.2) and (C.3). This, in itself, allows for the numerical exploration of equilibria, for example by the use of a tatômement scheme, see Topkis (1998) and Vives (1999). To proceed with the equilibrium analysis, recall that it is advantageous to re-sequence the products so that they are lexicographically ranked according to their supplier index \( (j) \), product index \( (k) \) and, lastly, retailer index \( (i) \). Let \( n \equiv |\{(j,k)\mid \text{ product } (i,j,k) \in N \text{ for at least one retailer } i\}| \) denote the number of distinct supplier/product combinations. Any
restricted wholesale price vector \( \bar{w} \) can be expanded onto the full price space \( \mathbb{R}^N_+ \) from the supplier/product space \( \mathbb{R}^n_+ \), via the transformation \( w = A^T \bar{w} \), where the \( n \times N \) matrix \( A \) is defined as follows:

\[
A_{jk,j'k'} = \begin{cases} 
1 & \text{if } j = j', k = k' \text{ and } (i',j',k') \in \mathcal{N} \\
0 & \text{otherwise. }
\end{cases}
\]

Let \( \bar{W} = \{ \bar{w} \geq 0 \mid A^T \bar{w} \in \mathcal{W} \} \). It is easily verified that on the polyhedron \( \bar{W} \), the demand functions \( \bar{D}(\cdot) \) are again affine, with

\[
\bar{D}(\bar{w}) = \bar{Q}(\bar{w}) = A[\Psi(R)a] - (ASA^T)\bar{w}.
\]

(\( \bar{D}(\bar{w}) \) denotes the aggregate induced demand for product \( k \) sold by supplier \( j \), across all of the retailers.) Moreover, since \( S = \Psi(R)R \) is positive definite, it is easily verified that the matrix \( \bar{S} = ASA^T \in \mathbb{R}^{n \times n} \) is positive definite as well. (Verification is immediate from the definition of positive definiteness; for any \( \bar{z} \in \mathbb{R}^n \) with \( \bar{z} \neq 0 \), \( \bar{z}^T \bar{S} \bar{z} = (\bar{z}^T A)(A^T \bar{z}) = z^T Sz \), with \( z = A^T \bar{z} \neq 0 \). Thus \( \bar{z}^T \bar{S} \bar{z} > 0 \).)

Unfortunately, while the vector of demand volumes \( \bar{D}(\bar{w}) \) can be obtained in closed form for all wholesale price vectors, including vectors \( \bar{w} \notin \bar{W} \), it is no longer true that the demand volumes \( \bar{D}(\bar{w}) = \bar{Q}(\bar{w}') \), with \( \bar{w}' \) the projection of \( \bar{w} \) onto \( \bar{W} \). As a consequence, the characterization of the equilibrium behavior in Theorem 2, no longer applies. However, the following partial characterization of the equilibrium behavior can be obtained if the competition among the suppliers is restricted to the price space \( \bar{W} \) on which the demand functions are affine. Recall, the interior of \( \bar{W} \) is the set of all wholesale price vectors under which all supplier/product combinations maintain a positive market share: We assume, without loss of generality, that the suppliers’ marginal cost rates satisfy the same type of restrictions as (C.1), i.e.,

\[
c_{ijk} = \bar{c}_{jk} \quad \text{for all retailers } i = 1, \ldots, I \text{ such that } (i, j, k) \in \mathcal{N}.
\]

(If (C.4) is violated, this, itself, provides a legal rationale, even within the context of the Robinson-Patman Act, for example, to use differentiated wholesale prices, as in our base model.) Under this cost rate vector \( \bar{c} \), the First Order Conditions of the game with affine demand functions (C.2) have the unique solution

\[
\bar{w}^*(\bar{c}) = \bar{c} + [\bar{S} + T(\bar{S})]^{-1} \bar{Q}(\bar{c}),
\]

see (15). Assume \( \bar{c} \) is such that \( \bar{w}^*(\bar{c}) \in \bar{W} \). In other words, assume \( \bar{c} \in \bar{C} = \{ \bar{c} \geq 0 \mid \Psi(\bar{S})\bar{Q}(\bar{c}) \geq 0 \} \). Then, \( \bar{w}^*(\bar{c}) \) is an equilibrium in the restricted competition game, and if \( \bar{c} \) is chosen in the interior of \( \bar{C} \), \( \bar{w}^*(\bar{c}) \) is the unique such equilibrium.
References


