Oligopoly Price Discrimination: The Role of Inventory Controls*

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Abstract

Inventory controls are the limits that firms place on sales at each of their prices and are used by airlines and other sellers to manage demand uncertainty. We argue that inventory controls also help firms to price discrimination over time, even when they face competition. In our model, firms first choose their capacity and then set prices in a sequence of advance-purchase markets. While a monopolist can easily price discriminate across these markets, we show that oligopoly firms with homogeneous products generally cannot. But inventory controls, which we model as changing the game from a sequential price game to a sequential price-and-quantity game, enable price discrimination over time and increase firm profits.

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1 Introduction

In many settings, including the airline, hotel, and entertainment industries, firms that sell in advance adjust their prices often as the event draws nearer. One reason to adjust prices is in response to new information about demand. Holding capacity fixed, a seller that learns demand is higher than expected will raise prices while a seller that experiences a sequence of low demand realizations will lower price. Another reason to adjust prices is because consumers preferences may be correlated with when they purchase (or arrive). If consumers who purchase closer to the event are less price sensitive, a seller wants to raise prices as the event approaches even if there is no new information about demand.

In airline markets, fares of purchased tickets generally rise during the last fifty days before departure, consistent with intertemporal price discrimination.[1] This price pattern has been documented in several academic papers, including Williams (2013), Lazarev (2013), and Puller, Sengupta, and Wiggins (2012), as well as in a recent series of reports produced for Expedia by the Airlines Reporting Corporation (2015), which suggests that fares are lowest 57 days before departure and increase dramatically within the last 21 days. Rising fares are also commonly observed in hotel pricing and event pricing.[2]

We analyze an oligopoly model in which firms face a sequence of heterogeneous markets, but have a common capacity. For example, airlines have one capacity for each flight but are free to adjust the price of seats on that flight as the departure time for a flight draws closer. We show that absent the use of inventory controls, there

[1]This pattern is also consistent with models of demand uncertainty and rigid prices such as Dana (1999), Prescott (1975), and Eden (1990).

[2]Note that the model can also be interpreted as a model on natural resource sales. Firms first invest in capacity, for example by drilling a well, and then sell that capacity over time. However, it is less natural in this case to think that demand becomes more inelastic over time.

are strong competitive forces, even in a duopoly setting, that prevent intertemporal price discrimination, at least when capacity costs are sufficiently high.

However, we also show that firms with market power can always exploit this type of intertemporal price discrimination when they use inventory controls, or more specifically can make commitments that limit their sales (on and off-the-equilibrium sales) in period 1. That is, we show that inventory controls facilitate price discrimination.

Our model is closely related to the multiperiod Cournot literature (including Van den Berg et al. (2012) and Anton, Biglaiser, and Vettas (2014)) and to the sequential capacity-then-price oligopoly literature (including Kreps and Scheinkman (1983) and Davidson and Deneckere (1986)), because we consider an initial capacity decision and a sequence of competitive markets subject to the common capacity constraint. We simplify the analysis by focusing on high capacity costs, which allows us to consider only pure strategy equilibria, and focusing on two sales periods. And because of our interest in price discrimination, we assume that the monopoly price is higher in the late market than in the early market. We model competition as either price competition – firms simultaneous set prices – or price and quantity competition – firms simultaneously set prices and quantities. The later assumption fits well with firms' choices of price and inventory control in our applications.

2 The Model

Consider an oligopoly with n firms selling a homogeneous good to a continuum of consumers who purchase a different times. The good can be thought of as a ticket, such as a reservation for a seat on an airplane, a room in hotel, or a seat in a theater or at a concert. In these examples, ticket sales take place in advance of consumption in a sequence of advance-purchase markets, though it makes no difference in the model if consumption takes place immediately or at some future time, particularly because we ignore time discounting. However, what is essential is that sales across the periods have a common capacity constraint. For simplicity, we focus on just two purchase periods.

In period 0, firms simultaneously choose their capacities, K^i, which allows the firms to sell K^i units of output for delivery at some future date. The cost of capacity is c per unit. In the two selling periods, periods 1 and 2, the firms simultaneously choose prices and consumers make their purchase decisions. Capacity is common to both selling periods, so each unit of capacity can be used to sell one ticket, either in period 1 or period 2, but not both. For simplicity we ignore any other costs associated with selling a ticket or delivering the output, so the cost of producing a unit of output, or a ticket, is just the cost of building the capacity.

This is a game of complete and perfect information, so capacities, prices and sales are all observable.[3] Firms know each others' capacities when setting their price in period 1, and firms know each others' residual capacities when setting their price in the period 2.

For simplicity, we treat the two advance-purchase selling periods as separate markets. That is, consumers do not choose when to make their purchase decision, but instead are exogenously assigned to either period 1 or period 2. Formally, we could have instead assumed that consumers who choose to purchase in the second

[3]Consider for example, the hotel industry. The capacity decision is made far in advance and hotels sell that capacity for many arrival and departure dates. In our model we simplify this by considering reservations for just one night and model the advance purchases for only that night. Similarly, in the airline industry the capacity decision is made 5 or 6 months ahead of the travel date when airlines schedule their airplanes and crews. And in the performance and related entertainment industries, the capacity decision is made when firms choose, or build, their performance venue. In each case sales for the event takes place sequentially over time.

period don't know their demand in the period one, and hence must buy in period two, and consumers who purchase in the early market know their demands and choose to purchase in period one because they rationally anticipate the firms' prices will be higher in period two (see, for example, Dana (1998) and Akan, Ata, and Dana (2015)).

Products are homogeneous, so consumers purchase at the lowest price available, as long as their valuation exceeds that price. If firms set different prices, then a firm can have positive sales if and only if all of the lower priced firms have sold all of their capacity (i.e., stocked out). If two or more firms charge the same price, then we assume that the firms divide the sales equally subject to their capacity constraints.

We denote the market demand function in period t by $D_t(p)$. When consumers have unit demand, then $D_t(p)$ is the total number of consumers with valuations greater than or equal to p. We assume demand is a continuous function (that is that the cumulative distribution of consumers' valuations is continuous). The associated inverse demands are denoted by $p_t(q)$. We assume that the monopolist's profit function, $(p_t(q) - c)q$, is strictly concave in q for $t = 1, 2$.

When firms charge different prices, the sales of the higher-priced firm depend on which consumers purchase from the low priced firm and which consumers are left to pay the higher price. The high-priced firm faces the spillover demand, after low-priced firm's output has been sold. Because products are homogeneous, the residual (or spillover) demand at any price p is a function of the underlying distribution of consumer demands, or the market demand, and the supply available at all lower prices. For simplicity, we write the residual demand function as $RD_t(p; p^{-i}, K^{-i})$ for $t = 1, 2$ where the arguments are the price, p, and the distribution of all other prices and capacities, (p^{-i}, K^{-i}).

The shape of the residual demand functions depends on the order in which consumers make their purchase decisions, or more generally on rationing rule, which describes which consumers are allocated the good when there is a shortage. In this paper we try not to assume a specific rationing rule. Instead we consider an even broader class of residual demand functions by assuming that the residual demand function satisfies simple properties that are consistent with a wide range of plausible rationing rules.

We assume that the residual demand function in period 1, $RD_1(p; p^{-i}, K^{-i})$, defined as the number of units a firm or firms can sell at price p if rivals are selling K^{-i} units at a price $p^{-i} < p$, satisfies the following four properties:

1. $RD_1(p; p^{-i}, K^{-i})$ is decreasing in p;
2. $\lim_{p \uparrow p_1^{-i}} RD_1(p; p^{-i}, K^{-i}) - \lim_{p \downarrow p_1^{-i}} RD_1(p; p^{-i}, K^{-i}) = K^{-i}$, when $\lim_{p \downarrow p_1^{-i}} RD_1(p; p^{-i}, K^{-i}) > 0$;
3. $D(p) \geq RD_1(p; p_1^{-i}, K^{-i}) \geq D(p) - \sum_{i | p_1^i < p} K^i$; and
4. $RD_1(p; p^{-i}, K^{-i})$ is continuous p, p^{-i}, and K^{-i}, except where $p = p^{-i}$.

We also implicitly assume the residual demand function is symmetric and depends only on the prices and capacities of rival firms, not on which firms offer those prices and capacities, and not on which firm's residual demand we are defining. Also our four conditions are written as if firm i has exactly one rival, or more precisely, as if all firm i's rivals charge the same price, p^{-i}, but we relax this assumption and generalize our four conditions in the Appendix and use these generalized conditions in all of our proofs.

The first condition is obvious. The second condition stipulates that the discontinuous jump in the residual demand at any price must be equal to the number of units offered by rivals at that price. The third condition stipulates that the residual demand is bounded above by the underlying demand and bounded below by the underlying demand less all of units sold at lower prices. The former bound would bind if the lower priced supply went to consumers with willingness to pay always below p and the latter bound would bind if the lower priced supply went to consumers with willingness to pay always above p. So the third condition is not at all restrictive. The fourth condition is natural, but is more restrictive. All four conditions are clearly satisfied for both the efficient rationing rule and the proportional rationing rule (see the Appendix).

To simplify our proofs, we assume that the price in the period 2 clears the market.

Assumption 1. *In period 2 every firm charges the market clearing price.*

The reason for Assumption 1 is to avoid the complexity of analyzing second period pricing subgames with mixed-strategy equilibria. If the residual capacities are large, then the equilibrium strategies in this pricing subgame may be a mixed-strategy equilibrium. However, under many reasonable conditions capacity will never be so large. Assumption 1 should be interpreted as stipulating that capacities are sufficiently small in any payoff-relevant second-period pricing subgame.

An obvious alternative to making Assumption 1 is assuming

$$\sum_{t=1,2} \hat{q}_t^{\infty}(c) < \hat{q}_2^{m}(0), \tag{1}$$

where $\hat{q}_t^n(c)$ denotes the symmetric, n-firm Cournot output in period t, written as a function of the capacity costs, c. That is, the total equilibrium competitive output in both periods is less than the output of a monopolist with zero costs in period 2. Firms won't choose capacity greater than the competitive capacity, so even if the firms choose to sell no output in period 1 (which is clearly not optimal), then the equilibrium price in the second period will be the market clearing price.[4] Firms cannot increase their individual profits by selling less at a higher price since even if they colluded their profits would be lower if they did so.

We let $\eta_t(p)$ denote the elasticity of demand associated with the demand function $D_t(p)$, so $\eta_t(p) = D_t'(p)p/D_t(p)$. We assume $|\eta_1(p)| \geq |\eta_2(p)|$ for all p, so demand is less elastic in period 2, so the monopoly price, p_t^m, is increasing over time.

Once again, the crucial differentiating feature of our model is that firms set capacity ex ante and then can sell that common capacity over two periods. That is, firms choose capacity in period 0, and then choose how much of that capacity to sell in each sales period by choosing their price. This implies that the allocation of each firm's capacity across the periods (or markets) is determined not only by their own price but also by their rivals' prices.

3 Benchmark Models

We contrast our model with several benchmarks. First, we discuss oligopoly capacity-then-price games in which price is set only once, that is Kreps and Scheinkman (1983) and Davidson and Deneckere (1986). Second, we consider a monopoly model with two sequential pricing periods. Our model has two pricing periods and more than one firm, so it is a blend of these two benchmark modelss.

Later in the paper we also consider a capacity-then-quantity oligopoly benchmark model in which firms choose their capacity at time 0 and then choose how much to sell in each in each period. In this model, each firm's price is affected by its rival's

[4] Condition (1) would clearly be less realistic if we generalized the model to include a large number of sales periods, but much weaker sufficient conditions are likely to exist and again the assumption is allow us to focus on pure strategies and not because we think the intuition isn't robust.

quantity choice each period, but its sales are unaffected. This serves as a benchmark for our model when we add inventory controls and is discussed in Section 5.

3.1 The Capacity-then-Price Benchmark with One Pricing Period

Before turning to the two-period analysis, it helps to recall the solution to the one-period problem, that is, when firms simultaneously choose capacity first and then, after observing each others' capacity, simultaneously set price in a single pricing period. Kreps and Scheinkman (1983) solved this game under the assumption that rationing is efficient, and showed that firms choose the Cournot capacity then set price to clear the market in the single sales period, so choosing quantity and the price yields the Cournot outcome. Davidson and Deneckere (1986) showed that when rationing is proportional, or random, the equilibrium capacity may not be the Cournot capacity. For some capacity costs, firms produce more than the Cournot capacity in the first period. And for other smaller capacity costs the pricing subgame is a mixed-strategy equilibrium. However firms still do choose the Cournot capacity when the firms' cost of capacity is sufficiently high. In this case, regardless of the rationing rule, firms choose price to sell all of their capacity in the pricing stage.

In particular, the cost of capacity must be so high that in equilibrium firms choose less capacity then a monopolist would choose if the monopolist had a zero cost of capacity, or $\hat{q}^{\infty}(c) < \hat{q}^{m}(0)$. This means that if the competitive-market capacity is less than the monopoly capacity for a firm with zero cost of capacity, then oligopoly firms will always choose the Cournot output. This is true not only for proportional and efficient rationing, but for any reasonable rationing rule.

3.2 The Monopoly Benchmark with Two Pricing Periods

The monopolist chooses its capacity K at time 0, and then chooses its price, or equivalently quantity, q_t, in periods $t = 1, 2$ subject to its capacity constraint. Because we have assumed away the commitment problem, and because there is no difference between choosing price or quantity each period, we can write the monopolist's pricing problem, given K, as

$$\max_{q_1, q_2} p_1(q_1)q_1 + p_2(q_2)q_2$$

subject to the capacity constraint, $q_1 + q_2 \leq K$.

The optimal prices are defined by the first order conditions,

$$p_t'(q_t)q_t + p_t(q_t) = \lambda,$$

or the simple monopoly pricing rules,

$$\frac{p_t^m - \lambda}{p_t^m} = -\frac{1}{\eta_t},$$

where $t = 1, 2$ and $\lambda \geq 0$ is the shadow cost of the capacity constraint. To characterize prices when capacity, K, is chosen optimally at time 0, we can simply set $\lambda = c$, where c is the cost per unit of capacity.

This implies the firm equates its marginal revenue across the T periods and that $p_1^m < p_2^m$ since $|\eta_1(p)| > |\eta_2(p)|$ for all p. The monopolist price discriminates across the two time periods by charging a higher price to consumers with less elastic demand.

4 The Capacity-then-Price Oligopoly Model

We now turn to solving the full model as described in Section 2. We solve the model for the subgame perfect pure strategy Nash equilibrium. Assumption 1 simplifies the three-stage game and makes it a two-stage game – in the second pricing period all firms set the market clearing price. So working backwards from this, we start by analyzing the first-period pricing subgame.

4.1 The Pricing Subgame

In the first-period pricing subgame firms have limited capacity and compete in prices. The equilibria of the pricing subgames (period 1 and period 2) are characterized by Proposition 1. The proof is in the Appendix.

Proposition 1. *Under Assumption 1, given any capacities, $K^i, i = 1, \ldots, n$, the equilibrium of the pricing subgame is either a uniform price equilibrium satisfying $p_1^i = p_2$ for all firms with positive sales in period 1, or an asymmetric price equilibrium in which just one firm sets a low first period price, $p_1^i < p_2$, and every other firm has zero sales in period 1.*

When a uniform price equilibrium exists, then no asymmetric price equilibrium exists.

When an asymmetric price equilibrium exists in which firm i sets a low price in period 1, then no uniform price equilibrium exists and this asymmetric price equilibrium is the only assymmetric price equilibrium in which firm i is the low-priced firm, however other asymmetric price equilibria may exist in which different firms are the low-priced firm in period 1.

Proposition 1 establishes that the equilibria of the pricing subgame is either a uniform price equilibrium in which all of the transactions prices are uniform across the two periods, or an asymmetric equilibria in which exactly one firm sets a low price in period 1 and all firms set a higher price in period 2. No symmetric non-uniform price equilibrium exists, that is, there is no equilibrium in which every firm charges a higher price in period 2 than in period 1.

Intuitively, pricing low in period 1 increases first period sales and decreases the capacity remaining for the second period leading to higher second period prices. If one firm sets a low price in period 1, the other firms will sell only in period 2. Individually firms want to increase their rivals first period sales, so there can be no equilibrium in which two firms set a low price in the first period.

The presence of free riding makes it less likely that the firms can profit by lowering its price in the first period. The incentive to deviate to a lower price is increasing in the deviating firm's capacity and in the elasticity of first period demand, and is decreasing in the size of first period demand. This is because deviating is only profitable if the firm has sufficient remaining capacity in period 2 to benefit from higher second period price, and if lowering the first period price has a sufficiently large impact on the second period price.

The following assumption guarantees that when capacity is symmetric, no firm wants to deviate from the symmetric price equilibrium and the asymmetric equilibrium does not exist (the symmetric equilibrium is the unique equilibrium). The assumption stipulates that the elasticity of demand in period 2 be sufficiently close to the elasticity of demand in period 1.

Assumption 2. *The elasticity of demand and capacities satisfy*

$$\frac{\eta_2(p)}{\eta_1(p)} > \max_i \frac{K^i}{\sum_{i=1}^n K^i} \tag{2}$$

for all p.

Assumption 2 requires that the elasticity of demand does not decrease in magnitude too quickly. For example, if capacity is symmetric, Assumption 2 states that the ratio of elasticities must be greater than $1/n$ or equivalently with two firms the demand in period 1 cannot be more than twice as elastic as in period 2. Or alternatively, equation (2) requires that if the demand is twice as elastic in period 1, then no firm can have a capacity share greater than 50%.

We now show that under Assumption 2, if the capacity choice is symmetric, then the pricing subgame has only a symmetric price equilibrium and prices are uniform. The proof is in thse Appendix.

Proposition 2. *Under Assumptions 1 and 2, the unique equilibrium of the price subgame is a uniform price equilibrium.*

Intuitively, deviating from a uniform price is profitable for a monopolist if it raises the second-period profit by more than it lowers the first-period profits. However, since rivals all free ride and sell in period 2, an oligopoly firm that deviates from the uniform price, by lowering its first-period price, earns at most $1/n$th of the second-period industry profits, so the oligopoly firm that deviates cannot increase its profit unless it can increase the second-period industry profits by at least n times the decrease in its first-period profit. So the first-period demand must be at least n times more elastic than the second-period demand for such a deviation to be profitable. Assumption 2 guarantees that such a deviation is not profitable.

4.2 The Capacity Choice

In many respects Proposition 2 is the most interesting result of the paper. It specifies that for any allocation of initial capacity satisfying Assumptions 1 and 2, oligopoly firms cannot price discriminate even though a monopolist clearly would.

We now ask what happens when firms choose their capacity optimally. We replace Assumption 2 with the following.

Assumption 3. *The elasticity of demand satisfies*

$$\frac{\eta_2(p)}{\eta_1(p)} > \frac{1}{n}. \tag{3}$$

Assumption 3 clearly states that Assumption 2 holds when capacities are symmetric.

Proposition 3. *Under Assumptions 1 and 3, there exists only one symmetric subgame perfect Nash equilibrium of the oligopoly capacity-then-price model and the equilibrium prices are uniform (i.e., $p_1 = p_2$).*

5 Inventory Controls

In this section we consider the capacity-then-price model with inventory controls. We begin this section by considering one additional benchmark. This benchmark is a model in which firms choose capacity first, and then choose quantity each period as opposed to price.

5.1 The Capacity-then-Quantity Model with Two Periods

At time 0 each firm i chooses its total capacity, K_i, and then in each period, $t = 1, 2$, each firm chooses how much of its capacity to allocate to that period, q_i^t.[5] As before

[5]This is a generalization of Van den Berg et al. (2012) to include the capacity decision, but again we assume that capacity costs are high. The find price is weakly increasing, but here we find price is strictly increasing because we assume that the monopoly price is strictly increasing. Without

we assume the second period prices clear the market, or equivalently that capacity costs are sufficiently high. While the game ends in period 1 under this assumption, notice that if it did not then we would need to be explicit about whether the period 1 quantities, or inventory controls, were observable. If they were unobservable and the game had many periods (not just 2), then we would need to consider the Bayesian Perfect Equilibria of the game. We discuss this more in the conclusion.

Let $\hat{q}_t^n(c)$ denote the symmetric, n-firm Cournot output in period t, written as a function of the capacity costs c. This is the output firms would produce if they were free to choose capacity each period. So $\hat{q}_t^n(c)$ depends only on n, c, and the period t demand, $d_t(p)$.

Proposition 4. *The unique pure-strategy, symmetric equilibrium quantity in the capacity-then-quantity game is $\hat{q}_t^n(c)$ for each firm and in each period t, and the total capacity chosen by each firm in period zero is $K^n = \sum_{t=1}^{T} \hat{q}_t^n(c)$. That is, firms produce the Cournot output in each period.*

Proof. Suppose not. So $q_t^j \neq \hat{q}_t^n(c)$ for some period t and some firm j. It follows that in any pure strategy equilibrium, the marginal revenue in the period t is not equal to c for every firm. Now consider period 1. Given other firms' strategies it is clear that every firm equalizes its marginal revenue in period 1 and period 2. So it follows that some firm j is not setting marginal revenue equal to c in both period 1 and period 2, which implies that in period 0 some firm j is not choosing its capacity optimally. □

This is not surprising. The presence of a common capacity constraint across the two markets does not change the firms' equilibrium quantity decisions in each period. However this is not the case when firms compete in price. Because of the common

that assumption prices would be uniform as is the case in Van den Berg et al. (2012) when capacity is sufficiently constrained.

constraint, firms in period one internalize the strategic effect of their pricing decisions on their rivals' behavior in period two.

5.2 The Capacity-then-Price-and-Quantity Model

The game with inventory controls is equivalent to a game in which firms first choose capacity and then in each period simultaneously choose price and quantity. The quantity is an upper bound on sales. Because there is no production cost associated with choosing quantity each period, we think it is more natural to call the quantity choice an inventory control. It is after all simple a managerial device used by the firm to limit its sales in period one.

Prices are uniform without inventory controls because firms can divert their rivals' sales from the future to the current period, and their own sales from the present to the future period, by increasing their current price. However, rival firms can block this by setting a cap on how many units they will sell at a given price. So for example, a firm can limit the number of units available at p_1 to exactly the number of units it expects to sell in period 1. So if a rival firm deviates to a higher price in period 1, the firm's own sales will be unchanged.

However there is a natural asymmetry. If a rival firm deviates to a lower price, then the firm will sell less even if the firm uses inventory controls since inventory controls cap sales, but don't prevent them from decreasing.

In this section we consider game in which firms choose capacity, K, in period 0, and then choose price and quantity in each period $t = 1, \ldots, T$. The quantity can be interpreted as a cap, or inventory control on sales at the announced price. To formally distinguish between the inventory control and the actual sales, we denote the inventory for firm i in period t by k_t^i.

Proposition 5. *Under the efficient rationing rule, the equilibrium prices and quantities that are obtained in the capacity-then-quantity model (see Proposition 4) are a subgame perfect Nash equilibrium of the quantity-price model with inventory controls.*

Proof. Consider the symmetric SPNE of the capacity-then-quantity game (Proposition 4). Suppose that in the quantity-price game with inventory controls firms simply announce the prices and inventory controls equal to their equilibrium capacity allocations in the capacity-then-quantity game.

Now consider a deviation in period 1. If the firm deviates in price, it will always choose an inventory control equal to the residual demand at that price (or larger). It never makes sense to sell at a lower price than the price that clears the market given the desired sales. So we can think of the firm as choosing either a quantity and setting the marketing clearing price. If the deviator chooses q and the rivals choose q^{-i} then regardless of the prices they set, under the efficient rationing rule the market clearing price is $p_1(q + (n-1)q^{-i})$. This is because the consumers who purchase their units are the consumers with the highest willingness to pay and this set is independent of the price they charge.

But this means that the price is the same as the price the firm gets when it deviates in the quantity it chooses in period 1 in the capacity-then-quantity model (see Proposition 4). So the profit function from a deviation is the same, so given any capacity, the subgame prices and quanities are the same, so the subgame perfect equilibrium of the capacity-then-price model with inventory controls must be the same as the subgame perfect equilbrium of the capacity-then-quantity model. □

The capacity-then-price model with inventory controls has other equilibria. In particular, the symmetric capacity choice, uniform price equilibrium (Proposition 3) may still be a subgame perfect equilibrium of the inventory control game. This is because firms need not announce inventory controls equal to their expect sales. They could instead announce non-binding inventory controls. For example, if all firms choose inventory controls that are infinite, i.e., $k_1^j = \infty$, then the uniform price equilibrium is still an equilibrium. No deviation in period 1 can increase a firm's profit.

Also, note that Proposition 5 assumes the efficient rationing rule because in this case the highest-profit equilibrium is the same as the capacity-then-quantity benchmark. Under other rationing rules, prices would again determined by equating the marginal revenue across the two periods, but the marginal revenue of each firm's residual demand will depart slightly from the Cournot marginal revenue.

Intuitively, however, other rationing rules will also imply non-uniform prices. Imagine the firm chooses a price and an inventory control that allows rival firms to sell k units at price p_1^* so rival firms sales are unchanged. We know that at the Cournot prices a price decrease has no effect on profits – the lost profits in period 1 are exactly offset by higher profits from the price increase in period 2. However, any rationing rule besides the efficient rationing rule will lead to a residual demand elasticity even greater in period 1. So under other rationing rules the same small price decrease must be more profitable than under efficient rationing, since the same price decrease will lead to a bigger price increase in period 2.

Proposition 6. *Under any reasonable rationing rule, in every symmetric subgame perfect equilibrium of the capacity-then-price model with inventory controls in which firms' choose inventory controls equal to their expected sales, price is higher in period 2 when* $|\eta_2| < |\eta_1|$.

6 Conclusion

We have shown that under reasonable general conditions, oligopoly firms are unable to price discriminate over time without using inventory controls. While a monopolist can exploit the tendency for less elastic consumers to purchase their tickets closely to the event time, an oligopolist cannot unless it can place limits on its sales.

We consider an advance purchase sequential pricing model and show that capacity constrained firms will compete in prices until prices are equalized across the pricing periods. Discriminatory prices are not sustainable because firms will raise their early prices in order to claim a larger share of sales at high prices later on.

An obvious limitation of the paper is that we consider only two pricing periods. It would be preferable to consider a large finite number of pricing periods or alternatively to consider a continuous time model with stochastic consumer arrivals or a fluid approximation of this model. However this adds considerable complexity. And potentially more problematic, it would require an explicit assumption that inventory controls were observable, or alternatively considerably more analysis to describe the model if past inventory controls were unobservable.

References

Airlines Reporting Corporation (2015). *Preparing for Takeoff: Air Travel Outlook for 2016*. Airlines Reporting Corporation.

Akan, Mustafa, Barış Ata, and James D. Dana Jr. (2015). "Revenue management by sequential screening". In: *Journal of Economic Theory* 159, pp. 728–774.

Anton, James J, Gary Biglaiser, and Nikolaos Vettas (2014). "Dynamic price competition with capacity constraints and a strategic buyer". In: *International Economic Review* 55.3, pp. 943–958.

Dana Jr., James D. (1998). "Advance-Purchase Discounts and Price Discrimination in Competitive Markets". In: *Journal of Political Economy* 106.2, pp. 395–422.

— (1999). "Equilibrium Price Dispersion under Demand Uncertainty: The roles of Costly Capacity and Market Structure". In: *RAND Journal of Economics* 30.4, pp. 632–660.

Davidson, Carl and Raymond Deneckere (1986). "Long-run competition in capacity, short-run competition in price, and the Cournot model". In: *The Rand Journal of Economics*, pp. 404–415.

Eden, Benjamin (1990). "Marginal cost pricing when spot markets are complete". In: *Journal of Political Economy*, pp. 1293–1306.

Kreps, David M. and Jose A. Scheinkman (1983). "Quantity precommitment and Bertrand competition yield Cournot outcomes". In: *The Bell Journal of Economics*, pp. 326–337.

Lazarev, John (2013). *The welfare effects of intertemporal price discrimination: an empirical analysis of airline pricing in US monopoly markets*. Tech. rep. working paper.

Prescott, Edward C. (1975). “Efficiency of the natural rate”. In: *Journal of Political Economy* 83.6, pp. 1229–1236.

Puller, Steven L., Anirban Sengupta, and Steven N. Wiggins (2012). “Does scarcity drive intra-route price dispersion in airlines?” In: *NBER Working Paper* 15555.

Van den Berg, Anita et al. (2012). “Dynamic Cournot duopoly with intertemporal capacity constraints”. In: *International Journal of Industrial Organization* 30.2, pp. 174–192.

Williams, Kevin R. (2013). *Dynamic airline pricing and seat availability*. Tech. rep. mimeo.

Appendix

Properties of the Residual Demand Function

The residual demand function facing a firm is defined by the underlying market demand, the prices and quantities offered by rivals, and the rationing rule or equivalently the allocation process that governs which consumers receive which prices. Sometimes the rationing assumption is motivated by the way in which consumers queue, but not necessarily.

The obvious rationing rules, because they have received the most attention from the economics literature, are efficient rationing and proportional rationing, but there are many other potential rules. Instead of choosing a rule and deriving the associated residual demands, in this paper we impose reason general restrictions on the shape of the residual demand. These conditions are satisfied by efficient rationing, proportional rationing, and other plausible rationing rules.

The residual demand function, $RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i})$ characterizes the number of units consumers are willing purchase at price p given the supply offered already by firms, which is represented by the sets or vectors $\mathbf{p}^{-i}$ and $\mathbf{K}^{-i}$, that is, the set of prices offered and the associated capacity available at each of those prices. For ease of exposition, we interpret $\mathbf{p}^{-i}$ as the set or vector of all the firms' price except firm i and $\mathbf{K}^{-i}$ as the set or vector of all the firms' capacities except firm i, but note we have assumed RD_1 does not depend on which firm offers which price, only on the total capacity offered at each price, and the residual demand function does depend on which firm chooses p, only on the distribution of other firms' prices.

The first set of properties that we impose on $RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i})$ is that it is non-negative, decreasing in p, weakly decreasing in each element of $\mathbf{K}^{-i}$ and weakly increasing in each element of $\mathbf{p}^{-i}$. We assume that $RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i})$ is no larger than $D_1(p)$, and also that the residual demand function at p does not depend on the distribution of prices higher than p. That is, it does not depend on any elements of $\mathbf{K}^{-i}$ or $\mathbf{p}^{-i}$ that correspond to prices greater than p. These properties are consistent with our interpretation of the residual demand, $RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i})$, as the number of remaining consumers whose willingness to pay is greater than p after every unit at a price less than or equal to p has been sold.

The second property is that the residual demand is discontinuous at the prices offered by other firms, and the jump in the residual demand must equal total sales by other firms at that price. That is, let the price offered by some firm j, $j \neq i$, be p^j. If $\lim_{p\downarrow p^j} RD_1(p^j;\mathbf{p}^{-i},\mathbf{K}^{-i})$ is strictly positive, then at p^j the residual demand falls by the capacity available from other firms at p^j.

$$\lim_{p\uparrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i}) - \lim_{p\downarrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i}) = \sum_{k|p^k=p^j} K^k. \tag{P2}$$

If $\lim_{p\downarrow p^j} RD_1(p^j,\mathbf{p}^{-i},\mathbf{K}^{-i}) = 0$, then

$$\lim_{p\uparrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i}) - \lim_{p\downarrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i}) \leq \sum_{k|p^k=p^j} K^k, \tag{4}$$

and the discontinuity is instead equal to total sales by all firms (other than firm i) that are charging p^j. Note also that $RD_1(p^j;\mathbf{p}^{-i},\mathbf{K}^{-i})$ must be in between $\lim_{p\uparrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i})$ and $\lim_{p\downarrow p^j} RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i})$ because we have already stipulate in our first set of properties that $RD_1(p;\mathbf{p}^{-i},\mathbf{K}^{-i})$ is decreasing in p.

The third property is that the residual demand at any price p is at least the original demand less the units sold at lower prices, and is no more than the underlying demand. That is, units sold at lower prices might have been allocated only to consumers with valuations higher than p, or only to consumers with valuations lower than p, or something in between. So

$$D_1(p) \geq RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i}) \geq \max\left\{0, D_1(p) - \sum_{i|p^i \leq p} K^i\right\}. \tag{P3}$$

In conjunction with our first property, this implies that $RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i}) = D_1(p), \forall p < \min_j p^j \in \mathbf{p}^{-i}$.

The fourth property is that the residual demand function is continuous, except at prices charged by other firms. That is $RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i})$, is continuous in p, $\mathbf{p}^{-1}$, and $\mathbf{K}^{-i}$, except where p is equal to one of the prices in the vector $\mathbf{p}^{-i}$.

These properties are all satisfied by the residual demand functions associated with efficient rationing. The residual demand function under efficient rationing is

$$RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i}) = \max\left\{0, D_1(p) - \sum_{j|p^j < p} K^j\right\},$$

if p is not charged by other firms, and residual demand function under efficient rationing is equal to firm i's share of the remaining demand, $D_1(p) - \sum_{j|p^j<p} K^j$, if more than one firm charges p, where the firms are assumed to have equal shares subject to their capacity constraints. Clearly this function satisfies the four properties above. And note that under efficient rationing the second inequality in property 3 binds for all p, as long as p is any price not charged by another firm.

The four properties are also satisfied by the residual demand functions associated with proportional rationing. The residual demand is

$$RD_1(p; \mathbf{p}^{-i}, \mathbf{K}^{-i}) = \max\left\{0, D_1(p) - \sum_{j|p^j < p} \frac{D_1(p)}{D_1(p^j)} K^j\right\},$$

if p is not charged by other firms, and residual demand function under proportional rationing is equal to firm i's share of the remaining demand, $D_1(p) - \sum_{j|p^j<p} \frac{D_1(p)}{D_1(p^j)} K^j$, if more than one firm charges p, where the firms are assumed to have equal shares subject to their capacity constraints. In this case, units sold at price p_1^j are allocated to consumers with valuations above p in proportion to their share of the total consumers willing to pay p. In other words, the goods are allocated randomly to every consumer who wants the good.

Proof of Proposition 1:

Define $p_L = \min_i p_1^i$, so that p_L is the lowest equilibrium price offered in period 1, and let $\hat{p}_2(q)$ denote the market clearing price in period 2 as a function of the remaining capacity, q. Note that the function p_2 is continuous and q. Also note that we suppress the period 1 label on price where it is unambiguous that we are referring to period 1 prices.

The proof proceeds as a series of eight claims.

1) In any equilibrium of the pricing subgame, $p_L \leq p_2$.

Suppose not, so $p_L > p_2$. And suppose that some firm has zero sales in period 1. Since $p_L > p_2$, then the firm with zero sales would be strictly better off setting a first period price $p \in (p_2, p_L)$. By deviating to $\hat{p}$, this firm increases its sales in period 1 and decreases its sales in period 2 by the same amount. The firm's profits are strictly higher because it sells at a higher price in period 1, and because it's deviation may increase the second period price, p_2, so any capacity it sells in period 2 will also be at a weakly higher price. Hence this is a contradiction.

Suppose not, so $p_L > p_2$, and now suppose instead that every firm has positive sales in period 1. It follows that every firm must be charging p_L. If the prices were not equal, then the firm charging the highest price in period 1 has positive sales only if all of the firms charging lower prices are selling all of their capacity, and the residual demand at the next highest price is positive. But in this case, the firm charging the next highest price can increase its price without reducing its sales, which implies its profit strictly increases, which is a contradiction.

More formally, suppose that $p_L > p_2$, that every firm has positive sales, and that some firm j is charging a price p^j above p_L. Let $\hat{K} = \sum_{i|p_1=p_L} K^i$ denote the total capacity at price p_L. Clearly $D_1(p_L) > \hat{K}$, because $RD_1(p^j; p_L, \hat{K}) > 0$ and $RD_1(p; p_L, \hat{K})$ is decreasing in p (Property 1), which implies $\lim_{\epsilon \to 0} RD_1(p_L + \epsilon; p_L, \hat{K}) > 0$. And $\lim_{\epsilon \to 0} RD_1(p_L + \epsilon; p_L, \hat{K}) = D_1(p_L) - \hat{K}$. By properties 2 and 3 of a residual demand function. And together these two conditions imply that $\lim_{\epsilon \to 0} RD(p_L + \epsilon; \hat{K} - K^i, p_L) > K^i$ for any i such that $p^i = p_L$. So there exists a strictly positive ϵ such a firm i charging p_L can deviate to a higher price, $p_L + \epsilon$, and still sell all of its capacity, which is a contradiction of profit maximization.

However, if all firms are charging p_L in period 1, and $p_L > p_2$, then any firm that has excess capacity in period 1 (that is, any firm for which $x^i < K^i$) could increase its profit by deviating to a first period price of $p_L - \epsilon$. By deviating, this firm sells strictly more in period 1, and strictly less in period 2. Letting i denote the deviating firm and x^i denote the deviating firm's equilibrium period 1 sales, the deviating firm's total profit is

$$(p_L - \epsilon) \min \left\{ D_1(p_L - \epsilon), K^i \right\} + \hat{p}_2(\cdot) \max \left\{ K^i - D_1(p_L - \epsilon), 0 \right\}$$

and as ϵ goes to 0, this is strictly greater than its profit at p_L, which is

$$p_L x^i + p_2 \left(K^i - x^i \right),$$

because $x^i < K^i$ (must hold for some i) and $x^i < D_1(p_L)$, so $x^i < \min\{D_1(p_L), K^i\}$, $p_L > p_2$ (by assumption), and $\lim_{\epsilon\to 0}\hat{p}_2(\cdot) = p_2$. So this is a contradiction, and the first claim holds.

2) In any equilibrium of the pricing subgame, when p_L is offered by two or more firms in period 1, then $p_L = p_2$.

Suppose not, so $p_L < p_2$ and p_L is offered by two or more firms. Suppose firm i is one of those firms. When firm i charges p_L its profit is $p_L x_i + p_2\left(K^i - x_i\right)$, where $x_i = \min\left\{RD_1\left(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), K^i\right\}$ is firm i's sales at p_L.

At a slightly higher price of $p_L + \epsilon$ the firm i's profit is

$$(p_L+\epsilon)\min\left\{RD_1\left(p_L+\epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), K^i\right\} \\ + \hat{p}_2(\cdot)\max\left\{K^i - RD_1\left(p_L+\epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right), 0\right\}, \quad (5)$$

which is clearly greater than the firm i's profit at p_L when $x^i = K^i$, since $p_L+\epsilon > p_L$ and $\hat{p}_2(\cdot) > p_L$ so all of firm i's sales are at a higher price and its sales volume doesn't change.

If on the other hand $RD_1(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j) < K^i$, so $x^i < K^i$, then the deviation is still profitable for firm i because

$$\lim_{\epsilon\to 0} RD_1^i\left(p_L+\epsilon; p_L, \sum_{j\neq i|p^j=p_L} K^j\right) \leq RD_1^i(p_L; p_L, \sum_{j\neq i|p^j=p_L} K^j) < K^i,$$

since RD is decreasing (property 1), and the limit of (5) as ϵ goes to 0 is

$$p_L \lim_{p \downarrow p_L} RD_1^i(p; p_L, \sum_{j \neq i | p^j = p_L} K^j) + p_2 \left(K^i - \lim_{p \downarrow p_L} RD_1^i(p; p_L, \sum_{j \neq i | p^j = p_L} K^j) \right),$$

so for sufficiently small ϵ it again follows that profits are higher because the firm sells more units at p_2 and fewer units at (or near) p_L and $p_2 > p_L$. So a deviation is profitable, which is a contradiction, so either $p_L = p_2$, or only one firm charges p_L.

3) If $p_L = p_2$, then the pricing equilibrium is a uniform price equilibrium. That is, all firms that have positive sales in period 1 set the same price in period 1 and period 2.

Suppose some firm j sets a price $p^j > p_L = p_2$ in period 1 and has strictly positive sales. The residual demand at p^j is strictly positive, which implies that the residual demand in a neighborhood of p_L must also be strictly positive. Therefore, if a firm, say firm i, deviated from p_L to any price $p_L + \epsilon$, a price in a neighborhood of p_L (but below any higher-priced firm's price) it would be able to sell all of its capacity at that price. This is because by property 2 and property 4 (continuity) imply that when firm i removes its capacity K^i at p_L it increases the residual demand in a neighborhood of p_L by K^i. So any firm charging p_L could strictly increase its profits by increasing their price since its total sales would not be affected.

4) There exists at most one uniform price equilibrium of the pricing subgame (the total sales and the transaction prices in each period are unique).

Given the capacity, the sales and volume of sales in a uniform price equilibrium are uniquely defined, because only one price satifies $D_1(p) + D_2(p) = \sum_i K^i$.

5) A uniform price equilibrium exists as long no firm wants to deviate to a lower price.

Consider the unique uniform price strategies: $p_L = p_2$ and no firm has positive sales at any period 1 price other than p_L.

Deviating to a higher price is never profitable. Setting a higher price in the period 1 price lowers industry profit – given the industry capacity and Assumption 2, industry profit is higher when the period 1 price is lower than the period 2 price, not higher – and the deviator's share of first period revenue falls and share of second period revenue rises, so the change in revenue for the deviator must be smaller than for other firms, so the deviator's revenue and profit must fall.

However deviating to a lower price might be profitable. Clearly though, such a deviation cannot be profitable unless $K^i > D_1(p)$ so that the firm has positive sales in period 2. Otherwise all of the deviating firm's sales would be at a lower price. And since $D(p)$ is decreasing, this implies $K^i > D_1(p_L)$ is a necessary condition for a deviation to a lower price to be profitable, and $K^i < D_1(p_L)$ for all i is a sufficient condition for a uniform price equilibrium to be the unique equilibrium of the pricing subgame.

6) When a uniform price equilibrium of the pricing subgame does not exist, then an assymmetric price equilibrium exists in which exactly one firm offers $p_L < p_2$ and all other firms have zero sales in period 1.

Suppose a uniform price equilibrium does not exist. So a deviation is profitable for some firm, and clearly then it must be profitable for the firm with the largest capacity. That firm looses the same profit it period one from a price decrease, but gains more from the price increase in period 2.

Let firm i denote the firm with the largest capacity and p_1^i denote the firm's profit maximizing deviation and $\hat{p}_2$ the resulting second period price. Then p_1^i and $\hat{p}_2$ clearly represent an assymmetric equilibrium. All firms except firm i sell only in period 2. Firm i sells in both periods. And no firm wants to undercut firm i in period 1 because it would sell more at the low price and less at the high price in period 2. And if it could increase the price and its profits by charging less than p_1^i then so could firm i in which case p_1^i is not firm i's profit maximizing price which is a contradiction.

7) There are at most n assymmetric price equilibria, and a uniform price equilibrium and an asymmetric price equilibrium don't exist at the same time.

An assymmetric price equilibrium may not be unique, but there is at most one assymmetric price equilibrium in firm i is the low priced firm in period one since firm i is the only firm choosing a price with positive sales and firm i is choosing that price to maximize its profits – this is essentially a single firm problem with all other firms taking p_2 as given. This implies that there are at most n asymmetric price equilibria.

And if a uniform price equilibrium fails to exist, an asymmetric price equilibrium exists (see 6 above).

For any candidate asymmetric price equilibrium either all firms other the low-priced firm prefer to free ride and sell only in period 2, or some firm has more capacity and is willing to undercut the low priced firm in period 1. But then that implies there is another asymmetric price equilibrium. So consider a candidate asymmetric price equilibrium in which no firm wants to undercut the low-priced firm. This asymmetric price equilibrium exists unless the low priced firm wants to deviate to some other price. But again, if that price deviation exists it defines another asymmetric price equilibrium unless that price is NOT lower than p_2. So an asymmetric price equilibrium exists unless the low priced firm wants to deviate to a price that is greater than or equal to p_2. However in that case a uniform price equilibrium exists.

Proof of Proposition 2:

Let K^i denote the firm i's capacity and $\tilde{p}$ denote the unique uniform price defined by $D_1(\tilde{p}) + D_2(\tilde{p}) = \sum_{i=1}^n K^i$.

Suppose $D_1(\tilde{p}) \geq \max_i K^i$, so first-period sales at $\tilde{p}$ exceeds any firm's capacity. Then a uniform price equilibrium at $\tilde{p}$ exists because any firm that cuts price in period 1 will have zero sales in period 2, so its profit is unambiguously lower, and a price increase clearly can't increase profits either.

Now suppose $D_1(\tilde{p}) < \max_i K^i$. Then by the same argument, for all i such that $K^i < D_1(\tilde{p})$, no deviation is profitable. For all i such that $K^i > D_1(\tilde{p})$, a deviating firm's profit function, when it deviates to a lower price, $p < \tilde{p}$, is

$$\pi(p; \tilde{p}, \mathbf{K}) = p \min\left\{K^i, D_1(p)\right\} + p_2 \left(\sum_{i=1}^n K^i - D_1(p)\right) \left(K^i - \min\left\{K^i, D_1(p)\right\}\right), \tag{6}$$

or equivalently, letting $q = \min\{K^i, D_1(p)\}$ denote the firm's sales in period one,

$$\hat{\pi}(q; \tilde{p}, \mathbf{K}) = \pi(p_1(q); \tilde{p}, \mathbf{K}) = qp_1(q) + p_2 \left(\sum_{i=1}^n K^i - q\right) \left(K^i - q\right). \tag{7}$$

So

$$\frac{d\hat{\pi}(q; \tilde{p}, \mathbf{K})}{dq} = p_1(q) + qp_1'(q) - p_2 \left(\sum_{i=1}^n K^i - q\right) - p_2' \left(\sum_{i=1}^n K^i - q\right) (K^i - q), \tag{8}$$

or

$$\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} = p_1(q)\left(1+\frac{1}{\eta_1(p_1(q))}\right) \\ - p_2\left(\sum_{i=1}^{n} K^i - q\right)\left(1+\frac{1}{\eta_2(p_2\left(\sum_{i=1}^{n} K^i - q\right))}\right)\frac{K^i - q}{\sum_{i=1}^{n} K^i - q}, \quad (9)$$

where η_1 and η_2 are the elasticity of demand. Lowering price is not profitable if $\frac{d\hat{\pi}(q;\tilde{p},\mathbf{K})}{dq} < 0$ near $q = D_1(\tilde{p})$ and clearly

$$\frac{d\hat{\pi}(D_1(\tilde{p});\tilde{p},\mathbf{K})}{dq} < p_1(D_1(\tilde{p}))\left(1+\frac{1}{\eta_1(p_1(D_1(\tilde{p}))}\right) \\ - p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right)\left(1+\frac{1}{\eta_2(p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right))}\right)\frac{K^i}{\sum_{i=1}^{n} K^i} \quad (10)$$

because $(K^i - q)/\left(\sum_{i=1}^{n} K^i - q\right) < K^i/\sum_{i=1}^{n} K^i$ and also because $p_1(D_1(\tilde{p})) = p_2(\sum_{i=1}^{n} K^i - D_1(\tilde{p})) = \tilde{p}$. So lowering price is not profitable if

$$\frac{1}{\eta_1(p_1(D_1(\tilde{p})))} - \frac{1}{\eta_2\left(p_2\left(\sum_{i=1}^{n} K^i - D_1(\tilde{p})\right)\right)}\frac{K^i}{\sum_{i=1}^{n} K^i} < 0, \quad (11)$$

or

$$\frac{\eta_1(\tilde{p})}{\eta_2(\tilde{p})} < \frac{\sum_{i=1}^{n} K^i}{K^i}, \quad (12)$$

so lowering price is not profitable for any firm i if Assumption 2 holds.

Proof of Proposition 3:

Under Assumptions 1 and 3, if a subgame perfect equilibrium exists in which every firm chooses K^* units of capacity, then by Proposition 2 the prices in the pricing subgame are equal in periods 1 and 2 and the pricing subgame has a unique equilibrium and it is a uniform pricing equilibrium. Moreover, for all firm capacities in a neighborhood of K^*, under Assumption 3, Proposition 2 implies the pricing subgame will have a uniform pricing equilibrium, and so the stage one profit function for firm i can be written as

$$\Pi^u(K^i; \mathbf{K}^{-i}) = \left(p_{1+2} \left(\sum_j K^j \right) - c \right) K^i, \tag{13}$$

where $\mathbf{K}^{-i}$ is the capacity of the other firms and $p_{1+2}(K)$ is the inverse total demand, or more precisely, $p_{1+2}(K)$ is implicitly defined by

$$D_1(p_{1+2}) + D_2(p_{1+2}) = K. \tag{14}$$

Firm i's capacity, K^i, maximizes firm i's profits only if $K^i = K^*$ is the solution to

$$\frac{\partial \Pi^u(K^i; K^*)}{\partial K^i} = p_{1+2}((n-1)K^* + K^i) - c + p'_{1+2}((n-1)K^* + K^i)K^i = 0, \tag{15}$$

which is concave, has unique solution, $K^i(K^*)$, which is decreasing in K^*, so equation (15) defines a unique symmetric solution K^*. (Note that formally the second argument of Π^u should actually be a vector with each element equal to K^*.) That is, if a subgame perfect equilibrium exists in which capacity is symmetric, it must be that all firms are choosing K^*, the unique solution to (15) satisfying $K^i = K^*$. Hence there exists at most one symmetric equilibrium of the capacity setting game, and it is easy to see that K^* must be exactly equal the Cournot quantity associated with n firms, cost c, and demand $D_1(p) + D_2(p)$.

Next we show that $K^i = K^*$ is always a best response when the rival firms all choose K^*. Clearly it is a local best response since K^* is a solution to (15) That is locally, the first and second order conditions hold.

Consider all $K^i < K^*$. For sufficiently small K^i, a uniform price equilibrium may not exist and $n-1$ assymmetric price equilibria may exist. However, firm i's profits are uniquely defined. There are $n-1$ such equilibria because anyone of the $n-1$ firms with capacity K^* could set the low price in period 1. Let $\hat{K}^i$ be the value of K^i below which only asymmetric price equilibria exist. Clearly profits decline from $K^i = K^*$ to $K^i = \hat{K}^i$ because $\Pi^u(K^i; K^*)$ is increasing $K^i < K^*$. Now consider firm i's profits in an asymmetric price equilibria:

$$\Pi^a(K^i; \mathbf{K}^{-i}) = \left(p_2 \left(\sum_j K^j - D_1(p) \right) - c \right) K^i, \tag{16}$$

where p maximizes

$$D_1(p)(p-c) + \left(p_2\left(K^i + (n-1)K^* - D_1(p)\right) - c\right)\left(K^* - D_1(p)\right) \tag{17}$$

Clearly the profits are continuous in K^i and profits decline as K^i falls. So K^* earn higher profits than any $K^i < K^*$.

Now consider and $K^i > K^*$. Again, the equilibrum of the pricing subgame may be an asymmetric price equilibrium, but in this case only one asymmetric price equilibrium exists so firm i's profit is again uniquely defined. This is because the firms that didn't deviate to a higher capacity strictly prefer to set the uniform price rather than deviating to a lower first period price, so there exist no asymmetric price equilibria in which these firms charge a low price in period 1. Only the firm that increased its capacity might be willing to set a lower price, so by Proposition 2 the asymmetric price equilibrium is unique. To see this, recall that equation (9) in the proof of Proposition 2 implies that under Assumption 3 no firm will deviate in price if it's capacity is less than $1/n$th of the industry capacity, so only the firm that deviates to a higher capacity can possibly exceed that threshold.

Consider a profitable deviation $K^i > K^*$ and the associated unique assymmetric price equilibrium of the subgame. The deviator's profit is

$$\max_{p_1} D_1(p_1)p_1 + p_2((n-1)K^* + K^i - D(p_1)).$$

But given this profit function, it follows that firm i can increase its profit by lowering K^i unless $K^i < K^*$.

To see this, first note that the uniform Cournot price must be below the monopoly price. The monopoly price in each period satisfies $(p_t^m - c)D_i'(p_t^m) + D_1(p_t^m) = 0$, or $(p_t^m - c)/p_t^m = 1/\eta_t(p_t^m)$, and the uniform Cournot price maximizes each firm's profit, or

$$\max_{p^u}(p^u - c)(D_1(p^u) + D_2(p^u) - (n-1)K^*)$$

so

$$(p^u - c)(D_1'(p^u) + D_2'(p^u)) + D_1(p^u) + D_2(p^u) = (n-1)K^*$$

or

$$(p^u - c)D_1'(p^u) + D_1(p^u) + (p^u - c)D_2'(p^u)) + D_2(p^u) = (n-1)K^*,$$

or

$$D_1(p^u)\left(\frac{p^u - c}{p^u}\eta_1(p^u) + 1\right) + D_2(p^u)\left(\frac{p^u - c}{p^u}\eta_2(p^u) + 1\right) = (n-1)K^*,$$

which using Assumption 2 (i.e., $\eta_1(p) > n\eta_2(p)$) implies that

$$D_1(p^u)\left(\left(\frac{p^u - c}{p^u}\right)\eta_1(p^u) + 1\right) + D_2(p^u)\left(\left(\frac{p^u - c}{p^u}\right)\eta_1(p^u)\frac{1}{n} + 1\right) > (n-1)K^*,$$

or

$$(D_1(p^u) + D_2(p^u)) \left(\left(\frac{p^u - c}{p^u} \right) \eta_1(p^u) \left(\frac{n+1}{n} \right) + 1 \right) > (n-1)K^*.$$

But this can only be true if

$$\left(\frac{p^u - c}{p^u} \right) \eta_1(p^u) \left(\frac{n+1}{n} \right) + 1 > 0$$

or

$$\frac{p^u - c}{p^u} < - \left(\frac{n}{n+1} \right) \frac{1}{\eta_1(p^u)}$$

But this implies that

$$\frac{p^u - c}{p^u} < \frac{1}{\eta_1(p^u)}$$

so it follows that $p^u < p_1^m$, the uniform price must be below the monopoly price in period 1.

It follows that firm i can increase its profit by lowing K^i in the capacity stage and selling the same quantity in period 2.

First, imagine that after cutting K^i the pricing subgame still has an asymmetric price equilibrium. Then firm 1 is still the low priced firm and the only seller in period 1. Firm i's profits in period 2 are unchanged but its profits in period 1 must be higher because $p_1^i < p_u < p_1^m$. Firm i is acting like a monopoly in period 1, so it wants to lower output and raise price. Firm i wants to lower its sales in period 1 as much as possible.

Second, imagine that after cutting K^i the pricing subgame only has a uniform price equilbrium. That is, imagine K^i is a capacity at which the asymmetric price equilibrium and the uniform price equilibrium both exist and have the same prices, so that if firm i lowers its capacity in the capacity stage, the pricing subgame has only a uniform price equilibrium. In that case lowering K^i is profitable because the profits are given by Π^u in equation (13) which is decreasing is K^i for $K^i > K^*$.

So K^* is a best response to K^* and a symmetric capacity equilibrium exists with uniform prices.

Proof of Proposition 6:

Consider a symmetric equilibrium in which the first period price, p_1^*, the capacity, K, and the first period inventory control, k, are the same for every firm, and in which the inventory control, k, binds (is equal to first period sales). In a symmetric equilibrium each firm's choice of price and inventary control maximizes its profit in period 1 given the symmetric capacities, K. We will show that this implies prices must increase from period 1 to period 2.

First, if rationing is efficient, then a uniform price is not an equilibrium. This was established in Proposition 5. In particular, we showed that if rivals are setting uniform prices (given K), then any firm can increase its profit by setting a lower price in period 1 and choosing an inventory control such that it's rivals still sell k units in period 1. That is, under the efficient rationing rule, a firm will do better choosn the Cournot best response to its rivals' inventory controls then setting the same price as its rivals. Formally, let $\tilde{q}(p_1^i; \mathbf{p}, \mathbf{k})$ be the maximum quantity firm i can sell at price $p_1^i < p$ such that the rivals will still sell k units. That is, $\tilde{q}$ is defined by $RD(p; (p, \ldots, p, p_1^i, p, \ldots, p), (k, \ldots, k, \tilde{q}(p_1^i; \mathbf{p}, \mathbf{k}), k, \ldots, k)) = k$. Proposition 5 shows that

$$\Pi(p, k; \mathbf{p}, \mathbf{k}, \mathbf{K}) < \tilde{\Pi}(p_1^i; \mathbf{p}, \mathbf{k}) = \Pi(p_1^i, \tilde{q}(p_1^i; \mathbf{p}, \mathbf{k}); \mathbf{p}, \mathbf{k})$$

for some $p_1^i < p$, where $\Pi(p, k; \mathbf{p}, \mathbf{k}, \mathbf{K})$ is profit as a function of the firm's price and inventory control, the rival firms' price and inventory control, and the capacities; and where $\tilde{\Pi}(p_1^i; \mathbf{p}, \mathbf{k})$ is the same profit function except that the firm is constrained to choose its inventory control equal to $\tilde{q}(p_1^i;\mathbf{p},\mathbf{k});\mathbf{p},\mathbf{k})$, or

$$\tilde{\Pi}(p_1^i;\mathbf{p},\mathbf{k}) = p_1^i\tilde{q}(p_1^i;\mathbf{p},\mathbf{k}) + p_2(nK-(n-1)k-\tilde{q}(p_1^i;\mathbf{p},\mathbf{k}))(K-\tilde{q}(p_1^i;\mathbf{p},\mathbf{k}))$$

That is, firms make more money if they to deviate to a lower price in period 1. We can also right this in terms of the derivative of the profit function, or

$$\begin{aligned} \left.\frac{d\tilde{\Pi}(p_1^i;\mathbf{p},\mathbf{k})}{dp_1^i}\right|_{p_1^i=p} &= \tilde{q}(\cdot) + p\tilde{q}'(\cdot) - p_2'(\cdot)\tilde{q}'(\cdot)(K-\tilde{q}(\cdot)) - p\tilde{q}'(\cdot) \\ &= \tilde{q}(\cdot) - p_2'(\cdot)\tilde{q}'(\cdot)(K-\tilde{q}(\cdot)) < 0, \quad (18) \end{aligned}$$

where $\tilde{q}$ is evaluated at $(p;\mathbf{p},\mathbf{k})$. But at $p_1^i = p$ we have $\tilde{q}(\cdot) = k$, so this becomes

$$p_2'(\cdot) < \frac{k}{\tilde{q}'(\cdot)(K-k)}.$$

But it is now easy to see that the same must be true under any rationing rule. For a small deviation near p, a price decrease increases profits because the only change is to $\tilde{q}'(p;\mathbf{p},\mathbf{k})$. For the efficient rationing rule $\tilde{q}'(\cdot) = D_1'(\cdot)$. However, for any other rationing rule $\tilde{q}'(\cdot) < D_1'(\cdot)$, so if a deviation from the uniform price is profitable under the efficient rationing rule it must be profitable under any rationing rule.