Economic Theory (2005)
DOI 10.1007/s00199-005-0031-0

RESEARCH ARTICLE

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Network externalities and long-run market shares

Received: 5 May 2004 / Accepted: 26 August 2005

Abstract We study a dynamic duopoly model with network externalities. The value of the product depends on the current and past network size. We compare the market outcome to a planner. With equal quality products, the market outcome may result in too little standardization (i.e. too many products active in the long run) but never too much. The potential inefficiency is non-monotonic in the strength of the network effect, being most likely for intermediate levels. When products differ in quality, an inferior product may dominate even when the planner would choose otherwise, but only if the discount factor is sufficiently large.

Keywords Network externalities · Duopoly · Evolution of standards · Market power

JEL Classification Numbers L1 · L5

1 Introduction

Network externalities generate a natural mechanism for the evolution of market shares. In markets with network externalities and a durable value of past sales, firms that have experienced high market share in the past have a large *installed base*. A large installed base makes their product relatively attractive to consumers through the network benefit it produced, meaning that the firm enjoys a competitive advantage. This advantage may lead to even higher market share in the future. The

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source of the network benefit can be either the availability of many complementary products and services (like add-ons, applications, repair service, publications) or directly the number of other users (for example in case of a communication product).

This intuition suggests that network externalities will tend to encourage divergence in market shares, in the sense that the market share of the leader will tend to grow relative to the follower. At the same time, the network externality also tends to make divergence of market shares socially attractive: from the point of view of a social planner it might be beneficial for consumers to focus on a given product in order to reap the network benefit. This paper studies the long-run market shares of firms in a network industry with two firms, and considers how the market outcome might differ from what a social planner would choose.

There are three main issues that we address. First, we are interested in the dynamics of market shares from the positive perspective: we ask when the market shares evolve to a monopoly (one standard) and when to a duopoly in the long run.

Second, from the normative point of view, the market outcome may be that there is *too little* disparity in market shares in the long run. One extreme case of this is when the market has multiple products with positive market share in the long run, while the planner would move to a monopoly. In this case we say that the market does not successfully "choose a standard." The concern that market forces may not provide enough incentive for a standard to emerge has prompted some governments, particularly in Europe, to legislate a particular standard, for instance for wireless communication, instead of allowing market forces to decide between possible products. On the other hand, there is also the opposite concern that network effects might give the market leader too strong a position. That view is implicit in some descriptions of Microsoft; the market is too skewed toward a single firm, and the network benefit of installed base keeps Microsoft's market share high. We ask under what conditions the market is less likely to choose a standard than the planner.

Third, a classic issue in network industries is if superior products are overlooked by the market when there are network effects.[1] Do network effects prevent superior standards from winning market share from inferior products, perhaps even when the social planner would prefer to change the standard?

Outline of the paper The model we study has two firms offering incompatible products. The customers are heterogeneous in their preferences over horizontally differentiated products; they also value the size of the network (past and current). We consider a variety of cases, so we provide now a short roadmap.

In Section 2 we present the general model and in Proposition 1 describe the first main force: market over-tightness in a static game (or, equivalently, with myopic firms). Equilibrium market shares are more balanced than a social planner would choose.

In Section 3 we consider symmetric products to focus on the first two questions above: the long-run market structure and its efficiency. We analyze the general model for the case of very low discount factors (Section 3.1) and very high ones

[1] The concern is that once a hardware or software platform gains dominance it may be difficult to replace it by new, superior products due to the advantage of the installed base. Similar claims are made for non-proprietary standards, like the QWERTY keyboard layout.

(Section 3.2). For intermediate levels we focus on a standard linear utility model. The main results are as follows:

- In the general myopic case the market diverges (chooses a standard) only if it is socially optimal to do so (Proposition 2). Therefore, in markets with frequent changes of products there is too little standardization.
- In the general model with very patient firms when the network effect is strong there is always an equilibrium where market shares diverge (Proposition 3). We also identify the second main force influencing pricing strategies: as leaders obtain a larger *future* benefit from increased market share today, they have incentives to price more aggressively. (This force is opposite to the market over-tightness, so the long-run market shares depend crucially on the discount factor.)
- For a general discount factor in the linear utility model we describe equilibrium evolution of the market shares and compare it with the socially efficient one (Propositions 4–7). If the network benefits are not too strong we can construct an equilibrium where the firms choose linear strategies (Proposition 4). We show that long-run market shares are never more skewed than the planner's. There are, however, cases of the reverse: the planner would like a single firm to eventually control the market, but competition leads to the market converging to equal shares. In other words, there is an important concern that the market will fail to successfully choose a standard.
- Moreover, our results in Section 3.3 allow us to relate the strength of the network externalities to observable implications. Stronger network effects lead to faster convergence, but to more unequal market shares.

In Section 4 we study the case of two products that differ in quality. We address the question if superior products are overlooked by the market when there are network effects. We provide two results for the general model:

- For low discount factors, if the market equilibrium leads to an inferior platform A eventually dominating a superior platform B, the planner would also have chosen to have everyone use A in the long run (Proposition 8). Under myopia, markets tend to over-value quality, not under-value it.
- For the case where firms are very patient we provide an opposing result: there always exist equilibria where quality is undervalued even in the long run (Proposition 9). In this case, it can be that the market diverges to the inferior product for some initial conditions, even though the planner would always choose the superior product in the long run.

Finally, in Section 4.3, we illustrate how the strength of the network benefits affects the equilibrium and socially optimal dynamics in the case of linear substitution costs.

Related literature In our model there is horizontal differentiation between products. In fact, we focus on the case where there is sufficient differentiation that demand for the products is downward sloping. This is essential for the model to have the tension we wish to study: either too much or too little difference in the market shares of the firms may emerge. In Katz and Shapiro (1985), for instance, with no horizontal product differentiation, it is always the case that the planner would choose to have a single firm service the entire market. The concern that markets might lead to market shares that are overly skewed is absent.[2]

Our results on the evolution of market share with different product qualities are in the spirit of work by Farrell and Saloner (1986). We have repeated price competition, however, whereas they have monopoly provision of one good and competitive provision of the other. The model of product competition we choose enables us to characterize what they call "excess inertia" as a function of the discount factor. The model has the appealing feature that it employs a rather standard set of assumptions. That is, there is price competition in each period, horizontally differentiated products, and overlapping generations of consumers with quasi-linear preferences; this is similar to the static model of Farrell and Saloner (1992), but without "converters." Our dynamic model allows for initial market share to impact the long run evolution of the system; in Farrell and Saloner (1986, 1992), the static model has planner and duopoly at equal market shares, since there is no past installed base to impact the outcome. This is critical, since we show cases where small differences in initial market share can lead to complete monopolization in the long run.

Recently, authors such as Economides and Flyer (1997), Dezso and Economides (2002), and Clements (2005) have studied market structure in network industries with product differentiation. Those papers model product differentiation as vertical; clearly the optimum arrangement would be for a single firm to sell the highest quality product to everyone. They compare monopoly and oligopoly to the planner's solution. They do not, however, incorporate explicit dynamics; rather, they study comparative statics.

Our model shares features in common with Argenziano (2004), who studies a duopoly model with imperfect information and network externalities. Our model has complete information and is dynamic. However, an important force in both models is that market leaders price higher than followers (despite identical marginal costs), leading to market shares that are inefficiently close.

Our paper is also related to Beggs and Klemperer (1992), who study an equilibrium in an overlapping-generations model with switching costs. The switching costs make the demands today depend on the sales in the previous period, and therefore allow for non-trivial dynamics. They consider a linear demand case, and construct Markov-perfect equilibria in linear strategies in a similar way that we do in Section 3.3. The main difference is that we incorporate network benefits and focus on how they influence the market outcomes.

The model also adds to the literature on industry dynamics. In work such as Jovanovic (1982), Ericson and Pakes (1995), Hopenhayn (1992), and Jovanovic and MacDonald (1994), changes in *productivity* drive changes in market share. In the model introduced in the next section, time periods are linked through durable network benefits of current sales. An individual choosing a product today is interested in the utility he can obtain from complementary products, which in turn depends on both new purchases today and installed base (past sales). Competition

[2] An example of a market with network effects and such form of heterogeneity is the market for operating systems: some customers prefer Macintosh operating systems (for example, to use graphics applications) and other customers prefer Windows operating systems (for example, to play the newest games). This example fits our model well, as the main source of network benefits is the availability of complementary products (applications). Clearly, there are other important forms of heterogeneity. For example, the customers may differ in the way they value the size of the network or differences in quality.

is endless (that is, it repeats itself in an infinity of time periods) and market leadership is endogenous. In that sense the model builds on a structure similar to the one introduced in Budd, Harris, and Vickers (1993), and Cabral and Riordan (1994).

2 Model

The model has an infinite horizon. There are two firms, denoted 0 and 1, located at the ends of a linear city. The city is populated by a uniform distribution of agents (or customers, or consumers) that live for one period. The mass of consumers born each period is fixed at 1.

2.1 Preferences, technology, and equilibrium

Customers of each firm's product form a network. Within that network the consumer surplus depends on the current number of people on the network and the last-period sales (the installed base). The interpretation is as follows. The customers benefit from using the product with complementary products. The number, quality, availability and prices of these complementary products vary favorably with the size of the network. Furthermore, the complementary products improve frequently, but are to some extent durable, hence the value depends not only on the current size of the network but also on the installed base. For simplicity we assume that only current complementary products and last-period ones are valuable to the customers. We allow this value to differ with vintage (but assume additive separability). We call the customers that bought the product last period 'old' and the current customers 'young' or 'new'.[3]

Formally, let $q_{i,t}$ be the quantity sold (realized or expected) by firm i in time t. Let $p_{i,t} \geq 0$ denote the price charged by firm i (we drop the time subscript when there should be no confusion). We take the indirect utility function to be quasi-linear, so that the utility of a customer that is located in a distance s from firm 0 that buys the product from firm i is:

$$U_i^D(s) + U^{Ny}(q_{i,t}) + U^{No}(q_{i,t-1}) - p_i \tag{1}$$

where the first element corresponds to the usual "transportation costs" used to reflect cost of substitution and represents the value of product i not taking into account the network effect. The second element represents the network value from new consumers, and the third element the network effect from old consumers.

[3] An example of such a product is a computer hardware platform or an operating system: the value of the product depends crucially on the available software/applications. In turn, the availability of those improves with the popularity of the given standard. We do not consider the utility benefit from future consumers. There are two possible interpretations given our hardware/software example. The first is that you will purchase new hardware in the following period, and therefore your current buying decision is not affected by the future size of the network; you will optimize on that in the future decisions. We have in mind cases like operating systems for computers, where computer purchases are made frequently, but software designed in the prior period is still useful. Tomorrow's network will effect tomorrow's purchasing decision. A second example is the case where software is purchased once and is not "forward compatible." A word processor is bundled with your hardware purchase, but tomorrow's buyers of hardware get software that creates incompatible files.

Each consumer chooses a product that maximizes his utility.[4] $U_0^D(s)$ is strictly decreasing in s, whereas $U_1^D(s)$ is strictly increasing in s (as s is the distance to firm 0). The value of new customers is such that $U^{Ny}(q_{i,t})$ and $q(\mathrm{d}U^{Ny}(q)/\mathrm{d}q)$ are strictly increasing. The second assumption makes the distinction between market and planner particularly sharp: as market share rises, the planner will have a greater and greater incentive to make it rise even faster, but the leading firm will have an incentive to raise the price, discouraging more consumers from buying.

Finally, $U^{No}(q_{i,t-1})$ is assumed to be increasing. We also assume that all these functions are differentiable. It is common to refer to $q_{i,t-1}$ as the *installed base* of product i.

Each of the two firms can produce the product at a constant marginal cost normalized to zero. The firm's strategy is a sequence of prices possibly conditional on the state of the industry $(q_{0,t-1})$ and history of the game. The strategy is Markov if the current price $p_{i,t}$ depends only on the current state of the industry.[5]

Given a strategy of firm j and the consumer behavior, the profit-maximization problem of firm i is to choose a (possibly history dependent) sequence of prices $\{p_{i,t}|h_t\}$ that maximizes

$$\sum_{t=1}^{\infty} \beta^{t-1} \pi_i(p_{i,t}, p_{j,t}, q_{0,t-1}, q_{1,t-1}) \tag{2}$$

where β is a common discount factor. The prices of the product determine the consumer behavior and yield demands:

$$q_{i,t}(p_{i,t},\ p_{1,t},\ q_{0,t-1},\ q_{1,t-1}) \tag{3}$$

These, in turn, determine next-period state of the industry and current profits:

$$\pi_i(p_{i,t}, p_{j,t}, q_{0,t-1}, q_{1,t-1}) = p_{i,t}\ q_{i,t} \tag{4}$$

As the consumers are non-strategic in the model, we define the equilibrium to specify the strategies of the firms only given the dynamic demands:

Definition 1 *A subgame-perfect equilibrium is a sequence of prices of the two firms $\{p_{i,t}|h_t\}$ such that, for all t:*

$$\{p_{i,t}|h_t\} = \arg\max_{\{p_{i,t}\}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} \pi_i(p_{i,\tau}|h_\tau, (p_{j,\tau}|h_\tau), q_{0,\tau-1})$$

The equilibrium is Markov-perfect if the strategies depend only on $q_{0,t-1}$, i.e. $\{p_{i,t}|h_t\}=\{p_{i,t}|q_{0,t-1}\}$.

[4] Throughout the paper we assume that the consumers value the good sufficiently highly that as long as the prices are finite, all buy one of the products: i.e. market is always covered. In this way we focus on the efficiency of the network sizes and allocation of consumers to firms, and keep inefficiency of market coverage aside.

[5] To define the game precisely we allow the strategies to depend on the whole history of the game. However, we analyze only the Markov-perfect equilibria in which firms choose to ignore all history other than the current state.

In the paper we consider only Markov-perfect equilibria. In a Markov-perfect equilibrium, the evolution of the system can be concisely described. Define $y = q_{0,t-1} - q_{1,t-1}$ (the prior period) and $x = q_{0,t} - q_{1,t}$ (the current period) as the differences in the market shares in the previous and current period, respectively, so that y and x are numbers in the range $[-1, 1]$. The equilibrium evolution of the system is described by $x(y)$ and we treat y as the state of the industry (it is a sufficient statistic for $q_{0,t-1}$).

We often compare the equilibrium outcome to the result from a social planner who maximizes total surplus. Specifically, the social planner assigns s_t^* young agents to good zero at time t to maximize total surplus:

$$\max_{\{s_t^*\}} \sum_{t=0}^{\infty} \beta^t S(s_t^*, s_{t-1}^*) \tag{5}$$

where

$$S(s_t^*, s_{t-1}^*) = \int_0^{s_t^*} U_0^D(x)\mathrm{d}x + s_t^* U^{Ny}(s_t^*) + s_t^* U^{No}(s_{t-1}^*)$$
$$+ \int_{s_t^*}^{1} U_1^D(x)\mathrm{d}x + (1-s_t^*)U^{Ny}(1-s_t^*) + (1-s_t^*)U^{No}(1-s_{t-1}^*)$$

If the network effect from current consumers is very strong, social surplus $S(s_t, s_{t-1})$ can be a convex function of s_t and the social planner would choose one of the extremes. We focus on a case where the network effect from current consumers is relatively weak compared to the cost of substitution across products so that the social surplus is a strictly concave function of s_t. Similarly, if the network effects (from current consumers) are too strong, we can get upward sloping demands: in that case we could expect multiplicity of equilibria with one of the firms taking the whole market. Instead, we assume that the network effects are not as strong, so that the demands are downward sloping. Finally, for simplicity we often assume that the demands are such that the system of first order conditions of the best response problems is sufficient to uniquely describe the equilibrium in the one-shot game (this assumption implies that the resulting market shares are interior, i.e. $x(y) \in (-1, 1)$ for all $y \in [-1, 1]$).

In contrast to the market equilibrium evolution of the system, $x(y)$, define $x^*(y)$ to be the evolution of the system arising from the planner's optimal choice. Given the assumptions on the utility functions, it is easy to verify that $x^*(y)$ is monotone.

Define $x_T^*(y)$ and $x_T(y)$ to be T iterates on the respective functions. For instance, $x_2(y) = x(x(y))$. If for some small $\varepsilon \geq 0$ $|\lim_{T\to\infty} x_T(y)| > 1-\varepsilon$, we say that the market diverges from the initial condition y.[6] Otherwise we say that it converges. Similarly we describe the dynamics of the planner's problem. Note that monotonicity of $x^*(y)$ implies that $\lim_{T\to\infty} x_T^*(y)$ exists.

[6] In the formal analysis of the equilibrium with myopic firms we need to assume $\varepsilon > 0$ for technical reasons as we use first-order conditions in the proof. For that we need to assume that the market outcome is interior for any initial conditions. It corresponds to having agents for whom the substitution cost is large compared to the network benefit regardless of the size of the network.

2.2 Myopic firms and market over-tightness

We now describe one of the two main economic forces in the general model. When $\beta = 0$, the model is a repetition of the simple static Hotelling model. As we show in this section, in that model there is generally *over-tightness*. In other words, the solution to the planner's problem would involve even more customers buying from the "leader" firm (one with a larger market share) than do in equilibrium. This is true in the transition to the steady state as well as in the steady state.[7]

Consider two competing firms setting equal prices. Decreasing a price has two effects: gaining more customers and reducing revenue from existing ones. If the utilities of customers are quasilinear in prices, then a unilateral small (absolute) decrease of price increases the demand by an amount independent of the identity of the firm (i.e. the slope of demand is the same for both firms). However, the decrease in revenues from existing customers will be larger for the leader than for the follower. Therefore, in general the larger firm will set a higher price.

In contrast, the planner has just the opposite incentive: he would set a *lower* price for the larger firm. Consider the market shares that would result from pricing at marginal cost. In a model without externalities, that yields efficient allocations. However, in our model the network externality in general makes this allocation inefficient: when at equal prices the market shares are unequal, then in general moving one customer between the networks increases total surplus. In particular, if the social marginal benefit of network size is strictly increasing in the size of the network (as the benefit of an extra customer is enjoyed by more and more members of the network), the socially optimal allocation of market shares is more skewed in favor of the larger network than the allocation resulting from equal prices.

Summing up, since the planner would set a relatively lower price for the larger firm as compared to the equilibrium, in a static game the market equilibrium yields more equal market shares than is socially desirable:

Proposition 1 *Suppose* $\beta = 0$ *and* $x(y) \in (-1, 1)$. *When* $x^*(y) > 0$, $x(y) \in (0, x^*(y))$. *When* $x^*(y) < 0$, $x(y) \in (x^*(y), 0)$. *When* $x^*(y) = 0$, $x(y) = 0$.

This simple result identifies an important force affecting the dynamics of market outcomes. For cases where agents are nearly myopic, the planner's choice is more asymmetric than the market equilibrium. For higher discount factors dynamic considerations provide counteracting incentives for the market leaders to price more aggressively.

3 Symmetric products and the development of standards

In this section we study the special case where the products are symmetric with respect to quality and substitution costs:

$$U_0^D(s) = U_1^D(1-s) \tag{6}$$

[7] Also note, that the result does not require symmetry of the demands.

3.1 Myopic firms

Since products are symmetric and the solution to the planner's problem is unique, $x^*(0) = 0$.[8] From Proposition 1, $x^*(y) \geq x(y) \geq 0$ if $x^*(y) \geq 0$, and $x^*(y) \leq x(y) \leq 0$ if $x^*(y) \leq 0$. Moreover, from the construction we can see that $x^*(y)$ is strictly increasing. Throughout this section we assume without loss of generality that at $t = 0$ product 0 is the leader i.e. $y > 0$.

We obtain the following characterization of the long-run market shares:

Proposition 2 *Suppose* $\beta = 0$, $x(y) \in (-1, 1)$ *for all* y *and* $\lim_{T\to\infty} x_T(y)$ *exists. Then for any initial* $y > 0$, $0 \leq \lim_{T\to\infty} x_T(y) \leq \lim_{T\to\infty} x_T^*(y)$. *(For initial* $y < 0$ *the inequalities are reversed).*

In words, in the myopic case with symmetric products, if the planner converges to equal market shares in the long run, the market does so as well. The market over-tightness in the myopic case prevents the market from erroneously choosing a dominant product (divergence) when the planner would prefer to see both products marketed (convergence).

The proposition implies several economically interesting facts. First, if the planner converges to equal market shares (a limit of zero), the market must also converge to equal market shares; in other words, the market shares are efficient in the long run. On the other hand, if the planner converges to something other than equal market shares, the market must converge to market shares that are closer to 50/50 than the planner would choose. There is inefficiency in the long-run market shares.

Additionally, if the market diverges, the planner must also diverge. However, it is still possible that the market converges to having both firms active in the long run while the planner would have diverged. In other words, there is a relevant concern that market forces may be insufficient to develop standards.[9]

3.2 Patient firms

We now consider the evolution of the market shares when the firms are very patient and construct a Markov-perfect equilibrium in which the firms diverge to only one of the firms being active in the long run. That allows us to illustrate the second major economic force. If the network benefit is increasing in the size of the network, then the leader gets a larger future benefit from an additional customer than does the follower. In particular, the future benefit from getting an extra customer today can be divided into two parts: first, it allows to obtain more marginal customers next period (small effect, a bit larger for the leader) and second, it allows to obtain a higher price from the inframarginal customers (and the leader has more of those, and this effect is large). Therefore, the leader has a stronger incentive to reduce

[8] It does not imply that the social planner problem has a unique steady state: the equation $x^*(y) = y$ can have solutions different than 0.

[9] We show such a case in Section 3.3, where we study a linear utility model for general β. The results show how the force identified in Proposition 2 generalizes to the model with forward-looking firms, and illustrates the possibility that market may converge to equal market shares when the planner would asymptotically have one firm control the entire market.

prices for future benefits. This effect is stronger for higher discount factors and for stronger network benefits.

Consider a model with $\beta = 1$. In that case we use the criterion that firms maximize

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \pi_i(p_{i,t}, p_{j,t}, q_{0,t-1}, q_{1,t-1}) \quad (7)$$

Define

$$G(q) \equiv U_0^D(q) + U^{Ny}(q) + U^{No}(q) - U_1^D(q) - U^{Ny}(1-q) - U^{No}(1-q) \quad (8)$$

This function specifies difference in prices necessary for firm 0 to keep the current market share equal to the last-period market share q. Note that $G(1/2) = 0$.

We wish to show that equilibrium with patient firms can lead to divergence if $G'(q) > 0$. When $G'(q) > 0$, if both firms charge the same price, the market share of firm zero will be rising if its installed base is greater than $1/2$, and falling if not. Moreover, when faced with a competitor charging a price of zero, the price that allows the leader to maintain constant market shares will be increasing in the lead of the leader. In particular, the highest price that can be maintained as a steady state will be $G\,(q = 1)$ for firm zero and $-G\,(q = 0)$ for firm one.

For consistency, we state the result in terms of $x(y)$. Recall that $y = 0$ corresponds to $q = 1/2$.

Proposition 3 *Suppose firms maximize (7) and they cannot post negative prices. If $G'(q) > 0$, there always exists a Markov-perfect equilibrium where* $\lim_{T\to\infty} x_T(y) = 1$ *for* $y > 0$ *and* $\lim_{T\to\infty} x_T(y) = -1$ *for* $y < 0$.

The nature of the constructed equilibrium is very simple. Start with firm 0 . When installed base is favorable ($q > 1/2$), price low to attain market share; when market share gets near 1, price at the maximum level that maintains the high market share. Firm 1 is doomed: even at a price of zero, they never gain any market share. The patience of the leader motivates him to keep his price low enough to maintain a large market share for the future.

On the other hand, if firm 1 follows the same strategy when $q < 1/2$, firm 0 is doomed to lose all market share in the limit, even if it lowers its price all the way to zero. Any strategy leads to eventually losing all market share, and hence all strategies, including pricing at zero have a payoff of zero.[10]

[10] Note that the assumption that prices are non-negative is not completely innocuous: we have normalized the marginal costs to be 0, so it means that we assume that the firms cannot charge below marginal costs. Maintaining the assumption that firms cannot charge negative prices, if marginal cost is positive, but small (smaller than $G\,(1)$) then the strategies described still form an equilibrium. If, however, the marginal costs are higher than $G\,(1)$, then we can construct other equilibria in which one of the firms obtains the whole market in the limit: loosely the strategies are as follows. For any installed base $q_i \neq 1$ firm i sets price 0 and firm j sets price equal to marginal cost, mc. That leads to a growth of installed base of firm i. Once $q_i = 1$ firm i sets price $mc + G\,(1)$ and firm j sets price mc. Note that with large positive marginal costs *any* firm may become a market leader regardless of the original state of the industry. The same reasoning applies if the firms can set negative (but bounded) prices, for example by bundling free additional services.

3.3 Linear utility

So far we have identified the two fundamental economic forces that affect pricing strategies of the firms and hence the long-run market shares. We now parameterize the strength of network effects to provide some comparative statics with respect to the strength of the network benefit. To study a model with general β analytically, we specialize to the case where the utility is linear. Specifically, the utility (of an agent with distance η_i to good i) from consuming product i is:

$$U_i = v - p_i - \tau\eta_i + A\left(\alpha q_{i,t-1} + (1-\alpha)q_{i,t}\right) \tag{9}$$

The first term is the total value v, without network benefit. The price is p_i. The cost of substitution is τ dollars per unit of distance η_i, where, for consumer of type s, $\eta_0 = s$ and $\eta_1 = 1 - s$. Note that our (maintained) assumption that all consumers buy something if the prices p_i are bounded, amounts to assuming that v is sufficiently large. Without loss of generality we assume that $\alpha \in (0, 1)$. It represents the relative importance of last-period and current-period complementary products for the customers.

3.3.1 Market equilibrium

Our first step is to construct a Markov-perfect equilibrium. Given current prices (p_0, p_1), the demands are:

$$D_i(p_i, p_{1-i}) = \frac{1}{2}\left(\frac{p_{1-i} - p_i}{\tau - A(1-\alpha)} + 1 + (1-2i)\frac{A\alpha}{\tau - A(1-\alpha)}y\right) \tag{10}$$

For the demands to be downward-sloping, we assume that $\tau/A > 1 - \alpha$.

We look for an equilibrium in linear strategies. Given that firm 1 chooses a linear pricing rule $p_{1,t} = r_0 + r_1 y$, we can write the best response problem of firm 0 as a problem of choosing x, as there is a one-to-one relationship between p_0 and x. We can define the total current profit as a function of x and y:

$$\pi_0(x, y) = \frac{1}{2}r_0 + \frac{1}{2}(r_0 - c)\,x - \frac{1}{2}cx^2 + d_3 y + d_3 yx \tag{11}$$

where $d_3 = \frac{1}{2}(r_1 + A\alpha)$ and $c = \tau - A(1-\alpha)$. The dynamic (best response) problem is:

$$V(y) = \max_{x\in[-1,1]} \left(\pi_0(x, y) + \beta V(x)\right) \tag{12}$$

We start by showing that (under some parameter restrictions) the best response to a linear strategy is a linear strategy. The second step is to show that there exists an equilibrium in linear strategies when the network effect is not too strong. Moreover, the equilibrium in linear strategies is unique.

Proposition 4 *If $\tau/A > (1 - 2\alpha(1-\beta)/(3-\beta))$ then the game has a Markov perfect equilibrium. Furthermore, there is a unique symmetric equilibrium in linear strategies. In this equilibrium market shares converge over time to equal shares. The equilibrium path of market shares is given by:*

$$x(y) = \gamma y$$

where $\gamma \in (0, 1)$ The speed of convergence is increasing in τ and decreasing in A, β and α. Furthermore, if $\tau/A < (1 - 2\alpha(1-\beta)/(3-\beta))$ there does not exist a symmetric Markov-perfect equilibrium in linear strategies with convergence.

In words, if the network effects are not too strong compared with the costs of substitution, then there exists an equilibrium in linear strategies in which both firms stay active in the long run. The system converges over time to equal market shares. The stronger the network effect relative to the substitution cost (A/τ) the slower the convergence. Also, the smaller the discount factor, the faster the convergence.[11] This illustrates the important force that we mentioned above: when the firms are forward looking they obtain a future network benefit from an additional consumer today. The larger the discount factor, the more incentives the firms have to decrease prices (to obtain future network benefit). With linear utilities the network benefit is proportional to the size of the network and hence the future benefit is larger for the leader than for the follower and the leader has larger incentives to decrease prices the larger the network benefit and higher discount factor. Therefore compared with the myopic pricing, the leader's market share will not decrease as fast.

When the network effect (A) is relatively small, we get interior solutions to the best response problem, which leads to convergence. While we have no general results for $\tau/A < 1 - 2\alpha(1-\beta)/(3-\beta)$, we can show that when $\tau/A < 1$, the derivative defined in Section 3.2 satisfies: $G'(q) = 2(A - \tau) > 0$, and by the result of Proposition 3, there is always an equilibrium with divergence for very patient firms.[12]

3.3.2 Efficient allocations

To evaluate market outcomes, we now consider the maximization of total surplus. Rewriting this in terms of x and y, the full dynamic problem becomes:

$$V(y) = \max_{x \in [-1,1]} \left[-\frac{1}{4} d_2 x^2 + \frac{1}{2} A\alpha x y + \frac{1}{4}(2A - \tau) + \beta V(x) \right] \tag{13}$$

where $d_2 = \tau - 2A(1-\alpha)$. We obtain the following characterization of the dynamics of optimal allocation:

[11] We are focusing in this paper on the evolution of market shares, but as the proof is constructive, one can also describe the price dynamics.

[12] Although we are not able to prove that in this case there does not exist Markov-perfect equilibria with convergence (in non-linear strategies) or even existence of Markov-perfect equilibria, we believe it is the case, at least if we focus on equilibria that are limits of a sequence of equilibria in finite truncated games, as the truncation period increases to infinity. Within this class we conjecture that one can even show that the equilibrium we constructed is unique.

Proposition 5 *If $\tau/A > (2-\alpha(1-\beta))$ then the optimal planner's allocation rule is:*

$$x^*(y) = \frac{2A\alpha}{d_2 + \sqrt{\left(d_2^2 - 4\beta A^2\alpha^2\right)}} y = \Gamma y$$

According to this rule, for any starting allocation y $\lim_{T\to\infty} x_T^*(y) = 0$. The speed of convergence is increasing in τ and decreasing in β and A. If $\tau/A > 2$ it is decreasing in α (and increasing otherwise).

Proposition 6 *If $\tau/A < (2-\alpha(1-\beta))$, then $\lim_{T\to\infty} x_T^*(y) \in \{-1, 1\}$ for $y \neq 0$.*

If the network effect is relatively small compared with the 'transportation costs' τ, it is socially optimal for both firms to serve customers; since the products are symmetric, the planner eventually has equal market shares for the two firms. The choice between convergence and divergence depends also on β : the more forward-looking the planner is, the more likely it is that he will choose divergence. Note that the divergence result implies that there exist no stable steady states for the planner other than -1 and 1. Monotonicity of $x_T^*(y)$ guarantees that the limit always exists, so the planner must diverge for any initial condition other than $y = 0$.

3.3.3 Market versus planner

We now compare the evolution of market shares in equilibrium with the optimal one. Note that Propositions 4, 5, and 6 establish complete characterization results only for the planner. In the case of the market, we only have results for the convergence case (i.e. sufficiently small A/τ). However our numerical calculations have established that for higher A/τ the market diverges. Therefore our description of that case has to be treated with the caveat that it is as an educated guess confirmed by numerical simulations and not by a formal proof. Moreover, in the case of small A/τ we have only shown that there exists a Markov-perfect equilibrium in which the market shares converge to equal ones. We did not establish uniqueness of this equilibrium.

The comparisons between the market outcomes and the socially optimal outcome are illustrated in Figure 1. They depend on the costs of substitution relative to the strength of the network benefit, τ/A.

When $\tau/A \in (1-2\alpha(1-\beta)/(3-\beta),\ 2-\alpha(1-\beta))$ the optimal outcome is divergence, while the market converges. When $\tau/A \in (1-\alpha, 1-2\alpha(1-\beta)/(3-\beta))$ then both the market equilibrium and planner diverge.[13] Finally, when $\tau/A > 2-\alpha(1-\beta)$, both market and planner converge. The following proposition shows that in this case market equilibrium converges faster than the social planner:

Proposition 7 *If $\tau/A > 2-\alpha(1-\beta)$ then for any $y \in (-1, 1)$ the allocation obtained as the solution to the social planner problem converges slower to* 0 *than the allocation in the market equilibrium.*

[13] In case $\tau/A < 1-\alpha$ the demand is upward-sloping and we do not study equilibria in that case: in general there are multiple equilibria in which the consumers coordinate their purchases on only one of the products.

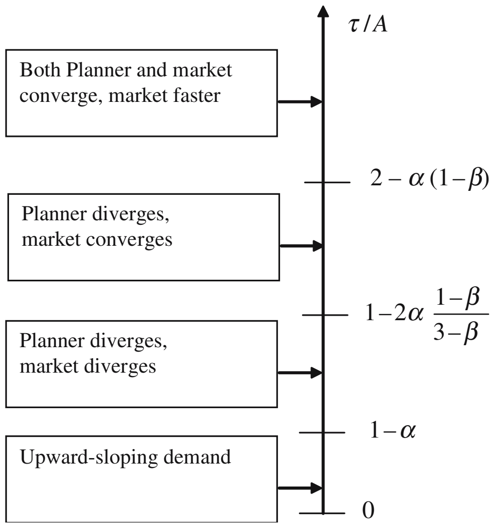

Fig. 1 Equilibrium versus planner long-run market shares

Summing up, in the symmetric model, we obtain that the market is less likely to select a standard than it is socially optimal to do. Although we do not have analytical results for the divergence cases, numerical simulations indicate that in case the market and the social planner diverge, it never happens that they select a different product. That turns out to be a special feature of the symmetric model. In the next section we show that when the products are asymmetric, the dynamic incentives may lead the market to select a socially suboptimal standard.

4 Asymmetric products: when does the market reward quality?

When the products differ in quality, a new issue emerges: instead of asking *whether* the market will select a standard, we may be more interested in *which* standard the market will select, the low or the high quality one.

The motivation for studying products of different quality comes from the concern of history-dependent selection of standards. For example, consider an operating system that enjoys market dominance. Suppose that a second firm improves its competing system and that absent network effects it becomes a better product for the median consumer. Will the new system be able to gain market share? We use the model to ask when inferior products can come to dominate, and whether the market undervalues or overvalues quality relative to the market share that the social planner would choose.

With only a minor modification from the previous section, the model can be used to study how long-run market shares are related to product quality in an industry with network effects.

In order to make the products differ, let

$$U_0^D(s) = U_1^D(1-s) - \delta \tag{14}$$

Without loss of generality, $\delta > 0$ i.e. product zero has lower quality. The difference in quality need not be through an additive constant; however, this specification facilitates comparison to the previous section.

4.1 Myopic firms

Unlike the symmetric case, where $x^*(0) = 0$, here we have $x^*(0) < 0$: from equal market shares, the myopic planner never chooses to give product zero more than half of the market. The market over-tightness result of Proposition 1 allows us to compare the market outcomes with the optimal ones:

Proposition 8 *Suppose $\beta = 0$, $x(y) \in (-1, 1)$ for all y and $\lim_{T\to\infty} x_T(y)$ exists. Then for any initial y,*
(a) $\lim_{T\to\infty} x_T^*(y) > 0$ implies $-1 \leq \lim_{T\to\infty} x_T(y) < \lim_{T\to\infty} x_T^*(y)$
(b) $\lim_{T\to\infty} x_T^*(y) < 0$ implies $\lim_{T\to\infty} x_T^*(y) \leq \lim_{T\to\infty} x_T(y) < 0$

The implication of part (b) is that, for any initial condition such that the planner chooses the superior product to have greater market share (in other words, the superior product is the market leader), the market also makes the superior product the market leader, albeit with somewhat less market share. On the other hand, according to (a), when the planner would choose to make the inferior product the market leader, there is no guarantee that the market will do the same. Furthermore, even if in the market the inferior product will be the leader in the long run, its lead will be smaller than the planner would choose.

This is an important sense in which the market may overrate quality: the 'superior' product may emerge as dominant even when the planner would have the inferior product dominate.

Note that, if $\lim_{T\to\infty} x_T(y) > 1 - \varepsilon$, it cannot be the case, by part (b), that $\lim_{T\to\infty} x_T^*(y) \leq 0$, and therefore by part (a) it must be the case that the planner also has its limit at 1:[14]

Corollary 1 *Suppose $\beta = 0$, $x(y) \in (-1, 1)$, and $\lim_{T\to\infty} x_T(y)$ exists. Then for any initial y, $\lim_{T\to\infty} x_T(y) \geq 1 - \varepsilon$ implies $\lim_{t\to\infty} x_T^*(y) \geq 1 - \varepsilon$.*

In words, if the market diverges to everyone using the inferior product (product zero), the planner does the same. Consider Figure 2, depicting market evolution as a function $x(y)$ for the market and $x^*(y)$ for the planner. The market over-tightness result implies two things about the picture. First, when $x^*(y)$ is zero, so is $x(y)$. Second, for $x(y) > 0$, $x^*(y) > x(y)$, with the reverse true for $x(y) < 0$.[15]

For initial $y < y_1$, both market and planner converge to product one. For initial $y > y_2$, both market and planner converge to product zero. For intermediate initial conditions, the planner converges to the inferior product, while the market converges to the superior product. In other words, for those cases, a myopic planner would eventually choose the inferior product, but the market diverges to everyone using the superior product.

[14] As mentioned before, for technical reasons we talk about divergence to market shares above $1 - \varepsilon$ for some small $\varepsilon > 0$: the argument in Proposition 1 requires interior equilibrium.

[15] Product 1 is of higher quality hence x (0) and x^* (0) are negative.

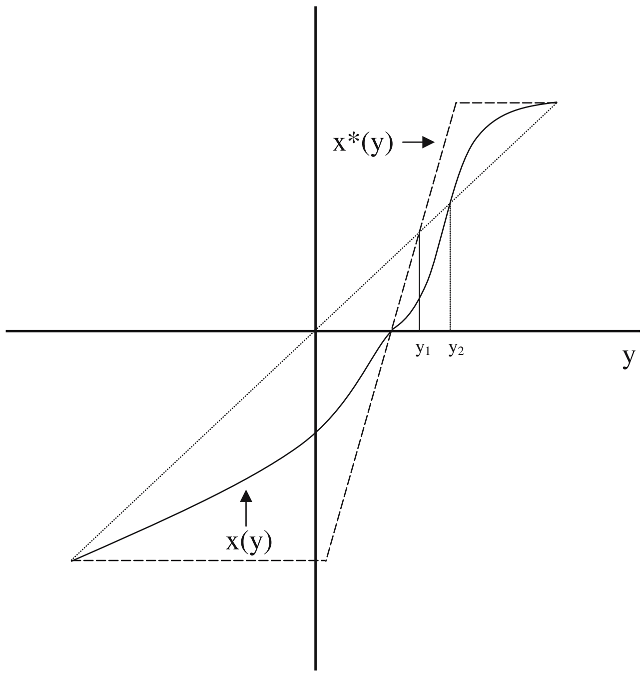

Fig. 2 Optimal and equilibrium dynamics, myopia

Under myopia, the concern that the market might overvalue quality is more of a concern than that it will undervalue quality. Market over-tightness is an important force which makes it difficult for the market to choose the inferior product if the planner would not. In the case of myopia, it guarantees that the market only chooses the inferior product if the planner would. In the following sections, we consider patient firms (and forward-looking firms with linear utility), to show that the second fundamental force – that the leader gains more in the future from increases in today's market share – can lead to cases where the market chooses the inferior product in the long run when the planner would not.

4.2 Patient firms

In this section we show that provided δ is small enough, for large β divergence to either product is possible, just as in Section 3.2. Note that, for any $\delta > 0$, a patient planner will never diverge to the inferior product, since long-run welfare is higher by δ when the superior product is chosen instead. In other words, for large β there is a natural concern that markets might diverge to an inferior product when the planner would not. Markets may undervalue quality.

We use the same criterion function as in Section 3.2. For any quality difference δ, define analogously to the case where products were of equal quality:

$$G(q) \equiv \delta + U_0^D(q) + U^{Ny}(q) + U^{No}(q) - U_1^D(q) - U^{Ny}(1-q) - U^{No}(1-q) \tag{15}$$

The $G(q)$ function again describes difference in prices that would leave the market shares unchanged at q. Suppose there exists some $\bar{q} \in (0, 1)$, such that $G(\bar{q}) = 0$. This assumption of a $\bar{q}$ between zero and one amounts to an assumption that δ is small enough in absolute value.

Define $\bar{y} = 2\bar{q} - 1$. We have the following analogous result to Section 3.2 (The nature of the constructed equilibrium is analogous to that in Proposition 3).

Proposition 9 *Suppose firms maximize (7) and cannot set negative prices. If* $G'(q) > 0$*, there always exists a Markov equilibrium where* $\lim_{T\to\infty} x_T(y) = 1$ *for* $y > \bar{y}$ *and* $\lim_{T\to\infty} x_T(y) = -1$ *for* $y < \bar{y}$

This proposition establishes the existence of a divergence equilibrium where, for some initial conditions, there is divergence to the inferior product, provided δ is not too extreme. Which product is chosen is determined solely by whether the initial condition is above or below $\bar{q}$. On the other hand, if the planner chooses to diverge when $\beta = 1$, it would always be to the superior product. When $|\delta|$ is sufficiently large so that no $\bar{q}$ between zero and one exists, the superior product can always eventually claim the whole market by charging a low enough price for long enough. As a result, the inferior product never has a chance.

4.3 Role of network externalities

The role of network externalities, via the size of A, is an important issue that cannot be addressed by the analytic results presented thus far. To study these issues we specialize to the case where the two products are asymmetric in quality, but the utilities are linear:

$$U_i = v_i - p_i - \tau\eta_i + A\left(\alpha q_{i,t-1} + (1-\alpha)q_{i,t}\right) \tag{16}$$

with $v_1 - v_0 = \delta > 0$, so that product 1 is of higher quality.

We do not solve for equilibria analytically; instead we solve the model numerically to illustrate some possible outcomes. Figures 3, 4, and 5 illustrate the market equilibrium $x(y)$ and planner's $x^*(y)$ for varying degrees of network externalities A. The illustration is for the case of $\beta = 0.5$, $\alpha = 0.5$, $\tau = 12$, and $\delta = 2$. In all figures the thin line is a 45° line, the horizontal axis represents y and the vertical axis shows $x(y)$ or $x^*(y)$.

For the case of relatively low network effects (Fig. 3, $A = 5$), both the planner and market choose convergence to a point where firm 1 controls the majority of the market. In the equilibrium the market converges to nearly equal market shares, but the planner has more unequal shares, as anticipated by the results for myopia. The market undervalues quality, in the sense that the high quality product gets too little market share.

When we increase network effects to moderate levels (Fig. 4, $A = 15$), we see more fundamental differences emerge. For any initial point, the planner eventually (and, in the example, quickly) diverges to a point where a single standard emerges. Whether the standard is the inferior or superior product depends on the initial market shares. The market, by contrast, converges to a point where both firms have positive market share. This is exactly the concern that prompts policy in Europe:

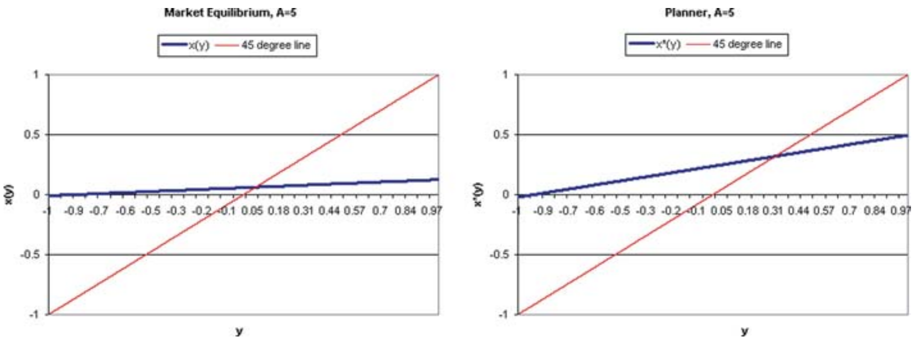

Fig. 3 Weak network effects

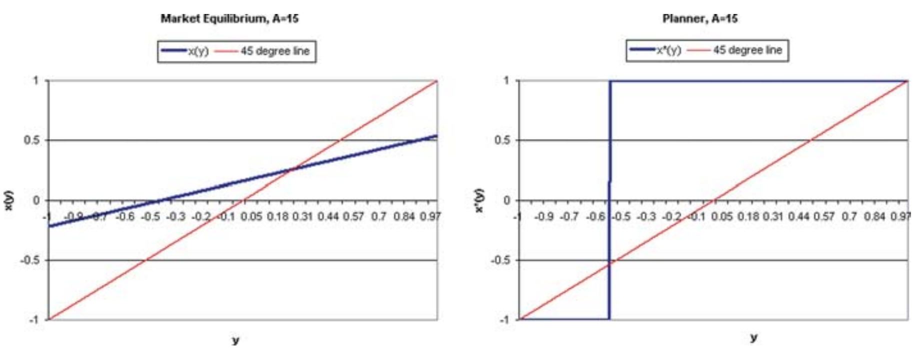

Fig. 4 Intermediate network effects

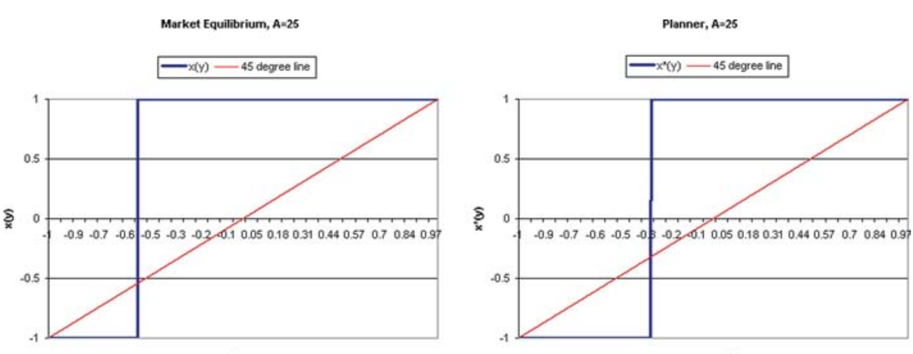

Fig. 5 Strong network effects

the market fails to successfully choose a standard. For low initial points, the market overvalues quality; otherwise the market undervalues it.

When A is raised to very high levels (Fig. 5, $A = 25$), we still have downward sloping demands, but A is sufficiently high that both market and planner diverge to a single standard. The threshold initial market share that determines which product emerges as the standard is different, however; there is a range where the market diverges to everyone using the superior product, but the planner would have everyone use the inferior product. The market overvalues quality here.

Differences between market and planner seem to be, in a sense, non-monotonic in the degree of network effects in this example. For small or large network effects, both the planner and market choose to converge or diverge. But for intermediate network effects, we see a fundamental difference between market and planner, as the market converges to both firms existing, but the planner would choose a single standard.

5 Discussion and conclusion

The model developed here provides a framework for studying dynamics of duopoly driven solely by network effects. We have shown how the model can be used to understand the evolution of network industries and the forces leading to disparity between market outcomes and the planner's optimum.

A main lesson of the analysis is that the discount factor is very important. When the discount factor is low, market over-tightness keeps market shares from becoming too skewed. In fact, it tends to lead to market shares being too equal. In the context of the evolution of standards, this implies that markets might do a bad job of choosing a standard, to the point where markets might lead to equal market shares when the planner would eventually have one firm control the entire market. On the other hand, when network effects are strong enough, patient firms will always have an equilibrium in which the market shares diverge; if anything, in that case, one might be worried that something that looks like a "standard" evolves even when the planner would offer multiple products in the long run.

The discount factor can be thought of in a variety of ways. Suppose that the two firms studied here produce a product that is likely to become obsolete relatively soon. In other words, there is a high probability in each period that the market will be superseded by a market for a new, improved product that renders the old product completely undesirable. In that case, one would not expect excessive divergence in market shares, relative to the planner's solution. In fact, one should be relatively concerned that in such a situation there might not be sufficient "standardization." On the other hand, for a market that is likely to remain viable for a long time, there is more reason to be concerned that a single firm may become more dominant than a planner would prefer.

Our model can be used as a benchmark to study several related questions. For example:

1. Suppose that the firms can invest in quality of their product, i.e. in v_i in equation (16). Will such investment be efficient? Will the firms overinvest or underinvest? How will incentives to investment change with the evolution of the market shares? (We expect that the leaders will underinvest and the followers will sometimes over- and sometimes under-invest).
2. Suppose that the firms can unilaterally spend resources to make the products more or less compatible. Which firms will invest in compatibility and which invest in non-compatibility? Suppose that the firms can sign contracts with side payments and make the products completely compatible at some fixed cost. How will their decisions compare with the socially optimal configura-

tion? (We expect that the leaders will be investing in non-compatibility and the followers in compatibility. Also, because the differences in networks provide the firms with additional product differentiation they will often underinvest in compatibility, even if they can make joint decisions with side payments).

3. Antitrust regulators sometimes consider breaking up monopolies. In network industries it is often argued, however, that such solutions are only temporary: any small initial differences will lead to one of the post-breakup firms dominating the market again. Our results for the myopic case show that if it is efficient to have two firms active in the industry, the market will never diverge. The question is to what extent this is true for forward-looking firms: do there exist initial conditions such that the market equilibrium with both firms active is locally stable when the social planner's problem is stable? (Our results for linear utility presented above provide an affirmative answer – even a global stability. We conjecture that the local stability can be obtained even in the general utility case).

A variety of extensions could be incorporated into the framework. We focus on the case of two firms here for expositional simplicity. Nonetheless, the basic framework could well be applied to industries with more than a simple "leader" and "follower." A more general approach would be to study an oligopoly of N firms with differentiated products. We leave that investigation for future research.

Surely this model does not contain all the nuances of cases such as competition between operating systems. However, it does shed light on some key forces at work as a result of the network effects, and provides a starting point for future study of dynamics of these markets.

6 Appendix

Dynamic programming lemma In the proofs of propositions we utilize the following technical lemma:

Lemma 1 *A dynamic programming problem*

$$V(y) = \max_{x\in[-1,1]} \{a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy + \beta V(x)\}$$

has a unique solution.
In the special case $a_1 = a_3 = a_4 = 0$, $a_5 > 0$, if and only if $a_2 < 0$ and $|a_2| \geq [(1+\beta)/2]a_5$, the optimal policy is interior for every $y \in [-1, 1]$ and the solution is $V(y) = b_0 + b_2 y^2$. The optimal policy function is then linear:

$$x^*(y) = \frac{-a_5}{a_2 - \sqrt{a_2^2 - \beta a_5^2}} y = gy$$

with $g \in [0, 1]$.

Proof of Lemma 1 Consider the mapping T:

$$TV(y) = \max_{x\in[-1,1]} \{a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy + \beta V(x)\}$$

where $V(x)$ is a continuous (and bounded) function $[-1, 1] \to R$ and $\beta \in (0, 1)$. Since T maps from continuous functions to continuous functions and satisfies the Blackwell's sufficiency conditions (see Stokey and Lucas 1989), it is a contraction mapping . Therefore it has a unique fixed point. We now analyze this fixed point.

Guess that $V(y)$ is a quadratic function: $V(y) = b_0 + b_1 y + b_2 y^2$. Then the problem becomes:

$$\begin{aligned} V(y) &= \max_{x \in [-1,1]} \{a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy + \beta(b_0 + b_1 x + b_2 x^2)\} \\ &= \max_{x \in [-1,1]} \{c_0 + c_1 x + c_2 x^2 + a_3 y + a_4 y^2 + a_5 xy\} \end{aligned}$$

where $c_n = a_n + \beta b_n$. If the optimal choice x^* is interior we get:

$$x^* = \frac{-(c_1 + a_5 y)}{2c_2} \tag{17}$$

and the solution is:

$$V(y) = \underbrace{\left(c_0 - \frac{1}{4}\frac{c_1^2}{c_2}\right)}_{b_0} + \underbrace{\left(-\frac{1}{2}c_1\frac{a_5}{c_2} + a_3\right)}_{b_1} y + \underbrace{\left(a_4 - \frac{1}{4}\frac{a_5^2}{c_2}\right)}_{b_2} y^2$$

Now consider the special case: $a_1 = a_3 = a_4 = 0$ and $a_5 > 0$. Conditional on x^* being interior the unique solution becomes: [16]

$$b_0 = \frac{a_0}{1-\beta},\ b_1 = 0,\ b_2 = \frac{1}{2\beta}\left(-a_2 - \sqrt{a_2^2 - \beta a_5^2}\right) \tag{18}$$

When is x^* interior? First, the S.O.C. holds if and only if $a_2 < 0$. Second, b_2 is well-defined if $a_2^2 - \beta a_5^2 \geq 0$. Third, the policy is interior for all states if $g = -a_5/2c_2 < 1$. The third condition implies the second and is equivalent to $|a_2| > (1 + \beta)\, a_5/2$. If any of those conditions is not satisfied, then the value function is not quadratic. □

Proof of Proposition 1 The proof proceeds in terms of market shares rather than market share differences. First, define s' to be the allocation that would result from setting both prices to zero (or any other low, equal prices). It is a solution to:

$$\begin{aligned} 0 = U_0^D(s') - U_1^D(s') + U^{Ny}(s') + U^{No}(q_{0,t-1}) \\ -U^{Ny}(1 - s') - U^{No}(q_{1,t-1}) \end{aligned} \tag{19}$$

Our assumption that the demands are downward sloping for all prices, implies that the left-hand side of this expression is strictly decreasing in s' [see below equation (25)]. Therefore, the solution is unique.[17]

[16] Solving the equation for b_2 yields two roots, but one of them leads to an exploding policy function, which violates the assumption that the solution is interior. The other one does satisfy this condition, so we have found a fixed point. As there is exactly one fixed point, we have the solution.

[17] If it exists, otherwise define $s' = 0$ if the expression is negative for all s' and $s' = 1$ if it is positive for all s'.

We now compare the socially-optimal allocation, s^*, and the market-equilibrium allocation, $\bar{s}$, to the s'. Note that the allocation resulting from pricing at marginal cost in general is not socially optimal because of the network externality. Define $\Phi(s)$ to be the derivative of the total surplus:

$$\begin{aligned}\Phi(s) \equiv\; & U_0^D(s) - U_1^D(s) + U^{Ny}(s) + U^{No}(q_{0,t-1}) \\ & - U^{Ny}(1-s) - U^{No}(q_{1,t-1}) \\ & + s\frac{\mathrm{d}U^{Ny}(s)}{\mathrm{d}q} - (1-s)\,\frac{\mathrm{d}U^{Ny}(1-s)}{\mathrm{d}q}\end{aligned} \tag{20}$$

The F.O.C. of the planner's problem defines the planner's solution: $\Phi(s^*) = 0$ (if it is interior, otherwise, $s^* = 1$ or 0). By assumption, s^* is unique.

Clearly, if $s' = 1/2$ then $s^* = 1/2$. Furthermore, if $s' > 1/2$ then (as we assumed that $q(\mathrm{d}U^{Ny}(q)/\mathrm{d}q)$ is increasing):

$$\Phi(s') = s'\frac{\mathrm{d}U^{Ny}(s')}{\mathrm{d}q} - \left(1 - s'\right)\frac{\mathrm{d}U^{Ny}(1-s')}{\mathrm{d}q} > 0 \tag{21}$$

and strict concavity of the social surplus implies $s^* > s'$. By the same argument, if $s' < 1/2$ then $s^* < s'$.

Second, consider the market equilibrium (recall we have assumed that in equilibrium both firms have positive market share and the market is covered).[18] Demand $s(p_0, p_1)$ is a solution to:[19]

$$U_0(s(p_0, p_1), s(p_0, p_1), q_{0,t-1}) - p_0 = U_1(s(p_0, p_1), 1 - s(p_0, p_1), q_{1,t-1}) - p_1 \tag{22}$$

where

$$U_0(q, q', q_{0,t-1}) = U_0^D(q) + U^{Ny}\left(q'\right) + U^{No}\left(q_{0,t-1}\right) \tag{23}$$

and U_1 is defined analogously. The first order conditions of profit maximization evaluated at the equilibrium prices (p_0^*, p_1^*) are:

$$\begin{aligned} -p_0^*\tfrac{\partial s(p_0^*, p_1^*)}{\partial p_0} &= s(p_0^*, p_1^*) \\ p_1^*\tfrac{\partial s(p_0^*, p_1^*)}{\partial p_1} &= 1 - s(p_0^*, p_1^*)\end{aligned} \tag{24}$$

From (22) and implicit function theorem we obtain:

$$\begin{aligned} \tfrac{\partial s(p_0^*, p_1^*)}{\partial p_0^*} &= \left(\tfrac{\mathrm{d}U_0(s,s,q_{i,t-1})}{\mathrm{d}s} - \tfrac{\mathrm{d}U_1(s,1-s,q_{i,t-1})}{\mathrm{d}s}\right)^{-1} < 0 \\ \tfrac{\partial s(p_0^*, p_1^*)}{\partial p_1^*} &= -\left(\tfrac{\mathrm{d}U_0(s,s,q_{i,t-1})}{\mathrm{d}s} - \tfrac{dU_1(s,1-s,q_{i,t-1})}{ds}\right)^{-1} = -\tfrac{\partial s(p_0^*, p_1^*)}{\partial p_0^*} > 0 \end{aligned} \tag{25}$$

Combining these expressions we get that the equilibrium demand of firm 0, $\bar{s}$, satisfies:

$$\frac{p_0^*}{p_1^*} = \frac{\bar{s}}{1-\bar{s}} \tag{26}$$

[18] If there are multiple interior equilibria, the result holds for every one of them.

[19] This is a condition for rational expectations, self-fulfilling equilibrium: the agents optimize given their expectations and their actions are such that the expectations are realized.

Suppose that $s' > 1/2$. If $p_0^* \leq p_1^*$ then we would obtain from (26) $\bar{s} \leq 1/2$. But at $p_0^* = p_1^*$ the demand would be $s = s'$ and from (25) at $p_0^* \leq p_1^*$ we get that $\bar{s} \geq s'$, a contradiction. Therefore in equilibrium $p_0^* > p_1^*$ and hence $\bar{s} \in (\frac{1}{2}, s')$.

Finally, consider the case $s' = 1/2$. If $p_0^* < p_1^*$, then from (26) $1/2 > \bar{s}$ and from (25) $\bar{s} > s'$, a contradiction. Similarly, if $p_0^* > p_1^*$ we get a contradiction. Therefore, $p_0^* = p_1^*$ and $\bar{s} = s^* = 1/2$.

Combining with our first result we find that $\bar{s} \in (1/2, s^*)$ if $s^* > 1/2$ and $\bar{s} = 1/2$ is $s^* = 1/2$. The argument for $s^* < 1/2$ is analogous. □

Proof of Proposition 2 Suppose $y > 0$. By Proposition 1, $x^*(y) \geq x(y) \geq 0$. By induction, if $x_T^*(y) \geq x_T(y) \geq 0$,

$$x_{T+1}^*(y) = x^*(x_T^*(y)) \geq x^*(x_T(y)) \geq x(x_T(y)) = x_{T+1}(y) \geq 0 \tag{27}$$

where the first inequality follows from the monotonicity of $x^*(y)$ and the second one from $x^*(y) \geq x(y)$. Therefore $x_T^*(y) \geq x_T(y) \geq 0$ for all T and so $\lim_{T\to\infty} x_T^*(y) \geq \lim_{t\to\infty} x_T(y) \geq 0$. □

Proof of Proposition 3 This proposition is a special case of Proposition 9 for $\delta = 0$. □

Proof of Proposition 4 Suppose firm 1 follows a linear strategy $p_1 = r_0 + r_1 y$. Then (using the same reasoning and notation as in Lemma 1) if the best response is interior then the value function is quadratic

$$V(y) = b_0 + \underbrace{\left(-\frac{1}{2}c_1\frac{a_5}{c_2} + a_3\right)}_{b_1} y + \underbrace{\left(-\frac{1}{4}\frac{a_5^2}{c_2}\right)}_{b_2} y^2 \tag{28}$$

The corresponding best-response Markov-perfect strategy is:

$$x^* = \frac{-c_1}{2c_2} - \frac{a_5}{2c_2}y \tag{29}$$

where $c_n = a_n + \beta b_n$, and:

$$a_1 = (r_0 - c)/2 \tag{30a}$$
$$a_2 = -c/2,\ a_4 = 0 \tag{30b}$$
$$a_3 = a_5 = d_3 = r_1/2 + A\alpha/2 \tag{30c}$$

(recall $c \equiv \tau - A(1-\alpha)$). Since x^* is a linear function of y and p_0 is a linear function of x, the best response is a linear pricing strategy.

We assume for now that the best response is interior and construct the equilibrium. We then verify that indeed under the assumption $\tau > A(1-2\alpha(1-\beta)/(3-\beta))$ the equilibrium is interior. Since we are looking for a symmetric equilibrium, if $p_1 = r_0 + r_1 y$ then $p_0 = r_0 - r_1 y$. The demands (10) imply $p_0 = p_1 - cx + A\alpha y$. Combining it with equation (29) yields the equilibrium condition:

$$r_0 - r_1 y = \left(r_0 + \frac{1}{2}c\frac{c_1}{c_2}\right) + \left(r_1 + \frac{1}{2}c\frac{a_5}{c_2} + A\alpha\right) y$$

for every y, or:

$$0 = \frac{1}{2}c\frac{c_1}{c_2} \tag{31}$$

$$-r_1 = r_1 + \frac{1}{2}c\frac{a_5}{c_2} + A\alpha \tag{32}$$

Note that $\tau > A(1 - 2\alpha(1-\beta)/(3-\beta))$ implies $c > 0$ and as the best response problem is concave $c_2 < 0$ (otherwise we cannot have an interior equilibrium). Therefore condition (31) implies $c_1 = 0$. That simplifies several expressions (for example, (29) , and $b_1 = a_3$) and yields $r_0 = c - \beta(r_1 + A\alpha)$ (by combining $c_1 \equiv a_1 + \beta_1 = 0$ and the expressions for a_1 and b_1).

We will focus on (32) to pin down the equilibrium evolution of market shares. From (28) we have $c_2 = 1/2(a_2 - d_1)$, where $d_1 = \sqrt{a_2^2 - \beta a_5^2}$. Define $z \equiv r_1 + A\alpha$. Substituting $a_5 = 1/2z$ to (32) and simplifying, we can express z in terms of the fundamentals A, α, β and τ only:

$$2z = \frac{z}{1 + \sqrt{1 - \beta\left(\frac{z}{c}\right)^2}} + A\alpha \tag{33}$$

Call z^* the solution to this equation. It can be easily verified that the solution is positive. From (29) and $c_1 = 0$ we have that the equilibrium market allocation is equal to:

$$x(y) = -\frac{a_5}{2c_2}y = \frac{2z^* - A\alpha}{c}y = \gamma y \tag{34}$$

Since $z^* > 0$, (33) implies $\gamma > 0$. We now show that if and only if $\tau > A(1 - 2\alpha(1-\beta)/(3-\beta))$ there is a unique solution $\gamma \in (0, 1)$ to the system (33) and (34) . Rewriting (33) we get

$$c^2 = \beta\frac{z^*(2z^* - A\alpha)^2}{3z^* - 2A\alpha}$$

Combining with (34):

$$z^* = \frac{2A\alpha}{3 - \beta\gamma^2}$$

Plugging it into (34) , γ is a solution to:

$$A\alpha\left(\frac{1 + \beta\gamma^2}{3 - \beta\gamma^2}\right) = \underbrace{\tau - A(1-\alpha)}_{c}\gamma \tag{35}$$

Now, in the range $\gamma \in (0, 1)$ the RHS of (35) is increasing and linear in γ. The LHS of (35) is increasing and convex in γ. At the extremes:

a) At $\gamma = 0$, $RHS = 0$ and $LHS = \frac{1}{3}A\alpha > 0$.
b) At $\gamma = 1$, $RHS = (\tau - A(1-\tau))$ and $LHS = A\alpha((1+\beta)/(3-\beta))$

We get that the sides cross for $\gamma < 1$ if

$$\tau > A\left(1 - 2\alpha\frac{1-\beta}{3-\beta}\right) \tag{36}$$

Also, they cross only once in this range because the RHS is linear and LHS is convex (and given the established values at $\gamma \in \{0, 1\}$). Finally, notice that if $\tau < A(1 - 2\alpha(1-\beta)/(3-\beta))$ then there is no crossing for $\gamma \in (0, 1)$. The reason is that the LHS is independent of τ and the slope of the RHS is increasing in τ. Therefore the first crossing (if exists) between these two functions shifts to the left in τ and is at $\gamma = 1$ at the threshold.

So we have established that in the assumed range there exists an equilibrium in linear strategies with convergence (as our guess that it is possible to find best-response strategies that lead to interior market shares turns out to be correct) and that outside this range there does not exist a symmetric equilibrium in linear strategies with convergence.

Finally, we can use equation (35) to establish the comparative statics: γ is increasing in β and decreasing in $(\tau - A)/A\alpha$. Therefore γ is decreasing in τ and increasing in A. The behavior with respect to α is more complicated: if $\tau > A$ then γ is increasing in α and it is decreasing otherwise. □

Proof of Proposition 5 The social planner's problem is:

$$V(y) = \max_{x\in[-1,1]}\left[-\frac{1}{4}d_2x^2 + \frac{1}{2}A\alpha xy + \frac{1}{4}(2A-\tau) + \beta V(x)\right] \tag{37}$$

From Lemma 1, the optimal policy function is:

$$x^*(y) = \frac{2A\alpha}{d_2 + \sqrt{\left(d_2^2 - 4\beta\,(A\alpha)^2\right)}}y = \Gamma y \tag{38}$$

As $d_2 > 2A\alpha$ the multiplier is less than 1 so indeed we have convergence and hence the optimal choice is always interior for any $y \in [-1, 1]$. The comparative statics follow directly from the above expression for Γ. □

Proof of Proposition 6 The cross partial derivative of the current surplus with respect to x and y is $\frac{1}{2}A\alpha > 0$. Thus the optimal policy $x^*(y)$ is increasing (regardless of τ). Since the planner's decision rule is monotone, it is always the case that $\lim_{T\to\infty} x_T^*(y)$ exists.

Consider the set $Y = \{y : \lim_{T\to\infty} x_T^*(y) \notin \{-1, 1\}\}$. By monotonicity and symmetry, it must be the case that $Y = [-a, a]$. Moreover, $y \in Y$ implies $x_T^*(y) \notin \{-1, 1\}$ for $T > 1$, in other words, the constraint that $y \in [-1, 1]$ never binds if we start from a point in Y. Finally $x_T^*(y) \in Y$ for all T and $\lim_{T\to\infty} x_T^*(y) \in Y$. By Proposition 5, since the constraints don't bind, if $d_2 > 0$ then $V(y)$ is quadratic in y on Y and the decision rule is $x^*(y) = \Gamma y$ for $\Gamma > 1$.

Suppose that $1 > a > 0$. $\Gamma > 1$ implies that $x_T^*(y)$ diverges from Y, a contradiction. Suppose $a = 1$. Then $\lim_{T\to\infty} x_T^*(y) = \pm 1$, a contradiction.

Finally, we have assumed in the model that the static total surplus is concave in x^*. In the linear utility case it corresponds to $d_2 > 0$ or $\tau > 2A(1-\alpha)$. If $d_2 < 0$ we can extend the divergence result easily. In this case static social surplus is convex and myopic maximization always selects x^* either 1 or -1 depending on the sign of y. If the planner could choose myopically x and y, the surplus would be maximized by setting both of them to 1 or -1. Such allocation cannot be improved upon by a forward-looking planner. Additionally, this allocation is achieved for $t > 1$ if the planner acts myopically in the first period. Therefore if $d_2 < 0$ the optimal plan for the planner is to diverge immediately to 1 if $y > 0$ and to -1 if $y < 0$. □

Proof of Proposition 7 First, by Propositions 5 and 4 both the planner problem and market equilibrium result in convergence. Using Proposition 5 we obtain a lower bound on Γ :

$$\Gamma \geq \frac{2A\alpha}{2d_2} > \frac{A\alpha}{\tau - A(1-\alpha)} = g^* \tag{39}$$

We now show that $\gamma \leq g^*$. From equation (33) $z^* > A\alpha$. Then from equation (34) , $\gamma \leq A\alpha/c = g^* < \Gamma$. So indeed the market equilibrium allocation converges faster than the social optimum. □

Proof of Proposition 8 (a) Denote by $\bar{y} \geq 0$ the initial condition such that $x^*(\bar{y}) = x(\bar{y}) = 0$. For $y < \bar{y}$, $\lim_{T\to\infty} x_T^*(y) \leq 0$, so we can focus on $y \geq \bar{y}$. Then $x_T^*(y) \geq \bar{y} \geq 0$ for all T. There are two possibilities. If $x_T(y) \geq \bar{y}$ for all T, then by the same argument as in Proposition 2, we have $x_T^*(y) \geq x_T(y)$ for all T, implying the result. If there is a T such that $x_T(y) < \bar{y}$, then $x_{\hat{T}}(y) < 0$ for all $\hat{T} > T$, again implying the result.

(b) Denote $\bar{T} = \sup\left\{T : x_T^*(y) \geq \bar{y}\right\}$ so that for all $T \leq \overline{T}$ we have $x_T^*(y) \geq \bar{y}$ and for all $T > \overline{T}$ we have $x_T^*(y) < \bar{y}$. By the same argument as in Proposition 2, we have $x_T^*(y) \geq x_T(y) \geq 0$ for all $T \leq \overline{T}$. Now, for $\overline{T}+1$ we get

$$\bar{y} > x_{\overline{T}+1}^*(y) \geq x^*\left(x_{\overline{T}}(y)\right) \tag{40}$$

That implies that $x_{\overline{T}+1}(y) < \bar{y}$ (as it is either negative or smaller than $x^*\left(x_{\overline{T}}(y)\right)$). By induction we can show that for any $T > \overline{T}+1$

$$x_T^*\left(y\right) \leq x^*\left(x_{T-1}\left(y\right)\right) < 0 \tag{41}$$

and by Proposition 1 $x_T\left(y\right) \in [x_T^*\left(y\right), 0)$, as claimed.

□

Proof of Proposition 9 This proposition generalizes Proposition 3. Define $\bar{x}(y)$ implicitly by the system

$$\begin{aligned} \delta + U_0^D(q) + U^{Ny}(q) + U^{No}(q_{-1}) &= U_1^D(q) + U^{Ny}(1-q) + U^{No}(1-q_{-1}) \\ \bar{x} &= 2q - 1 \\ y &= 2q_{-1} - 1 \end{aligned} \tag{42}$$

The interpretation is that $\bar{x}(y)$ is the evolution of market share at equal prices. Recall that $\bar{q}$ is the defined as $G\left(\bar{q}\right) = 0$ and $\bar{y} = 2\bar{q} - 1$. We now construct an equilibrium where $p_0(q_{t-1}) = 0$, for $q_{t-1} \leq \bar{q}$ and $p_1(q_{t-1}) = 0$, for $q_{t-1} \geq \bar{q}$. We have the following three claims:

Claim 9.1 $\lim_{T\to\infty} \bar{x}_T(y) = 1$ for $y > \bar{y}$ and $\lim_{T\to\infty} \bar{x}_T(y) = -1$ for $y < \bar{y}$.

Proof Follows from the fact that $\bar{x}(y)$ is continuous and $\bar{x}(y) > y$ for $y > \bar{y}$, $\bar{x}(y) < y$ for $y < \bar{y}$.

Claim 9.2 Given the postulated strategies: $p_0(q_{t-1}) = 0$ for $q_{t-1} \leq \bar{q}$ and $p_1(q_{t-1}) = 0$ for $q_{t-1} \geq \bar{q}$, if $p_1(q_{t-1})$ is continuous and the strategies are such that $\lim_{T\to\infty} x_T(y) = -1$ for $y < \bar{y}$, then the optimal strategy $p_0(q_{t-1})$ is such that $\lim_{T\to\infty} x_T(y) = 1$ for $y > \bar{y}$.

Proof By the previous claim, such a strategy is feasible for firm zero if $p_0(q_{t-1})$ is close enough to zero. Any best response $p_0(q)$ must be continuous since $p_1(q)$ is continuous. As a result, $x(y)$ is continuous. Next notice that $x(y) > \bar{y}$ for $y > \bar{y}$; if $x(y) \leq \bar{y}$, then firm 0 would obtain payoff 0 which is clearly dominated by charging $G(y)$.

Suppose that $\lim_{T\to\infty} x_T(y) \neq 1$ for some y. Since $x(y) > \bar{y}$ for all $y > \bar{y}$, and $x(y)$ is continuous, there are only two possible cases for what can happen to the evolution of the system on $y > \bar{y}$. Either $\lim_{T\to\infty} x_T(y) = \bar{y}$ for all $y > \bar{y}$, or there exists some $\hat{y} \in (\bar{y}, 1)$ such that $x(\hat{y}) = \hat{y}$.

Either case implies a contradiction. If $\lim_{T\to\infty} x_T(y) = \bar{y}$, $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \pi_i = 0$. This is dominated, for any $y > 0$ by pricing at $G((y+1)/2) > 0$ and maintaining market share y, which yields $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \pi_i = yG((y+1)/2) > 0$. In other words, a strategy with $\lim_{T\to\infty} x_T(y) = \bar{y}$ cannot be a best response to the zero pricing strategy pursued by firm 1.

The remaining possibility is that there is a $\hat{y}$ such that $x(\hat{y}) = \hat{y}$. Starting from $\hat{y}$ the market share remains at $\hat{y}$ forever, and hence $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \pi_i = \hat{y}G((\hat{y}+1)/2)$. However, pricing at zero for a finite number of periods increases market share to $\tilde{y} > \hat{y}$. That results in zero profits for a finite number of periods followed by pricing at $G((\tilde{y}+1)/2) > G((\hat{y}+1)/2)$ and profits $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \pi_i = \tilde{y}G((\tilde{y}+1)/2) > \hat{y}G((\hat{y}+1)/2)$. Hence it could not have been optimal to choose a strategy that led to $x(\hat{y}) = \hat{y}$.

Claim 9.3 Given the postulated strategies, for a continuous $p_0(q_{t-1})$ and strategies such that $\lim_{T\to\infty} x_T(y) = 1$ for $y > \bar{y}$, then the optimal strategy $p_1(q_{t-1})$ is such that $\lim_{T\to\infty} x_T(y) = -1$ for $y < \bar{y}$. Proof is identical to the previous claim.

So we have shown that, if the firms price at zero over a certain range, the firms find it optimal to choose strategies that lead to divergence. We complete the construction of the equilibrium by showing that it is optimal for the firms to choose $p_0(q_{t-1}) = 0$ for $q_{t-1} \leq \bar{q}$ and $p_1(q_{t-1}) = 0$ for $q_{t-1} \geq \bar{q}$, as asserted. Notice that, for any $q_{t-1} \leq \bar{q}$, the payoff for firm 0 is equal to zero regardless of the price; the firm is eliminated, asymptotically, even when it prices at the lowest possible price (0). Charging a higher price leads to the same outcome, in the limit. The same is true for firm 1 for $q_{t-1} \geq \bar{q}$. □

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