Unemployment Insurance with Hidden Savings

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Abstract

This paper studies the design of unemployment insurance when neither the searching effort nor the savings of an unemployed agent can be monitored by the principal. If the principal could monitor the savings, then the optimal unemployment insurance would leave the agent savings-constrained and therefore could not be implemented with hidden savings. With a constant absolute risk-aversion (CARA) utility function, we obtain a closed form solution of the optimal contract. Under the optimal contract, the agent is neither saving nor borrowing constrained. Counter intuitively, his consumption declines faster than implied by Hopenhayn and Nicolini (1997). We also show that, an increasing benefit during unemployment and a constant tax during employment implement the efficient allocation.

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1 Introduction

Unemployment insurance must balance the benefits of insurance against the concern that a too-generous program will discourage search effort. Card, Chetty, and Weber (2007) provide evidence that search effort varies according to an agent's financial situation. They find that exogenously richer agents take longer to find a job but do not find higher wages on their next job. In other words, it seems that wealth is one important determinant in the search effort decision.

Since neither search effort nor wealth are readily observable to the planner (the principal), it is natural to wonder how these information frictions might affect the optimal unemployment system. We introduce hidden savings into an environment similar to the Hopenhayn and Nicolini (1997) version of the model of Shavell and Weiss (1979). We show that the addition of hidden savings leads to faster consumption declines during an unemployment spell than the declines in a model with observable savings. Moreover, with hidden savings, agents with relatively high initial insurance claims have the fastest rate of consumption decline, eventually having lower claims than those agents who started out receiving less. We measure the amount of subsidy that results from remaining unemployed and show that the subsidy rises over the course of the spell.

Hidden savings is naturally relevant in repeated moral hazard models like this one. When savings can be monitored, as in Hopenhayn and Nicolini (1997) or the repeated moral hazard model of Rogerson (1985), the optimal policy leaves the agent savings-constrained: his marginal utility is lower today than tomorrow. By making an agent poor in the future, it encourages the agent to search harder for a job.

Our paper is related to both Werning (2001, 2002) and Abraham and Pavoni (2004), who use the first-order condition approach to study models with hidden savings and borrowing. Briefly speaking, the first-order approach studies a relaxed problem, which replaces the incentive constraints in the original problem with some first-order conditions of the agent. In these papers, the first-order condition for the type that has never deviated in previous periods and thus has zero hidden wealth is imposed.

Werning (2001) acknowledges that imposing first-order conditions may not be sufficient to ensure incentive compatibility. Furthermore, Kocherlakota (2004) shows that when the disutility function is linear, the agent's problem is severely non-convex and the first-order condition cannot be sufficient.[1] When the first-order approach is invalid, to solve the problem recursively, the number of state variables would be infinite, making even numerical computations intractable. We overcome this problem by focusing on a special case of constant absolute risk-aversion (CARA) utilities and linear disutilities. In this case we conjecture and verify the countable set of constraints that bind. With this in hand, it is straightforward to explicitly solve for the principal's optimum. It has the interesting feature that the incentive constraints of the searching agent never bind. Instead, the binding incentive constraint in any period is the one for the agent who has always shirked, and meanwhile saved.

The basic intuition for this structure of binding incentive constraints relates to the way in which shirking and saving interact. When an agent shirks, he increases the odds of continuing to be unemployed. The unemployed state involves lower consumption, so he wants to save in preparation for the greater probability of this low outcome. The agent who saves the most is the one who has always done maximum shirking, and who knew he would never become employed. Given that he has saved the most, he is best equipped to do additional shirking, which is, again, complementary with saving. This example shows, in two ways, the sense in which the first-order approach is not appropriate for this kind of problem: First, the complementarity between shirking and saving can make the first-order condition for effort insufficient for optimality, and second, since the binding incentive constraint is not for the agent who always searches, it is not enough to look at the always-searching agent's optimality condition in the first place.

The contract we study always implements, at an optimum, a one-time lottery over always searching as hard as possible, or always searching the least, which we refer to as

[1] The first-order approach remains a practical and useful method to solve hidden savings problems when the disutility function is sufficiently convex. In particular, Werning numerically verifies the incentive compatibility of the solution obtained with the first-order approach for a large range of convex disutility functions.

not searching. The interesting case is when the agent searches. The reason for keeping the agent on the Euler equation in this case relates to the savings-constrained nature of the optimal contract in Hopenhayn and Nicolini (1997). There, making the agent poor in the future generates incentives to search today. When the agent can freely save, it is no longer possible to keep the marginal utility of consumption high tomorrow, but there is still no reason to make it fall over time. In that case, the same logic as in the case with observable savings would suggest lowering today's marginal utility and raising tomorrow's in order to increase search incentives today and make the contract deliver utility more efficiently. As a result, the agent is on the Euler equation. As a corollary, our solution would also be the solution to a case where the agent had access to both hidden borrowing and hidden savings.

In the state-of-the-art model of optimal unemployment insurance in Shimer and Werning (2005), the focus is on the wage-draw aspect of the Shavell and Weiss (1979) structure.[2] The fundamental trade-off is between search time and quality of match (in terms of wages). They find that the optimal contract has constant benefits (for CARA utility, and approximately so for constant relative risk-aversion (CRRA) utility), and keeps the agent on the Euler equation. The reason their agents are not borrowing-constrained is different from ours, however. In the CARA model, the agent's reservation wage is independent of wealth; therefore, there is no incentive benefit from distorting the first-order condition, and there is the usual resource cost to the principal from the distortion. In that sense their model has no "wealth effects" of the kind emphasized by Card, Chetty, and Weber (2007). Our model, on the other hand, does have wealth effects; richer agents have less incentive to give effort than poorer agents. Hopenhayn and Nicolini (1997) show that there is an incentive benefit in making the agent savings-constrained; in our model such a constraint is impossible, so the optimal contract moves

[2]It is worth noting that empirical evidence in favor of both the wage-draw view and the search theoretic view of unemployment exists. We have stressed the evidence, such as Card, Chetty, and Weber (2007), that emphasizes search. However, recent research suggests that actual time used in search by the unemployed is very small, suggesting that compensating the disutility of search may not be as important a feature of unemployment systems.

to the Euler equation.

The case where search effort is a monetary cost and utility is CARA is studied in Werning (2001). Monetary search cost implies that poor agents have no greater incentives than rich agents, since the cost and benefit of search are both proportional to wealth. The main insight of this model is similar to Shimer and Werning (2005): agents are not borrowing-constrained and a constant benefit sequence is needed to implement the optimal allocation.

Our results contrast with Kocherlakota (2004), who studies a version of Hopenhayn and Nicolini (1997) with hidden savings, and finds that agents are borrowing-constrained in the optimal contract. Moreover, he finds that the consumption sequence of agents is constant while they are unemployed and jumps to a higher, constant level when they are employed. A key difference between our problem and the one that Kocherlakota (2004) studies is that he focuses on interior effort levels, while our optimal contract does not implement interior effort levels.

The paper proceeds as follows. In the next section we introduce the basic model. In Section 3 we show that the optimal contract either implements high effort forever or no effort forever. In Section 4 we solve for the optimal contract to implement high effort. In Section 5 we show the important characteristics of the optimal contract, and compare them to the case without hidden savings. We then conclude. All proofs are contained in the Appendix.

2 The Problem

In this section we describe an unemployment insurance model in the spirit of Shavell and Weiss (1979). There is a risk-neutral principal and a risk-averse agent. The preferences of the agent are

$$E\sum_{t=0}^{\infty}\beta^t\left[u(c_t)-a_t\right],$$

where c_t and $a_t \in [0,1]$ are consumption and search effort at time t, $u(c) = -\exp(-\gamma c)$ is a constant absolute risk-aversion (CARA) utility function, $\beta < 1$ is the discount factor, and E is the expectations operator. An agent can be employed or unemployed; he begins life unemployed. The choice of a_t affects the probability of becoming employed for an unemployed agent. Specifically, if an agent is unemployed in period t, then the probability of his becoming employed in period $t+1$ is πa_t, and the probability of his staying unemployed is $1-\pi a_t$. We assume $\pi \in (0,1)$, which implies that the agent might not find a job even if he exerts full effort.[3] If an agent is employed in period t, he stays employed in period $t+1$ with probability one. Thus, being employed is an absorbing state.

The agent's employment status is observable to others, but his choice of a_t is unobservable. As well, the agent can secretly hold a non-negative amount of assets. The principal can observe neither the consumption nor the savings of the agent. A contract σ in this environment specifies three sequences of real numbers, $(\{c_t^U\}_{t=0}^{\infty}, \{c_t^E\}_{t=1}^{\infty}, \{a_t\}_{t=0}^{\infty})$. Given such a contract, an agent who is unemployed in period t receives compensation c_t^U from the principal. If an agent became employed for the first time in period t, then his compensation from the principal in period $s \geq t$ is c_t^E. Thus, once an agent is employed, his compensation is constant over time. It is simple to show that, because the principal and agent have the same discount factor, this smooth compensation is efficient in this economy, so, to save on notation, we impose that feature now. The contract will also recommend an effort level a_t, and, given that the principal cannot observe a_t, a_t will be designed to satisfy incentive constraints. To simplify notation, we will let $U_t = -u(c_t^U)$ and $E_t = -u(c_t^E)$. Notice that U_t and E_t are proportional to the marginal utility of consumption, where the constant of proportionality is γ.

Without loss of generality, we require that there be no incentive for the agent to save if he follows the *recommended* strategy $\{a_t\}_{t=0}^{\infty}$. This is true if and only if all the first-order conditions are satisfied,

$$U_t \geq (a_t\pi)E_{t+1} + (1 - a_t\pi)U_{t+1}, \text{ for all } t \geq 0. \tag{1}$$

We refer to this condition as the Euler equation. Note that once the agent chooses a

[3] If full effort led to sure employment, the principal could simply recommend full effort and promise grave punishment if a job were not found.

different search effort $\tilde{a}_t \neq a_t$, equation (1) may be violated, and the agent holds positive assets.

In the literature, a deviation on effort combined with hidden savings is called a *joint deviation* by the agent. Hidden savings is an additional friction that makes the contract more difficult to solve than a traditional dynamic contracting problem without hidden savings or borrowing, since it adds a constraint to the problem of the principal. We therefore must introduce some notation to describe all the incentive constraints.

We must consider a variety of possible deviations. For instance, suppose $\{\tilde{a}_t\}_{t=0}^{\infty}$ is the agent's search strategy. If there is a finite $\bar{t}$, such that $\tilde{a}_t = a_t, \forall t \geq \bar{t}$, then $\{\tilde{a}_t\}_{t=0}^{\infty}$ is called a *finite-deviation* strategy. Otherwise it is called an *infinite-deviation* strategy. Let $V(\{\tilde{a}_t\}_{t=0}^{\infty})$ be an agent's ex ante utility if he uses strategy $\{\tilde{a}_t\}_{t=0}^{\infty}$ and privately saves. The incentive compatibility (I.C.) constraints will be written as

$$V(\{a_t\}_{t=0}^{\infty}) \geq V(\{\tilde{a}_t\}_{t=0}^{\infty}), \text{ for all } \{\tilde{a}_t\}_{t=0}^{\infty}. \tag{2}$$

However, in much of the following discussion, we will discuss only incentive constraints for finite-deviation strategies. This is without loss of generality, since the payoff of an infinite-deviation strategy can be approximated arbitrarily closely by those of finite-deviation strategies.

Let D_t denote the discounted (to period 0) disutility of efforts $\{a_s\}_{s=t}^{\infty}$ conditional on not finding a job at the beginning of period t,

$$D_t = \sum_{s=t}^{\infty} \beta^s (\Pi_{k=t}^{s-1}(1 - a_k \pi)) a_s.$$

This can be generated recursively by $D_t = \beta^t a_t + (1 - a_t \pi) D_{t+1}$; D_0 would be the total discounted disutility for an agent. Since disutility is linear and one unit of labor is compensated with $\beta \pi w/(1-\beta)$ units of wage income, $\beta \pi w D_0/(1-\beta)$ is the expected wage income that the principal can obtain. The expected cost for the principal is equal to the discounted value of consumption goods delivered to the agent, minus the expected wage income,

$$C(\sigma) = \sum_{t=0}^{\infty} \beta^t \left(\Pi_{s=0}^{t-1}(1 - a_s \pi)\right) \left[c_t^U + \beta a_t \pi \frac{c_{t+1}^E}{1-\beta}\right] - \frac{\beta \pi w}{1-\beta} D_0.$$

The problem of the principal is to choose a contract σ to minimize $C(\sigma)$, subject to the I.C. constraints (2), the Euler equation (1), and the delivery of some level of initial promised utility $\bar{V}$.

3 Optimal Effort

In this section we take up the question of what levels of effort the principal will implement. We show that the principal can do best by offering an initial lottery over full effort forever ($a_t = 1, \forall t \geq 0$ is denoted by $\{1\}_{t=0}^{\infty}$), or no effort forever ($a_t = 0, \forall t \geq 0$ is denoted by $\{0\}_{t=0}^{\infty}$). Along the way we show that, at the optimal recommended effort level, the agent is always on the Euler equation; in other words, equation (1) holds with equality. Thus, no shirkers will be borrowing-constrained, since their efforts are below the maximum, implying that they have incentives to hold a positive amount of savings. This result is relevant for two reasons. First, it helps us with the analysis of the next section; second, it shows that our results for the model with hidden savings carry over to the case where the agent has access to opportunities for both hidden saving and borrowing. The ability of the principal to make the agent borrowing-constrained is never used.

We first show that in any period where either full effort or no effort is implemented, the Euler equation holds with equality.

Lemma 1 *Let σ be an optimal contract. If $a_t \in \{0, 1\}$, then*

$$U_t = a_t \pi E_{t+1} + (1 - a_t \pi) U_{t+1}. \tag{3}$$

When $a_t = 0$, the result is simple; there is no reason not to let $U_t = U_{t+1} = E_{t+1}$, since there is no effort to induce. The more interesting case is that of $a_t = 1$. For an agent with no hidden wealth, the benefit of eliminating the borrowing constraint is analogous to the "inverse Euler equation" logic of Hopenhayn and Nicolini (1997) and Rogerson (1985): by moving consumption from $t+1$ to t (and thereby raising marginal utility tomorrow relative to today), the incentive to give effort rises, since greater marginal utility tomorrow makes finding a job more beneficial. In the CARA case, for an agent with no hidden savings, the algebra is especially simple: the marginal benefit of effort is proportional to $\beta\pi(-E_{t+1} + U_{t+1})$, so if the principal reduces consumption tomorrow uniformly by δ and raises it today by $\epsilon < \beta\delta$, leaving unchanged the utility for the agent who chooses the recommended effort ($a_t = 1$), the marginal benefit of effort increases by a factor of e^{δ}. Of course, in addition, the principal saves resources for the given level of delivered utility by reducing the intertemporal distortion.

The lemma must also consider whether smoothing consumption for the agent with no hidden wealth would violate incentive constraint for agents with hidden savings. In that case, consider a plan of some level of period t hidden wealth s_{t-1}/β, an action in period t denoted $\tilde{a}_t$, and a level of wealth in period $t+1$, s_t/β. The proof of the lemma shows that the proposed modification to the contract reduces the payoff to all combinations of s_{t-1}/β, $\tilde{a}_t$, and s_t/β.

To see the logic, consider first an agent with $s_t/\beta > 0$. Since this agent is already on the Euler equation, he must be made worse off by receiving less expected discounted resources. On the other hand, consider an agent with hidden savings at t who saves nothing for $t+1$. This agent consumes the entire allocation c_t^U in period t, plus his wealth s_{t-1}/β; from period $t+1$ onward, he is just like the agent who works the recommended amount. As a result, this agent gains less from the proposed change; his marginal utility in period t is lower than the agent who starts with no hidden wealth, and therefore he benefits less from moving consumption forward from $t+1$ to t.

Next, we introduce a result from Kocherlakota (2004), that with linear disutility, interior effort can only be implemented when the agent is borrowing-constrained:

Lemma 2 *(Kocherlakota (2004)) Let σ be an incentive compatible contract. If $a_t \in (0,1)$, then $U_t > a_t\pi E_{t+1} + (1 - a_t\pi)U_{t+1}$.*

A key feature of the linear disutility model is that, for any consumption sequence across states and dates, the agent is indifferent between all effort levels. Intuitively, linear disutility makes the model analogous to the case of effort being either zero or one, and $a_t \in (0,1)$ representing a mixed strategy. With the opportunity to also save, the agent has a further option: shirk and save. This option is attractive unless the contract makes saving unattractive by making the marginal utility high today. With observable savings, the agent's value function is linear in a_t, since the disutility function is linear. However, with hidden savings, when the agent is on the Euler equation, there is a second-order effect. Whenever the agent lowers a_t, he would save to smooth consumption, which makes the value function convex in a_t. This second-order effect cannot be made unprofitable when the agent is on the Euler equation.

A closely related feature in the linear disutility model is that implementing interior effort implies that, for the given consumption sequence, the principal is indifferent between implementing any level of effort a. So, starting from $a_t \in (0,1)$ and fixing consumption across states, the principal (and the agent) should be willing to replace the recommended a_t with $a_t = 1$. Since the agent is borrowing-constrained at $a_t < 1$, he is further borrowing-constrained at $a_t = 1$ (since his expected marginal utility of consumption tomorrow is lower when he searches harder). The *principal* then has a profitable sort of double deviation: move to $a_t = 1$ *and* adjust the consumption sequence to return the agent to the Euler equation. This must make the principal strictly better off by the line of argument in Lemma 1. We therefore have

Corollary 1 *It is never optimal for the principal to implement interior effort $a_t \in (0,1)$.*

At this point we know that any optimal effort sequence uses only ones and zeros. However, it turns out that the principal would always prefer to implement a lottery over a sequence of all ones and all zeros rather than implement a sequence that contains both ones and zeros.

Lemma 3 *Any sequence of effort levels $\{a_t\}_{t=0}^{\infty}$ where $a_t = 0$ for some t and $a_{t'} = 1$ for some t' is dominated by a simple lottery with two outcomes: either $\{1\}_{t=0}^{\infty}$, or with the complementary probability, $\{0\}_{t=0}^{\infty}$.*

For instance, consider the sequence $\{a_t\}_{t=0}^{\infty} = (0, 1, 1, 1, ...)$. Alternately, the principal could make the agent search hard immediately, with probability β, and let the agent rest forever with probability $1-\beta$. If savings were observed, the outcome would be no better or worse for both principal and agent. However, by choosing the lottery, the principal eliminates the possibility of hidden savings before search begins in period $t=1$, which is possible under the sequence that starts with a period of zero effort. Eliminating this possibility eliminates a profitable deviation for the agent, and is therefore beneficial for the principal. More generally, the proof shows that periods where the agent is not searching are costly because of the hidden savings possibilities they generate.

We are now ready to state the main result of this section, which gathers the previous results. It states that whenever the principal implements any effort, he implements high effort forever.

Proposition 1 *Suppose the principal has access to a one-time lottery before time* 0*. For any outcome of the lottery where* $a_t > 0$ *for any* t*, the principal implements* $a_t = 1$ *for all* $t \geq 0$*. Moreover,* $U_t = \pi E_{t+1} + (1-\pi)U_{t+1}$*.*

A useful point is that, the incentive to save for a shirker is always greater than it is for an agent searching hard, so the Euler equation also holds with equality for shirkers. In the next section we take up the issue of how to implement full effort forever. The Euler equation will be an aid in tackling that problem.

4 Implementing Full Effort Forever

We now focus on the problem of implementing full effort forever,

$$\min_{\{c_t^U\}_{t=0}^{\infty},\{c_t^E\}_{t=1}^{\infty}} \quad C(\sigma) \tag{4}$$

$$s.t. \quad U_t = \pi E_{t+1} + (1-\pi)U_{t+1},$$

$$\bar{V} = V(\{1\}_{t=0}^{\infty}),$$

$$V(\{1\}_{t=0}^{\infty}) \geq V(\{\tilde{a}_t\}_{t=0}^{\infty}), \text{ for all } \{\tilde{a}_t\}_{t=0}^{\infty}. \tag{5}$$

We solve problem (4) in two steps. First, we use the CARA assumption to solve the shirker's problem in closed form; that is, for any sequence $\{\tilde{a}_t\}$, we compute the expected discounted utility $V(\{\tilde{a}_t\}_{t=0}^{\infty})$ for that plan after optimally undertaking hidden saving. Second, we conjecture that the binding incentive constraints are for agents who shirk up to t ($\tilde{a}_s = 0$, for all $0 \leq s \leq t-1$), and search hard thereafter ($\tilde{a}_s = 1$, for all $s \geq t$). These agents are referred to as the all-shirking-up-to-t types. We solve a relaxed problem where we impose only the incentive constraints for those deviations, and show that the solution satisfies equation (5) and is therefore a solution to the full problem (4).

4.1 Utility after deviation

Suppose an agent chooses a finite deviation strategy $\{\tilde{a}_t\}_{t=0}^{\infty}$, and engages in hidden saving. In general, such a problem is intractable. However, with the CARA utility assumption, we can solve that problem in closed form:

Lemma 4 *Suppose an agent chooses actions $\{\tilde{a}_t\}_{t=0}^{\infty}$, where $\tilde{a}_t = 1$, for all $t \geq \bar{t}$. Then the agent's discounted utility from consumption after hidden saving is*

$$-\frac{U_0^{1-\beta}\left[\tilde{a}_0\pi E_1 + (1-\tilde{a}_0\pi)U_1^{1-\beta}\left[\tilde{a}_1\pi E_2 + (1-\tilde{a}_1\pi)U_2^{1-\beta}\left[...\left[\tilde{a}_{\bar{t}-1}\pi E_{\bar{t}} + (1-\tilde{a}_{\bar{t}-1}\pi)U_{\bar{t}}\right]^{\beta}...\right]^{\beta}\right]^{\beta}\right]^{\beta}}{1-\beta}.$$

4.2 Showing which constraints bind: a relaxed problem

We conjecture that the binding constraints are the ones for all-shirking-up-to-t types. The intuition comes from the fact that shirking and saving are complements, so that shirking at time t is best combined with prior saving. Prior saving, in turn, is most valuable when shirking guarantees that employment will not occur.

To show that these constraints are in fact the relevant ones, consider the relaxed problem

$$\min_{\sigma} \quad C(\sigma) \tag{6}$$

$$s.t \quad U_t = \pi E_{t+1} + (1-\pi)U_{t+1}, \text{ for all } t \geq 0, \tag{7}$$

$$\bar{V} = -\frac{U_0}{1-\beta} - \frac{1}{1-\beta(1-\pi)}, \tag{8}$$

$$\bar{V} \geq -\frac{\left(\Pi_{s=0}^{t-1} U_s^{(1-\beta)\beta^s}\right) U_t^{\beta^t}}{1-\beta} - \frac{\beta^t}{1-\beta(1-\pi)}, \text{ for all } t \geq 1. \quad (9)$$

First, we show in the relaxed problem that all the I.C. constraints bind.

Lemma 5 *In the optimal solution to problem (6), all the I.C. constraints bind.*

The relaxed problem has an equal number of variables and constraints; however, to understand why all the constraints bind, it is useful to explain the intuition that these constraints are independent. With change of variables $x_t = c_{t-1}^U - c_t^U = \log(U_t/U_{t-1})/\gamma$, the constraints in equation (9) can be rewritten as

$$\sum_{s=1}^{t} \beta^s x_s = \frac{1}{\gamma} \log\left(-\bar{V} - \frac{\beta^t}{1-\beta(1-\pi)}\right) - \frac{1}{\gamma} \log\left(-\bar{V} - \frac{1}{1-\beta(1-\pi)}\right).$$

Heuristically, we could pick a period $T = 10$ and divide the constraints into two groups, the "early" incentive constraints with $t \leq T$, and the "late" ones ($t > T$), and then pick one from each group. For example, we pick the incentive constraint for $t = 10$ from the first group and $t = \infty$ from the second group, and consider the constraints:

$$\sum_{s=1}^{10} \beta^s x_s \geq \frac{1}{\gamma} \log\left(-V - \frac{\beta^{10}}{1-\beta(1-\pi)}\right) - \frac{1}{\gamma} \log\left(-V - \frac{1}{1-\beta(1-\pi)}\right), \quad (10)$$

$$\sum_{s=1}^{\infty} \beta^s x_s \geq \frac{1}{\gamma} \log(-V) - \frac{1}{\gamma} \log\left(-V - \frac{1}{1-\beta(1-\pi)}\right). \quad (11)$$

If inequality (10) binds and inequality (11) is slack then, since x_s ($s > 10$) does not appear in inequality (10), $x_s = 0$ in the solution, which means that consumption is flat in later periods, a contradiction to the fact that the principal wants to implement high effort in all time periods. If (11) binds and (10) is slack, then the principal knows that the shirking type will shirk in all time periods, and thus would reach unemployment state U_t with probability one, while the equilibrium type would reach U_t with smaller and smaller probabilities as t gets bigger. Therefore, for efficiency reasons, the principal would like the distortion x_t to be increasing, and minimize the punishment in the early periods that the searcher hits relatively often.[4] This implies that

$$\sum_{s=1}^{10} \beta^s x_s < (1-\beta^{10}) \left[\frac{1}{\gamma} \log(-V) - \frac{1}{\gamma} \log \left(-V - \frac{1}{1-\beta(1-\pi)} \right) \right].$$

But the concavity of the function $\log(\cdot)$ implies that

$$\begin{aligned} & \log \left(-V - \frac{\beta^{10}}{1-\beta(1-\pi)} \right) - \log \left(-V - \frac{1}{1-\beta(1-\pi)} \right) \\ > \ & (1-\beta^{10}) \left[\log(-V) - \log \left(-V - \frac{1}{1-\beta(1-\pi)} \right) \right], \end{aligned}$$

contradicting (10). The proof extends this intuition to show that, if any constraint did not bind, the solution would imply an infeasible choice for that t.

Now we verify that the solution to the relaxed problem satisfies all the incentive constraints in the full problem.

Lemma 6 *If in a contract, for all $t \geq 0$,*

$$-\frac{\left(\Pi_{i=0}^{t-1} U_i^{(1-\beta)\beta^i}\right) U_t^{\beta^t}}{1-\beta} - \frac{\beta^t}{1-\beta(1-\pi)} \geq -\frac{\left(\Pi_{i=0}^{t} U_i^{(1-\beta)\beta^i}\right) U_{t+1}^{\beta^{t+1}}}{1-\beta} - \frac{\beta^{t+1}}{1-\beta(1-\pi)},$$

then all finite-deviation incentive constraints are satisfied.

Consider the beginning of period t, when an agent has not been able to find a job in the previous periods. Fix the saving decisions of agents up to t. At this moment, there are potentially many such types of agents. The equilibrium type has 0 wealth, and the other deviating types have saved various amounts and thus start with positive wealth. Among them, the type that has never exerted any effort is the richest, since he knew he would achieve this state with probability one, and has saved to prepare accordingly. Without further deviation, these agents' consumption behavior is simply to consume the claimed goods, plus the interest payment from the secretly accumulated wealth.[5] Now these agents are contemplating a one-period deviation, which is to lower

[4] In fact, in the proof we show that the sequence of x_s is decreasing when all the constraints bind in the relaxed problem.

[5] This depends on the assumption of CARA utilities.

a_t from 1 to 0 and follow the suggested strategy $a_s = 1$, for all $s \geq t+1$. Doing this, all the types have reduced the disutility by the same amount, which is 1. However, they value the transition from the employment state (with claim c^E_{t+1}) to the unemployment state (with claim c^U_{t+1}) differently. Because they have different wealth levels, and their true consumption is the sum of claimed goods and interest payments from wealth, the reduction in terms of utility is inversely proportional to the wealth level. The richer the agent is, the less he suffers from this transition. Thus the equilibrium type suffers the most, while the all-shirking type suffers the least and has a comparative advantage in taking this deviation. Therefore, if we know that the all-shirking type would not take this one-step deviation, then the other types are not willing either, since they gain even less.

We call the above story an *ex post* story, because it is as if the agent considered an ex post deviation, given the optimal saving rules when deviation was not anticipated. Of course, in this kind of problem, deviation combined with revised savings is typically optimal. An agent will increase his savings if he expects to deviate in the future. However, to understand why the logic works in spite of this, consider instead a small reduction in effort. Since all the types have saved appropriate amounts to satisfy their intertemporal Euler equations, the envelope theorem tells us that, for such changes, the benefit to revised savings is second order, and the transition from the employment state to the unemployment state is first order and is the dominant force; we can therefore always say that the all-shirking type has a comparative advantage at shirking by a small amount. As a result, it must be the case that the type that always shirks finds additional shirking most attractive.

In the next section we return, then, to the relaxed problem to develop some details of the optimal contract.

5 Analysis of the Optimal Contract

5.1 Decline of consumption over an unemployment spell

We can analyze how consumption changes over an unemployment spell by looking at the constraints in the relaxed problem. We have that

Proposition 2 *U_t/U_{t-1} is decreasing in t, and*

$$\lim_{t\to\infty} \frac{U_t}{U_{t-1}} = \lim_{t\to\infty} \exp(\gamma x_t) = \exp\left(\frac{(1-\beta)}{(-\bar{V})\beta(1-\beta(1-\pi))}\right) > 1. \tag{12}$$

Since U_t is proportional to the negative of utility from consumption, utility from consumption is falling at a decreasing rate, eventually reaching a constant rate of decline. In order to achieve this, consumption must fall by a constant amount in the long run.

A useful comparison is to the case of observable savings.

Proposition 3 *If savings are observed, as in Hopenhayn and Nicolini (1997), then*

$$\lim_{t\to\infty} (U_t - U_{t-1}) = \frac{(1-\beta)^2}{\beta(1-\beta(1-\pi))}. \tag{13}$$

In the observable savings case, the long-run decline in *utility* from consumption is a constant amount, implying slower and slower declines in *consumption* in the long run. This shows how unobservable savings can make an important difference in the optimal policy. When savings are observed, consumption is falling more and more slowly (because, in the long run, high marginal utility makes small consumption declines translate into large utility declines), but when consumption is unobserved, it cannot, since the prospective shirker has more and more hidden wealth.

Another interesting difference between the two cases is that the eventual decline in consumption depends on the initial promised utility when savings are unobserved. This difference gives rise to rank reversals in consumption levels over the spell for agents who start with different promised utilities. It is well known that in the case of observable savings, a larger promised $\bar{V}$ implies higher consumption at every node. That is not the case with unobserved savings. An agent with higher promised utility gets greater consumption at the start of the spell, but that extra consumption makes saving more tempting. In order to undo the temptation to shirk and save, the rate of consumption decline must be faster for the agent who consumes more initially; eventually his consumption is less than an agent with lower initial promised utility.

Proposition 4 *Consider two initial promised utility levels $\bar{V}^a > \bar{V}^b$. Then consumption is initially higher under $\bar{V}^a$ but is eventually lower forever.*

One interesting fact worth mentioning is that, for large t, although the equilibrium type from $\bar{V}^a$ is poorer than that from $\bar{V}^b$, the all-shirking-up-to-t type from $\bar{V}^a$ is always richer. In other words, our results are consistent with the monotonicity in Hopenhayn and Nicolini (1997): monotonicity is maintained for the wealth levels that are most likely to consider one deviation, and whose incentive constraints bind. In Hopenhayn and Nicolini (1997), there is only one wealth level, that of zero wealth, and the agent is indifferent between shirking and giving effort at $t+1$. Here the critical wealth level belongs to the all-shirking-up-to-t type, who is ex ante indifferent between his strategy and one more time period of shirking. Notice that, since an all-shirking-up-to-t type from $\bar{V}^a$ always obtains an ex ante utility equivalent to the equilibrium type, he has secretly saved so much wealth so that even if he faces lower future consumption claims, he is able to live better than an all-shirking-up-to-t type from $\bar{V}^b$.

5.2 Benefits over the spell

At the beginning of period t, we consider two types of agents, one who finds a job at period t, and the other who finds a job at $t+1$. Relative to the first type, the second type has an additional bad unemployment shock. We compare the discounted value of net subsidy to these two types of agents, as is done in Shimer and Werning (2005). The subsidy to the first type is $(c_t^E - w)/(1-\beta)$, while it is $c_t^U + \beta(c_{t+1}^E - w)/(1-\beta)$ for the second type. The subsidy to the second type is generally higher, and the difference is

$$c_t^U + \beta\frac{c_{t+1}^E - w}{1-\beta} - \frac{c_t^E - w}{1-\beta}.$$

We can think of the above as the insurance for the unemployment risk that an agent faces at period t. It is well known that in Shimer and Werning (2005), this is a constant. In our model, it is strictly increasing in t.

Proposition 5 $c_t^U + \beta \frac{c_{t+1}^E - w}{1-\beta} - \frac{c_t^E - w}{1-\beta}$ *is strictly increasing in* t.

There is a sense, then, that the subsidy increases over the spell. Although many transfer sequences, combined with asset markets, can achieve the optimum, we follow Shimer and Werning (2005) and pin down our benefit sequence by having an employment tax of zero. Since, for benefits b_t, the optimal savings problem of the agent implies

$$c_t^U + \frac{\beta c_t^E}{1-\beta} - \frac{c_{t+1}^E}{1-\beta} = w - b_t, \text{ for all } t,$$

we have immediately that the benefits are increasing.

Intuitively, the rising benefit sequence is related to the rising benefit sequence that Shavell and Weiss (1979) find for the case where job-finding probabilities are exogenous but borrowing and lending are allowed. There, the single force is consumption smoothing: higher marginal utilities in later periods of the unemployment spell dictate higher shifting benefits to those states. In their model the outcome is a very extreme jump from zero benefits to full benefits (equal to the wage) at some two-period interval. Here, however, due to moral hazard, a wedge between consumption when employed and unemployed must always be maintained.

6 Conclusion

Our paper is a first step toward solving the mechanism design problems with hidden savings. We have shown a case under which hidden savings (and, without loss of generality, borrowing) leads to faster utility declines over an unemployment spell than for the contract in Hopenhayn and Nicolini (1997). This result contrasts with the hidden savings model of Kocherlakota (2004), where the rate of decline of utility over a spell of unemployment is zero. Further, our contract has the feature that an agent with an initially higher consumption claim, due to higher promised utility, may eventually have a lower claim than the agent with initially low consumption. This non-monotonicity comes from the nature of the structure of the binding incentive constraints.

Our mode of attack relies on determining the binding incentive constraints. We use the notion that types who shirk also want to save. That motive is central to kocherlakota's argument about why first-order approaches to these problems might fail. We exploit this feature to prove that the type that shirks the most, and saves the most, will be the hardest agent from which to get effort. Therefore that type has the binding incentive constraints. While the particular results depend on our functional form assumptions, we think the result points to what can happen in general when non-implemented types have the binding constraints. The double deviation incentives make such a structure of binding constraints natural and lead to results that differ from those in the literature. In particular, the principal must fight the possibility of shirking and saving by making the rate of consumption decline more dramatic. This is exaggerated further for agents who get high initial claims and therefore may be most prone to saving.

The results here suggest that the richest type plays a role in determining the optimal contract, since it is that richest type whose incentive constraint binds. If this type of problem could be studied recursively, an artificial type would need to be added to the state space in addition to the promised utility of the equilibrium type. Further development of this idea is the task of future research.

7 Appendix

Proof of Lemma 1: We first show that if $a_t = 1$, then $U_{t+1} > E_{t+1}$. Let B_{t+1} denote the discounted (to period $t+1$) utility of consumption conditional on not finding a job at the beginning of $t+1$,

$$B_{t+1} = \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} \left(\Pi_{k=t+1}^{s-1}(1-a_k\pi)\right) \left[-U_s + \beta a_s \pi \frac{-E_{s+1}}{1-\beta}\right].$$

Recall that D_{t+1}/β^{t+1} is the discounted (to period $t+1$) disutility conditional on not finding a job at the beginning of $t+1$. Equation (1) implies that $B_{t+1} \geq -U_{t+1}/(1-\beta)$. The I.C. condition requires that an agent's benefit of using effort outweighs the cost,

$$\begin{aligned} 1 &\leq \beta\pi\left(\frac{-E_{t+1}}{1-\beta} - \left(B_{t+1} - \frac{D_{t+1}}{\beta^{t+1}}\right)\right) \\ &\leq \beta\pi\left(\frac{-E_{t+1}}{1-\beta} - \left(\frac{-U_{t+1}}{1-\beta} - \frac{D_{t+1}}{\beta^{t+1}}\right)\right) \\ &\leq \beta\pi\frac{U_{t+1}-E_{t+1}}{1-\beta} + \frac{\beta\pi}{1-\beta+\beta\pi} \\ &< \beta\pi\frac{U_{t+1}-E_{t+1}}{1-\beta} + 1, \end{aligned}$$

where the third inequality follows from that $D_{t+1}/\beta^{t+1} \leq 1/(1-\beta+\beta\pi)$. Thus $U_{t+1} > E_{t+1}$.

Second, we show that if $a_t = 1$, then $U_t = \pi E_{t+1} + (1-\pi)U_{t+1}$. By contradiction, suppose

$$U_t > \pi E_{t+1} + (1-\pi)U_{t+1}. \tag{14}$$

For $\epsilon > 0$, there is a unique $\delta(\epsilon) > 0$, such that

$$\begin{aligned} & u(c_t^U + \epsilon) + \beta\left[\pi u(c_{t+1}^E - \delta(\epsilon)) + (1-\pi)u(c_{t+1}^U - \delta(\epsilon))\right] \\ = \; & -U_t + \beta\left[\pi(-E_{t+1}) + (1-\pi)(-U_{t+1})\right]. \end{aligned} \tag{15}$$

Thus $\delta'(\epsilon) = u'(c_t^U + \epsilon)/\beta[\pi u'(c_{t+1}^E - \delta(\epsilon)) + (1-\pi)u'(c_{t+1}^U - \delta(\epsilon))]$. Equation (14) implies that there is a small $\bar{\epsilon} > 0$, such that $\delta'(\epsilon) > 1/\beta$, for all $\epsilon \in [0, \bar{\epsilon}]$. Modify the contract by choosing $\tilde{c}_t^U = c_t^U + \bar{\epsilon}$, $\tilde{c}_{t+1}^E = c_{t+1}^E - \delta(\bar{\epsilon})$, $\tilde{c}_{t+1}^U = c_{t+1}^U - \delta(\bar{\epsilon})$. After the modification,

(i) the principal saves resources, because $\bar{\epsilon} < \beta\delta(\bar{\epsilon})$;

(ii) the equilibrium type would not secretly save, and his promised utility is unchanged, given the definition of $\delta(\epsilon)$;

(iii) all the deviators are (weakly) worse off. At the beginning of t, consider a deviator who starts with savings $s_{t-1}/\beta \geq 0$, and exerts effort $\{\tilde{a}_s\}_{s=t}^{\infty}$. Let B_1 and B_2 be the deviator's utility from consumption before and after the modification, respectively.

$$
\begin{aligned}
B_1 &= \max_{s_t \geq 0} \left\{ u(s_{t-1}/\beta + c_t^U - s_t) + \beta[\tilde{a}_t \pi v^E(s_t/\beta + c_{t+1}^E) + (1-\tilde{a}_t\pi) v^U(s_t/\beta + c_{t+1}^U)] \right\}, \\
B_2 &= \max_{s_t \geq 0} \{ u(s_{t-1}/\beta + c_t^U + \bar{\epsilon} - s_t) + \beta[\tilde{a}_t \pi v^E(s_t/\beta + c_{t+1}^E - \delta(\bar{\epsilon})) \\
&\quad +(1-\tilde{a}_t\pi) v^U(s_t/\beta + c_{t+1}^U - \delta(\bar{\epsilon}))]\},
\end{aligned}
$$

where $v^E(\cdot), v^U(\cdot)$ are the deviator's value functions starting from period $t+1$. Then

$$
\begin{aligned}
B_2 &= B_1 + \int_0^{\bar{\epsilon}} \{ u'(s_{t-1}/\beta + c_t^U + \epsilon - s_t(\epsilon)) - \beta[\tilde{a}_t \pi (v^E)'(s_t(\epsilon)/\beta + c_{t+1}^E - \delta(\epsilon)) \\
&\quad +(1-\tilde{a}_t\pi)(v^U)'(s_t(\epsilon)/\beta + c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} d\epsilon, \qquad (16)
\end{aligned}
$$

where $s_t(\epsilon)$ is the optimal savings in the maximization problem. If $s_t(\epsilon) > 0$, then

$$
\begin{aligned}
& u'(s_{t-1}/\beta + c_t^U + \epsilon - s_t(\epsilon)) \\
= \ & [\tilde{a}_t \pi (v^E)'(s_t(\epsilon)/\beta + c_{t+1}^E - \delta(\epsilon)) + (1-\tilde{a}_t\pi)(v^U)'(s_t(\epsilon)/\beta + c_{t+1}^U - \delta(\epsilon))], \text{ thus} \\
& \{ u'(s_{t-1}/\beta + c_t^U + \epsilon - s_t(\epsilon)) - \beta[\tilde{a}_t \pi (v^E)'(s_t(\epsilon)/\beta + c_{t+1}^E - \delta(\epsilon)) \\
& +(1-\tilde{a}_t\pi)(v^U)'(s_t(\epsilon)/\beta + c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} \\
= \ & u'(s_{t-1}/\beta + c_t^U + \epsilon - s_t(\epsilon))(1 - \beta\delta'(\epsilon)) \\
< \ & 0.
\end{aligned}
$$

If $s_t(\epsilon) = 0$, then

$$
\begin{aligned}
& \{ u'(s_{t-1}/\beta + c_t^U + \epsilon) - \beta[\tilde{a}_t \pi (v^E)'(c_{t+1}^E - \delta(\epsilon)) + (1-\tilde{a}_t\pi)(v^U)'(c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} \\
\leq \ & \{ u'(s_{t-1}/\beta + c_t^U + \epsilon) - \beta[\tilde{a}_t \pi u'(c_{t+1}^E - \delta(\epsilon)) + (1-\tilde{a}_t\pi)u'(c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} \\
\leq \ & \{ u'(s_{t-1}/\beta + c_t^U + \epsilon) - \beta[\pi u'(c_{t+1}^E - \delta(\epsilon)) + (1-\pi)u'(c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} \\
\leq \ & \{ u'(c_t^U + \epsilon) - \beta[\pi u'(c_{t+1}^E - \delta(\epsilon)) + (1-\pi)u'(c_{t+1}^U - \delta(\epsilon))]\delta'(\epsilon)\} \\
= \ & 0,
\end{aligned}
$$

where the first inequality follows from the deviator's non-negative savings (and thus having higher marginal utility of consumption than the equilibrium type) at $t+1$, the second inequality follows from $\tilde{a}_t \leq 1$ and $c_{t+1}^E > c_{t+1}^U$, as shown in the first part of this proof, and the third inequality follows from $s_{t-1}/\beta \geq 0$. Therefore, equation (16) implies $B_2 \leq B_1$.

To summarize, if inequality (14) holds, then the principal could modify σ to save resources without violating any incentive constraints, which contradicts the optimality of σ. Using a similar argument, we can show that if $a_t = 0$, then $U_t = U_{t+1}$.

Third, the shirker is not savings-constrained as well. Since $U_t = \pi E_{t+1} + (1-\pi)U_{t+1}$ and $E_{t+1} < U_{t+1}$, shirking ($\tilde{a}_t < 1$) implies

$$U_t < \tilde{a}_t \pi E_{t+1} + (1 - \tilde{a}_t \pi) U_{t+1}.$$

Therefore, the shirker always secretly saves and chooses his consumption to restore the Euler equation. *Q.E.D.*

Proof of Lemma 3: The proof follows from **Lemma 3.1-3.5** in the following. Let $\{a_t\}_{t=0}^{\infty}$, $a_t \in \{0,1\}$, be an effort sequence that the principal wants to implement. And for $s > 0$, let $(\{a_t\}_{t=0}^{s-1}, \{1\}_{t=s}^{\infty})$ denote a modified sequence where the effort starting from s is always 1. $\sigma^*(\{a_t\}_{t=0}^{\infty})$ (the solution to a relaxed problem (20) in the following) can be approximated by $\sigma^*(\{a_t\}_{t=0}^{s-1}, \{1\}_{t=s}^{\infty})$ arbitrarily well, if s is large enough. Lemma 3.5 states that for all s, $\sigma^*(\{a_t\}_{t=0}^{s-1}, \{1\}_{t=s}^{\infty})$ is dominated by lotteries with two outcomes, thus $\sigma^*(\{a_t\}_{t=0}^{\infty})$ is also dominated by a lottery with two outcomes. Since the optimal incentive compatible contract is always dominated by $\sigma^*(\{a_t\}_{t=0}^{\infty})$, we obtain the result. *Q.E.D.*

Let $\{a_t\}_{t=0}^{\infty}$, $a_t \in \{0,1\}$, be an effort sequence that the principal wants to implement. Let n_t ($t \geq 1$) denote the total number of time periods when high effort is recommended before t, i.e., $n_t = \#\{s : a_s = 1, 0 \leq s \leq t-1\}$. Define $x_t = c_{t-1}^U - c_t^U = \log(U_t/U_{t-1})/\gamma$. Then $c_t^U = c_0^U - \sum_{s=1}^{t} x_s$. If $a_t = 1$, then equation (3) yields

$$\begin{aligned} c_{t+1}^E &= c_t^U - \frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x_{t+1})}{\pi}\right) \\ &= c_0^U - \sum_{s=1}^{t} x_s - \frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x_{t+1})}{\pi}\right); \end{aligned}$$

while if $a_t = 0$, equation (3) implies that $x_{t+1} = 0$, and c_{t+1}^E does not need to be specified.

The cost function $C(\sigma)$ can be decomposed as

$$
\begin{aligned}
C(\sigma) &= \sum_{t=0}^{\infty} \beta^t \left(\Pi_{s=0}^{t-1}(1-a_s\pi)\right) \left[c_t^U + \beta a_t \pi \frac{c_{t+1}^E}{1-\beta}\right] - \frac{\beta\pi w}{1-\beta} D_0 \\
&= \sum_{t=0}^{\infty} \beta^t \left(\Pi_{s=0}^{t-1}(1-a_s\pi)\right) \left[c_0^U - \sum_{s=1}^{t} x_s + \beta a_t \pi \frac{c_0^U - \sum_{s=1}^{t} x_s - \frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x_{t+1})}{\pi}\right)}{1-\beta}\right] \\
&\quad - \frac{\beta\pi w}{1-\beta} D_0 \\
&= \frac{c_0^U}{1-\beta} - \sum_{t=1}^{\infty} \frac{\beta^t}{1-\beta}(1-\pi)^{n_t} \left[(1-a_{t-1}\pi)x_t + a_{t-1}\pi \frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x_t)}{\pi}\right)\right] \\
&\quad - \frac{\beta\pi w}{1-\beta} D_0 \\
&= \frac{c_0^U}{1-\beta} - \sum_{t=1}^{\infty} \frac{\beta^t}{1-\beta}(1-\pi)^{n_t} \chi_{\{a_{t-1}=1\}} \left[(1-\pi)x_t + \pi \frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x_t)}{\pi}\right)\right] \\
&\quad - \frac{\beta\pi w}{1-\beta} D_0,
\end{aligned}
$$

where χ is the indicator function. Define a function $f(x) = -(1-\pi)x - \pi\frac{1}{\gamma}\log\left(\frac{1-(1-\pi)\exp(\gamma x)}{\pi}\right)$. Since $f(0) = 0$, $f'(x) = -1 + \frac{\pi}{1-(1-\pi)\exp(\gamma x)}$, $f'(0) = 0$, $f''(x) = \pi(1-\pi)\gamma(1-(1-\pi)\exp(\gamma x))^{-2}\exp(\gamma x) > 0$, we see that f is strictly convex and has a unique minimizer at $x = 0$. Thus

$$
C(\sigma) = \frac{c_0^U}{1-\beta} + \sum_{t=1}^{\infty} \frac{\beta^t}{1-\beta}(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}} f(x_t) - \frac{\beta\pi w}{1-\beta} D_0. \tag{17}
$$

Next consider a shirker who shirks in all periods up to a given t ($\tilde{a}_s = 0$, for all $0 \le s \le t-1$), and follows recommended efforts from then on ($\tilde{a}_s = a_s$, for all $s \ge t$). Using the same proof as in Lemma 4, we obtain

$$
V(\{\tilde{a}_s\}_{s=0}^{\infty}) = -\frac{\left(\Pi_{s=0}^{t-1} U_s^{(1-\beta)\beta^s}\right) U_t^{\beta^t}}{1-\beta} - D_t. \tag{18}
$$

We study a relaxed problem where only shirkers in the above are considered,

$$
\begin{aligned}
\min_{\sigma} \quad & C(\sigma) \\
s.t. \quad & U_t = \pi E_{t+1} + (1-\pi) U_{t+1}, \text{ if } a_t = 1, \\
& U_t = U_{t+1}, \text{ if } a_t = 0,
\end{aligned}
$$

$$\bar{V} = -\frac{U_0}{1-\beta} - D_0,$$

$$\bar{V} \geq -\frac{\left(\Pi_{s=0}^{t-1} U_s^{(1-\beta)\beta^s}\right) U_t^{\beta^t}}{1-\beta} - D_t, \text{ for all } t \geq 1.$$

Denote the optimal solution to the above by $\sigma^*(\{a_t\}_{t=0}^{\infty})$. The incentive constraint $-\frac{U_0}{1-\beta} - D_0 \geq -\frac{\left(\Pi_{s=0}^{t-1} U_s^{(1-\beta)\beta^s}\right) U_t^{\beta^t}}{1-\beta} - D_t$ is equivalent to

$$\sum_{s=1}^{t} \beta^s x_s \quad \geq \quad \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_t\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right). \tag{19}$$

Using equations (17) and (19), the relaxed problem is

$$\min_{\{x_t\}_{t=1}^{\infty}} \quad \sum_{t=1}^{\infty} \frac{\beta^t}{1-\beta} (1-\pi)^{n_t} \chi_{\{a_{t-1}=1\}} f(x_t) \tag{20}$$

$$s.t. \quad \sum_{s=1}^{t} \beta^s x_s \geq \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_t\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right), \text{ for all } t \geq 1, \tag{21}$$

$$x_t = 0, \text{ if } a_{t-1} = 1. \tag{22}$$

Lemma 3.1 *For $s < t$, if $a_s = 1$ and $a_t = 1$ ($D_s > D_{s+1} \geq D_t > D_{t+1}$), and $\frac{D_s - D_{s+1}}{\beta^{s+1}} \geq \frac{D_t - D_{t+1}}{\beta^{t+1}}$, then*

$$\frac{\frac{1}{\gamma}\log(\frac{U_0}{1-\beta} + D_0 - D_{s+1}) - \frac{1}{\gamma}\log(\frac{U_0}{1-\beta} + D_0 - D_s)}{\beta^{s+1}}$$
$$> \quad \frac{\frac{1}{\gamma}\log(\frac{U_0}{1-\beta} + D_0 - D_{t+1}) - \frac{1}{\gamma}\log(\frac{U_0}{1-\beta} + D_0 - D_t)}{\beta^{t+1}}.$$

Proof of Lemma 3.1: This follows from the strict concavity of function $\log(\cdot)$. *Q.E.D.*

Lemma 3.2 *If there is an $s \geq 0$, such that $a_t = 1$, for all $t \geq s$, then $\sigma^*(\{a_t\}_{t=0}^{\infty})$ satisfies*

$$\sum_{i=1}^{s} \beta^i x_i \quad = \quad \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right).$$

Proof of Lemma 3.2: In the optimal solution, $x_t \geq 0$, for all $t \geq 1$. We also know that

$$\sum_{t=1}^{\infty} \beta^t x_t = \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right). \tag{23}$$

By contradiction, suppose

$$\sum_{i=1}^{s} \beta^i x_i > \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right). \tag{24}$$

Then we will show that

$$\beta^{s+1} x_{s+1} > \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s+1}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right). \tag{25}$$

To see this, let s^* ($1 \le s^* \le s$) be the time index such that

$$\sum_{i=1}^{s^*-1} \beta^i x_i = \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s^*-1}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right), \tag{26}$$

$$\sum_{i=1}^{t} \beta^i x_i > \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_t\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right), s^* \le t \le s. \tag{27}$$

Therefore, $\beta^{s^*} x_{s^*} > \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s^*}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s^*-1}\right)$. For $\epsilon \neq 0$, changing x_{s^*} by ϵ/β^{s^*} and x_{s+1} by $-\epsilon/\beta^{s+1}$ will keep $\beta^{s^*} x_{s^*} + \beta^{s+1} x_{s+1}$ unchanged. Equations in (27) imply that no constraints in (21) will be violated after the change if ϵ is sufficiently small. Thus the first-order condition for a minimum at $\epsilon = 0$ is

$$(1-\pi)^{n_{s^*}} f'(x_{s^*}) + (1-\pi)^{n_{s+1}} f'(x_{s+1}) = 0,$$

which, with the convexity of f, implies that $x_{s+1} > x_{s^*}$. Thus

$$\begin{aligned} x_{s+1} &> x_{s^*} \\ &> \frac{\frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s^*}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s^*-1}\right)}{\beta^{s^*}} \\ &> \frac{\frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s+1}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right)}{\beta^{s+1}}, \end{aligned}$$

where the last inequality follows from $(D_{s^*-1} - D_{s^*})/\beta^{s^*} = (\beta^{s^*-1} + (1-\pi)D_{s^*} - D_{s^*})/\beta^{s^*} \ge (1/\beta - 1)/(1 - \beta(1-\pi)) = (D_s - D_{s+1})/\beta^{s+1}$, and Lemma 3.1. Equation (25) is proved. Therefore equations (24) and (25) imply

$$\sum_{i=1}^{s+1} \beta^i x_i > \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s+1}\right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta}\right).$$

Thus, using the same proof we used for equation (25) and by induction, we obtain

$$\beta^{t+1}x_{t+1} > \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}+D_0-D_{t+1}\right)-\frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}+D_0-D_t\right), \text{ for all } t\geq s,$$

which contradicts (23). *Q.E.D.*

Lemma 3.3 *If there is an $s\geq 0$, such that $a_t=1$, for all $t\geq s$, then $\sigma^*(\{a_t\}_{t=0}^{\infty})$ satisfies, for all $t\geq s+1$,*

$$\beta^t x_t \quad = \quad \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}+D_0-D_t\right)-\frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}+D_0-D_{t-1}\right). \tag{28}$$

Thus x_t is strictly decreasing in t when $t\geq s+1$.

Proof of Lemma 3.3: For any $s'\geq s$, the conditions in Lemma 3.1 ($a_t=1$, for all $t\geq s'$) are satisfied, which implies

$$\sum_{i=1}^{s'}\beta^i x_i \quad = \quad \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}+D_0-D_{s'}\right)-\frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}\right), \text{ for all } s'\geq s,$$

which implies (28). When $s\geq t+1$, $(D_{t-1}-D_t)/\beta^t=(\beta^{t-1}/(1-\beta(1-\pi))-\beta^t/(1-\beta(1-\pi)))/\beta^t=(1/\beta-1)/(1-\beta(1-\pi))$, thus it follows from equation (28) and Lemma 3.1 that x_t is strictly decreasing in t, when $t\geq s+1$. *Q.E.D.*

Lemma 3.4 *If there is an $s\geq 0$, such that $a_t=1$, for all $t\geq s$, then $\sigma^*(\{a_t\}_{t=0}^{\infty})$ satisfies*

$$\frac{\sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}f(x_i)}{\sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}} > f(x_{s+1}), \text{ when } \sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}>0.$$

Proof of Lemma 3.4: Define $\tilde{x}_t=(\log(U_0/(1-\beta)+D_0-D_t)-\log(U_0/(1-\beta)+D_0-D_{t-1}))/(\gamma\beta^t)$, for $t\leq s$. If $a_{t-1}=0$, then $\tilde{x}_t=0$; otherwise, Lemma 3.1 implies $\tilde{x}_t>x_{s+1}$, for $t\leq s$. Equation (21) implies

$$\sum_{i=1}^{t}\beta^i x_i \quad\geq\quad \sum_{i=1}^{t}\beta^i\tilde{x}_i, \text{ for all } t\leq s,$$

which yields

$$\sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}x_i\geq\sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}\tilde{x}_i>\left(\sum_{i=1}^{s}\beta^i(1-\pi)^{n_i}\chi_{\{a_{i-1}=1\}}\right)x_{s+1}.$$

Therefore, the convexity of f and $f'(x)$ being positive (when $x > 0$) imply

$$\frac{\sum_{i=1}^{s} \beta^i (1-\pi)^{n_i} \chi_{\{a_{i-1}=1\}} f(x_i)}{\sum_{i=1}^{s} \beta^i (1-\pi)^{n_i} \chi_{\{a_{i-1}=1\}}} \geq f\left(\frac{\sum_{i=1}^{s} \beta^i (1-\pi)^{n_i} \chi_{\{a_{i-1}=1\}} x_i}{\sum_{i=1}^{s} \beta^i (1-\pi)^{n_i} \chi_{\{a_{i-1}=1\}}} \right) > f(x_{s+1}).$$

Q.E.D.

Lemma 3.5 *If there is an $s \geq 1$, such that $a_{s-1} = 0$ and $a_t = 1$, for all $t \geq s$, then $\sigma^*(\{a_t\}_{t=0}^{\infty})$ is dominated by a lottery with two outcomes: either $\sigma^*(\{1\}_{t=0}^{\infty})$, or with the complementary probability, $\sigma^*(\{0\}_{t=0}^{\infty})$. Notice that $\sigma^*(\{0\}_{t=0}^{\infty})$ is trivially incentive compatible and by Lemma 5 and Lemma 6, $\sigma^*(\{1\}_{t=0}^{\infty})$ satisfies equation (5) as well.*

Proof of Lemma 3.5: Lemma 3.2 and 3.3 yield that

$$\begin{aligned}
\sum_{i=1}^{s-1} \beta^i x_i &= \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s-1} \right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} \right), \\
x_s &= 0, \\
\beta^t x_t &= \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_t \right) - \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{t-1} \right), \text{ for all } t \geq s+1.
\end{aligned}$$

Now construct a new contract $\tilde{\sigma}$ to implement effort sequence $\{\tilde{a}_t\}_{t=0}^{\infty}$, where

$$\begin{aligned}
\tilde{a}_t &= a_t, \text{ if } t < s-1, \\
\tilde{a}_t &= 1, \text{ if } t \geq s-1.
\end{aligned}$$

This implies that $\{\tilde{D}_t\}_{t=0}^{\infty}$ is

$$\begin{aligned}
\tilde{D}_0 &= D_0 + (1-\pi)^{n_{s-1}} \frac{\beta^{s-1} - \beta^s}{1-\beta(1-\pi)}, \\
\tilde{D}_t &= D_t + (1-\pi)^{n_{s-1}-n_t} \frac{\beta^{s-1} - \beta^s}{1-\beta(1-\pi)}, \text{ for all } t < s-1, \\
\tilde{D}_t &= \frac{\beta^t}{1-\beta(1-\pi)}, \text{ for all } t \geq s-1.
\end{aligned}$$

Define $\{\tilde{x}_t\}_{t=0}^{\infty}$ by

$$\begin{aligned}
\tilde{x}_t &= x_t, \text{ for all } t \leq s-1, \\
\tilde{x}_t &= x_{t+1}, \text{ for all } t \geq s.
\end{aligned}$$

We will verify that $\{\tilde{x}_t\}_{t=0}^{\infty}$ satisfies (21) in the relaxed problem where $\{\tilde{a}_t\}_{t=0}^{\infty}$ is implemented. For $t \leq s-1$,

$$\begin{aligned}
\sum_{i=1}^{t} \beta^i \tilde{x}_i = \sum_{i=1}^{t} \beta^i x_i &\geq \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_t\right) - \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}\right) \\
&\geq \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + \tilde{D}_0 - \tilde{D}_t\right) - \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}\right),
\end{aligned}$$

where the last inequality follows from $\tilde{D}_0 - \tilde{D}_t \leq D_0 - D_t$. For $t \geq s$, we need to show

$$\begin{aligned}
\sum_{i=1}^{s-1} \beta^i x_i + \sum_{i=s}^{t} \beta^i \tilde{x}_i &= \sum_{i=1}^{s-1} \beta^i x_i + \sum_{i=s}^{t} \beta^i x_{i+1} \\
&= \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{s-1}\right) - \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}\right) + \\
&\quad \frac{1}{\beta}\left(\frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + D_0 - D_{t+1}\right) - \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right)\right), \\
&\geq \frac{1}{\gamma} \log\left(\frac{U_0}{1-\beta} + \tilde{D}_0 - \tilde{D}_t\right) - \frac{1}{\gamma}\log\left(\frac{U_0}{1-\beta}\right).
\end{aligned}$$

Since $a_{s-1} = 0, D_s = D_{s-1}$, it is equivalent to show that

$$\log\left(\frac{U_0}{1-\beta} + D_0 - D_{t+1}\right) \geq (1-\beta)\log\left(\frac{U_0}{1-\beta} + D_0 - D_s\right) + \beta \log\left(\frac{U_0}{1-\beta} + \tilde{D}_0 - \tilde{D}_t\right),$$

which follows from the concavity of $\log(\cdot)$ and

$$\begin{aligned}
D_0 - D_{t+1} &\geq D_0 - \beta \tilde{D}_t \\
&\geq D_0 - \beta\tilde{D}_t - (1-\beta)D_s + (1-\pi)^{n_{s-1}} \frac{(1-\beta)\beta^s}{1-\beta(1-\pi)} \\
&= (1-\beta)(D_0 - D_s) + \beta\left(D_0 + (1-\pi)^{n_{s-1}} \frac{\beta^{s-1} - \beta^s}{1-\beta(1-\pi)} - \tilde{D}_t\right) \\
&= (1-\beta)(D_0 - D_s) + \beta(\tilde{D}_0 - \tilde{D}_t),
\end{aligned}$$

where the second inequality follows from $D_s = \beta^s/(1-\beta(1-\pi))$. Thus $\tilde{\sigma}$ satisfies the I.C. constraints in the relaxed problem.

Contract $\tilde{\sigma}$ is more efficient than $\sigma^*(\{a_t\}_{t=0}^{\infty})$ in the sense that $\tilde{\sigma}$ lowers the cost of distortion per unit of wage income generated,

$$\begin{aligned}
& \frac{\sum_{t=1}^{\infty}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}f(x_t)}{\sum_{t=1}^{\infty}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}} \\
= & \frac{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}f(x_t)+\sum_{t=s+1}^{\infty}\beta^t(1-\pi)^{n_t}f(x_t)}{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}+\sum_{t=s+1}^{\infty}\beta^t(1-\pi)^{n_t}} \\
> & \frac{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}f(x_t)+\sum_{t=s+1}^{\infty}\beta^{t-1}(1-\pi)^{n_t}f(x_t)}{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}+\sum_{t=s+1}^{\infty}\beta^{t-1}(1-\pi)^{n_t}} \\
= & \frac{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}f(x_t)+\sum_{t=s}^{\infty}\beta^t(1-\pi)^{\tilde{n}_t}f(\tilde{x}_t)}{\sum_{t=1}^{s-1}\beta^t(1-\pi)^{n_t}\chi_{\{a_{t-1}=1\}}+\sum_{t=s}^{\infty}\beta^t(1-\pi)^{\tilde{n}_t}},
\end{aligned}$$

where the inequality follows from Lemma 3.3. Hence $\sigma^*(\{a_t\}_{t=0}^{\infty})$ is dominated by a lottery with two outcomes: with probability $D_0/\tilde{D}_0$, the principal uses contract $\tilde{\sigma}$ to implement efforts $\{\tilde{a}_t\}_{t=0}^{\infty}$ and deliver promised utility $-U_0/(1-\beta)-\tilde{D}_0$; and with probability $(1-D_0/\tilde{D}_0)$, the principal implements $\{0\}_{t=0}^{\infty}$ and delivers utility $-U_0/(1-\beta)$. This lottery provides the same ex ante utility, but lowers the cost function $C(\sigma)$.

If there are still zero efforts remaining in $\{\tilde{a}_t\}_{t=0}^{\infty}$, we could repeat the above procedure and move forward the tail sequence of full efforts one step further. Eventually the first outcome in the lottery will specify $a_t = 1$, for all $t \geq 0$. *Q.E.D.*

Proof of Lemma 4: Denote the agent's savings by $\{s_t^U\}_{t=0}^{\infty}$. At the beginning of period $\bar{t}$, since he will follow strategy $\{a_t = 1\}_{t=\bar{t}}^{\infty}$, the optimal consumption is the sum of the goods claimed from the principal and the interest payment of $s_{\bar{t}-1}^U$. To see this, notice that $U_t = \pi E_{t+1} + (1-\pi)U_{t+1}$ and CARA utilities imply, for all $t \geq \bar{t}$,

$$u'(c_t^U + (1/\beta - 1)s_{\bar{t}-1}^U) = \pi u'(c_{t+1}^E + (1/\beta-1)s_{\bar{t}-1}^U) + (1-\pi)u'(c_{t+1}^U + (1/\beta-1)s_{\bar{t}-1}^U).$$

An unemployed agent's period $\bar{t}$ value function taking $s_{\bar{t}-1}^U$ as a state variable is

$$V_{\bar{t}}(s_{\bar{t}-1}^U) = -\frac{\exp(-\gamma c_{\bar{t}}^U)}{1-\beta}\exp(-\gamma(1/\beta-1)s_{\bar{t}-1}^U) = -\frac{U_{\bar{t}}}{1-\beta}\exp(-\gamma(1/\beta-1)s_{\bar{t}-1}^U).$$

Now suppose we know an unemployed agent's value function at period $t < \bar{t}$ is $V_t(s_{t-1}^U) = -x\exp(-\gamma(1/\beta-1)s_{t-1}^U)/(1-\beta)$, where x is a parameter. We can calculate the value function $V_{t-1}(s_{t-2}^U)$ as follows. With probability $\tilde{a}_{t-1}\pi$, the agent will find a job next period and has a value function $-E_t \exp(-\gamma(1/\beta-1)s^U_{t-1})/(1-\beta)$, and with probability $(1-\tilde{a}_{t-1}\pi)$, he is still unemployed and has a value function $-x\exp(-\gamma(1/\beta-1)s^U_{t-1})/(1-\beta)$. The agent's optimal savings problem at period $t-1$ is

$$\begin{aligned} V_{t-1}(s^U_{t-2}) &= \max_{c_{t-1},s^U_{t-1}} -\exp(-\gamma c_{t-1}) + \beta[\tilde{a}_{t-1}\pi(-\frac{E_t}{1-\beta}\exp(-\gamma(1/\beta-1)s^U_{t-1})) \\ &\quad +(1-\tilde{a}_{t-1}\pi)(-\frac{x}{1-\beta}\exp(-\gamma(1/\beta-1)s^U_{t-1}))] \\ &\quad s.t. \quad c_{t-1}+s^U_{t-1} = s^U_{t-2}/\beta + c^U_{t-1}. \end{aligned} \tag{29}$$

The first-order condition is

$$\exp(-\gamma c_{t-1}) = [\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)x]\exp(-\gamma(1/\beta-1)s^U_{t-1}). \tag{30}$$

Substituting equation (30) into equation (29) yields $c_{t-1} = (1/\beta-1)s^U_{t-2} + (1-\beta)c^U_{t-1} + \beta\log(\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)x)/(-\gamma)$. Thus

$$\begin{aligned} V_{t-1}(s^U_{t-2}) &= \frac{-\exp(-\gamma c_{t-1})}{1-\beta} \\ &= -\exp(-\gamma(1/\beta-1)s^U_{t-2})(\exp(-\gamma c^U_{t-1}))^{1-\beta}\left[\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)x\right]^{\beta}/(1-\beta) \\ &= -\frac{U^{1-\beta}_{t-1}\left[\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)x\right]^{\beta}}{1-\beta}\exp(-\gamma(1/\beta-1)s^U_{t-2}). \end{aligned}$$

Using the above formula recursively, we have the agent's utility in period $t=0$ (assuming that $s^U_{-1}=0$)

$$\frac{-U_0^{1-\beta}[\tilde{a}_0\pi E_1 + (1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2 + (1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\tilde{a}_{\bar{t}-1}\pi E_{\bar{t}} + (1-\tilde{a}_{\bar{t}-1}\pi)U_{\bar{t}}]^{\beta}...]^{\beta}]^{\beta}]^{\beta}}{1-\beta}.$$

Q.E.D.

Proof of Lemma 5: It is a special case of Lemma 3.2. *Q.E.D.*

Proof of Lemma 6: Consider a finite-deviation strategy $\{\tilde{a}_t\}_{t=0}^{\infty}$, where $\tilde{a}_t = 1, \forall t \geq \bar{t}+1$ for some $\bar{t}$. We first show that setting $\tilde{a}_{\bar{t}} = 1$ and keeping other efforts unchanged will weakly improve an agent's utility, i.e.,

$$\begin{aligned} &-\frac{U_0^{1-\beta}[\tilde{a}_0\pi E_1 + (1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2 + (1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\tilde{a}_{\bar{t}}\pi E_{\bar{t}+1} + (1-\tilde{a}_{\bar{t}}\pi)U_{\bar{t}+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta}}{1-\beta} \\ &-\left[\tilde{a}_0 + \beta(1-\pi\tilde{a}_0)\tilde{a}_1 + ... + \beta^{\bar{t}}\left(\Pi_{s=0}^{\bar{t}-1}(1-\pi\tilde{a}_s)\right)\left(\tilde{a}_{\bar{t}} + \beta(1-\pi\tilde{a}_{\bar{t}})\frac{1}{1-\beta(1-\pi)}\right)\right] \end{aligned}$$

$$
\begin{aligned}
\leq\ & -\frac{U_0^{1-\beta}[\tilde{a}_0\pi E_1+(1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2+(1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\pi E_{\bar{t}+1}+(1-\pi)U_{\bar{t}+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta}}{1-\beta} \\
& -\left[\tilde{a}_0+\beta(1-\pi\tilde{a}_0)\tilde{a}_1+...+\beta^{\bar{t}}\left(\Pi_{s=0}^{\bar{t}-1}(1-\pi\tilde{a}_s)\right)\left(1+\beta(1-\pi)\frac{1}{1-\beta(1-\pi)}\right)\right]. \qquad (31)
\end{aligned}
$$

But Lemma 6.1 in the following states that

$$
\begin{aligned}
& \frac{U_0^{1-\beta}[\tilde{a}_0\pi E_1+(1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2+(1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\tilde{a}_{\bar{t}}\pi E_{\bar{t}+1}+(1-\tilde{a}_{\bar{t}}\pi)U_{\bar{t}+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta}}{1-\beta} \\
& -\frac{U_0^{1-\beta}[\tilde{a}_0\pi E_1+(1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2+(1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\pi E_{\bar{t}+1}+(1-\pi)U_{\bar{t}+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta}}{1-\beta} \\
\geq\ & \left(\Pi_{s=0}^{\bar{t}-1}(1-\pi\tilde{a}_s)\right)(1-\tilde{a}_{\bar{t}})\left[\frac{\left(\Pi_{s=0}^{\bar{t}}U_s^{(1-\beta)\beta^s}\right)U_{\bar{t}+1}^{\beta^{\bar{t}+1}}}{1-\beta}-\frac{\left(\Pi_{s=0}^{\bar{t}-1}U_s^{(1-\beta)\beta^s}\right)U_{\bar{t}}^{\beta^{\bar{t}}}}{1-\beta}\right] \\
\geq\ & \left(\Pi_{s=0}^{\bar{t}-1}(1-\pi\tilde{a}_s)\right)(1-\tilde{a}_{\bar{t}})\frac{\beta^{\bar{t}}-\beta^{\bar{t}+1}}{1-\beta(1-\pi)},
\end{aligned}
$$

which implies equation (31). By backward induction, an agent can further set $\tilde{a}_{\bar{t}-1}=1$ and (weakly) improve his utility. In this way, we prove that setting $\tilde{a}_t=1, \forall t\geq 0$ is the optimal strategy. *Q.E.D.*

Lemma 6.1 *If* $0<U_0<U_1<...<U_{t+1}$ *and* $\tilde{a}_t<1$, *then*

$$
\begin{aligned}
& U_0^{1-\beta}[\tilde{a}_0\pi E_1+(1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2+(1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\tilde{a}_t\pi E_{t+1}+(1-\tilde{a}_t\pi)U_{t+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta} \\
& -U_0^{1-\beta}[\tilde{a}_0\pi E_1+(1-\tilde{a}_0\pi)U_1^{1-\beta}[\tilde{a}_1\pi E_2+(1-\tilde{a}_1\pi)U_2^{1-\beta}[...[\pi E_{t+1}+(1-\pi)U_{t+1}]^{\beta}...]^{\beta}]^{\beta}]^{\beta} \\
\geq\ & \left(\Pi_{s=0}^{t-1}(1-\pi\tilde{a}_s)\right)(1-\tilde{a}_t)\left[\Pi_{s=0}^{t}U_s^{(1-\beta)\beta^s}U_{t+1}^{\beta^{t+1}}-\Pi_{s=0}^{t-1}U_s^{(1-\beta)\beta^s}U_t^{\beta^t}\right].
\end{aligned}
$$

Proof of Lemma 6.1: First we state an inequality that will be used later:

$$
(x+a(z-y))^{\beta}\geq x^{\beta}+a(z^{\beta}-y^{\beta}),\ \text{for all } 0<x\leq y\leq z, 0\leq a\leq 1. \qquad (32)
$$

This can be easily shown by the concavity of function x^{β}. By induction, since $\pi E_{t+1}=U_t-(1-\pi)U_{t+1}$,

$$
\begin{aligned}
& [\tilde{a}_t\pi E_{t+1}+(1-\tilde{a}_t\pi)U_{t+1}]^{\beta} \\
=\ & [\tilde{a}_t(U_t-(1-\pi)U_{t+1})+(1-\tilde{a}_t\pi)U_{t+1}]^{\beta} \\
=\ & [U_t+(1-\tilde{a}_t)(U_{t+1}-U_t)]^{\beta} \\
\geq\ & U_t^{\beta}+(1-\tilde{a}_t)(U_{t+1}^{\beta}-U_t^{\beta}),
\end{aligned}
$$

where the inequality follows from inequality (32). We can proceed to the second step,

$$\begin{aligned}
& [\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)U_t^{1-\beta}(U_t^\beta + (1-\tilde{a}_t)(U_{t+1}^\beta - U_t^\beta))]^\beta \\
\geq\ & [\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)U_t + (1-\tilde{a}_{t-1}\pi)(1-\tilde{a}_t)(U_t^{1-\beta}U_{t+1}^\beta - U_t)]^\beta \\
\geq\ & [\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)U_t]^\beta + (1-\tilde{a}_{t-1}\pi)(1-\tilde{a}_t)(U_t^{(1-\beta)\beta}U_{t+1}^{\beta^2} - U_t^\beta).
\end{aligned}$$

Again, since $\tilde{a}_{t-1}\pi E_t + (1-\tilde{a}_{t-1}\pi)U_t \leq U_t \leq U_{t+1}$, the last inequality can be justified by inequality (32). We can use the same technique and prove the lemma by induction. *Q.E.D.*

Proof of Proposition 2: Lemma 3.3 states

$$\beta^t x_t = \frac{1}{\gamma}\log\left(-\bar{V} - \frac{\beta^t}{1-\beta(1-\pi)}\right) - \frac{1}{\gamma}\log\left(-\bar{V} - \frac{\beta^{t-1}}{1-\beta(1-\pi)}\right).$$

Therefore $\lim_{t\to\infty} x_t = (1-\beta)/(\gamma(-\bar{V})\beta(1-\beta(1-\pi))) > 0$. Recall that $U_t/U_{t-1} = \exp(\gamma x_t)$. *Q.E.D.*

Proof of Proposition 3: In Hopenhayn and Nicolini (1997), if the principal always implements high efforts, then $\{U_t\}_{t=0}^\infty$ is monotonically increasing. We will show

$$\lim_{t\to\infty} U_t = \infty. \tag{33}$$

By contradiction, suppose that for some $B > 0$, $\lim_{t\to\infty} U_t = B$. Inverse Euler equation

$$\frac{1}{U_t} = \frac{\pi}{E_{t+1}} + \frac{1-\pi}{U_{t+1}} \tag{34}$$

implies that $\lim_{t\to\infty} E_t = B$. Let W_t^U be the promised utility for an agent who has not found a job at the beginning of t. Recall that in Hopenhayn and Nicolini (1997), incentive constraints are all binding, implying

$$W_t^U = \sum_{s=t}^\infty \beta^{s-t}U_s \geq -\frac{B}{1-\beta}, \tag{35}$$

which violates the incentive constraint $\beta\pi(-E_t/(1-\beta) - W_t^U) \geq 1$ for exerting effort for large t.

Secondly we show that

$$\limsup_{t\to\infty}(U_{t+1} - U_t) < \infty. \tag{36}$$

Equation (34) and $U_t < U_{t+1}$ imply that $E_t < U_t < U_{t+1}$. Thus

$$\begin{aligned} \frac{U_{t+1}-U_t}{1-\beta} &< \frac{U_{t+1}-E_{t+1}}{1-\beta} < \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} U_s - \frac{E_{t+1}}{1-\beta} \\ &= W^U_{t+1} - \frac{E_{t+1}}{1-\beta} = \frac{1}{\beta\pi}, \end{aligned}$$

where the last equality follows from the binding incentive constraint.

Thirdly, we have

$$W^U_{t+2} - \frac{E_{t+2}}{1-\beta} = \frac{1}{\beta\pi}, \tag{37}$$

$$U_{t+1} + \beta W^U_{t+2} - \frac{E_{t+1}}{1-\beta} = W^U_{t+1} - \frac{E_{t+1}}{1-\beta} = \frac{1}{\beta\pi}. \tag{38}$$

Substituting equation (37) into equation (38) yields

$$(1-\beta)U_{t+1} + \beta E_{t+2} - E_{t+1} = \frac{(1-\beta)^2}{\beta\pi}. \tag{39}$$

Substituting equation (34) into equation (39) yields

$$(U_{t+2} - U_{t+1}) = \frac{A_t(U_{t+1} - U_t) - \frac{(1-\beta)^2}{\beta\pi}}{B_t}, \tag{40}$$

where $A_t = (U_t + (U_{t+1} - U_t))/(\pi U_t + (U_{t+1} - U_t))$, $B_t = (\beta(1-\pi)U_{t+1})/(\pi U_{t+1} + (U_{t+2} - U_{t+1}))$. Rewrite equation (40) as

$$(U_{t+2} - U_{t+1}) - C = \frac{A_t}{B_t}\left[(U_{t+1} - U_t) - C\right] + \frac{(A_t - B_t)C - \frac{(1-\beta)^2}{\beta\pi}}{B_t},$$

where $C = (1-\beta)^2/(\beta(1-\beta(1-\pi)))$. Equations (33) and (36) imply that $\lim_{t\to\infty} A_t = 1/\pi$ and $\lim_{t\to\infty} B_t = \beta(1-\pi)/\pi$. Therefore $\lim_{t\to\infty} A_t/B_t = 1/(\beta(1-\pi)) > 1$ and $\lim_{t\to\infty}((A_t - B_t)C - (1-\beta)^2/(\beta\pi))/B_t = 0$. For any $\epsilon > 0$ ($\epsilon < (1/(\beta(1-\pi)) - 1)/4$), there is a t^*, such that if $t \geq t^*$,

$$\frac{A_t}{B_t} \geq \frac{1}{\beta(1-\pi)} - \epsilon, \text{ and } \left|\frac{(A_t - B_t)C - \frac{(1-\beta)^2}{\beta\pi}}{B_t}\right| \leq \epsilon^2.$$

So if $|U_{t+1} - U_t - C| \geq \epsilon$,

$$\begin{aligned} |U_{t+2} - U_{t+1} - C| &\geq \left(\frac{1}{\beta(1-\pi)} - \epsilon\right)|U_{t+1} - U_t - C| - \epsilon^2 \\ &\geq \left(\frac{1}{\beta(1-\pi)} - 2\epsilon\right)|U_{t+1} - U_t - C| \\ &\geq \frac{\frac{1}{\beta(1-\pi)} + 1}{2}|U_{t+1} - U_t - C|. \end{aligned}$$

By induction, $|U_{t+2} - U_{t+1} - C|$ will grow at a geometric rate, contradicting equation (36). Therefore when $t \geq t^*$, $|U_{t+2} - U_{t+1} - C| \leq \epsilon$. Thus equation (13) holds. *Q.E.D.*

Proof of Proposition 4: If $\bar{V}^a > \bar{V}^b$, then it follows from equation (8) that $-U_0^a > -U_0^b$ and from equation (12) that $\lim_{t\to\infty} U_t^a/U_{t-1}^a > \lim_{t\to\infty} U_t^b/U_{t-1}^b$. *Q.E.D.*

Proof of Proposition 5: Lemma 3.3 states that $x_t > x_{t+1} > 0$, for all $t \geq 1$. We will show that both $c_t^E - c_t^U$ and $c_t^E - c_{t+1}^E$ are strictly decreasing in t.

Equation (3) implies

$$\exp(-\gamma(c_{t-1}^U - c_t^U)) \quad = \quad \pi \exp(-\gamma(c_t^E - c_t^U)) + (1-\pi),$$

which implies that $c_t^E - c_t^U$ is decreasing. Equation (3) also implies

$$\begin{aligned}\frac{\exp(-\gamma c_{t+1}^E)}{\exp(-\gamma c_t^E)} &= \frac{\exp(-\gamma c_t^U) - (1-\pi)\exp(-\gamma c_{t+1}^U)}{\exp(-\gamma c_{t-1}^U) - (1-\pi)\exp(-\gamma c_t^U)} \\ &= \exp(\gamma x_t)\left(1 + (1-\pi)\exp(\gamma x_{t+1})\frac{\exp(\gamma(x_t - x_{t+1})) - 1}{1 - (1-\pi)\exp(\gamma x_t)}\right).\end{aligned}$$

Since $\exp(\gamma x_t)$, $(x_t - x_{t+1})$, and $1/(1-(1-\pi)\exp(\gamma x_t))$ are all decreasing in t, we conclude that $c_t^E - c_{t+1}^E$ is decreasing in t. Finally

$$c_t^U + \beta\frac{c_{t+1}^E - w}{1-\beta} - \frac{c_t^E - w}{1-\beta} = w - (c_t^E - c_t^U) - \frac{\beta(c_t^E - c_{t+1}^E)}{1-\beta}$$

is strictly increasing in t. *Q.E.D.*

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