Online Appendices for "Understanding the Puzzling Risk-Return Relationship for Housing"

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Appendix I. A Housing-Consumption-Based Asset Pricing Model

In this appendix, I present a simple consumption-based asset pricing model that incorporates housing to illustrate the idea presented in Section 3.1. The model consists of a large number of infinitely-lived households that enjoy both housing services h_t and a nondurable, numeraire consumption good c_t. Given the discount factor β, preferences are

$$E\left(\sum_{t=0}^{\infty} \beta^t u(c_t, h_{t-1})\right) \tag{1}$$

where $E(\cdot)$ indicates the expectation conditional on the information at time 0. The period utility, $u(c_t, h_{t-1})$, is separable, increasing and concave in both c_t and h_{t-1}. There are two assets in the economy. One is a bond s_t, which represents a risk-free financial asset. The riskless return at time t is denoted as r_t^f. Following the convention, I assume that the bonds are in positive net supply. The other is housing capital h_t, which represents a real asset. One unit of housing stock a household purchased at the end of $t-1$ produces one unit of housing services at the beginning of t. Thus households consume h_{t-1} at time t.[1] Housing assets trade at the price P_t and depreciate at the rate δ. Let H_t denote the aggregate supply of housing at time t. For convenience, the model assumes away transaction costs.

There are N_t identical households. All households enter period 0 with initial bond endowment s_0 and housing stock h_0. At time t, each household is endowed with labor income y_t, which is defined in units of perishable non-durable consumption. The budget constraint is given by

$$c_t + P_t h_t + s_t = y_t + s_{t-1}(1 + r_t^f) + P_t h_{t-1}(1 - \delta) \tag{2}$$

The stochastic house price and labor income are the key sources of uncertainty in this model, with properties to be specified later. Markets are incomplete in the sense that we close down other financial asset markets, including insurance markets. To finance future durable and non-durable

[1] The timing convention, although non-standard, is innocuous (see Piazzesi, Schneider and Tuzel, 2003). It is made to allow an analytical solution that characterizes the risk-return relationship.

consumption and insure against house price risk and income risk, households can transfer wealth over time through bond and housing holdings. The low correlation between housing and existing financial assets suggests that housing markets are highly incomplete and that adding other assets does not help lower the risk associated with house prices.

An equilibrium is a collection of processes (c_t, s_t, h_t) and price process $\{r_t^f, P_t\}$ such that (i) (c_t, h_t, s_t) solve the household's problem of maximizing utility (1) subject to the budget constraint (2) and (ii) markets clear for bonds, housing and non-durable consumption.

Following the standard capital asset pricing models with fixed asset supply, I first consider an endowment economy where housing supply, H_t, is exogenous and fixed. The Lagrangian for the household's problem is

$$L = E\left(\sum_{t=0}^{\infty}\left(\beta^t u(c_t, h_{t-1}) - \mu_t\left[c_t + s_t + P_t \cdot h_t - (1 + r_t^f)s_{t-1} - P_t h_{t-1}(1-\delta)\right]\right)\right) \tag{3}$$

where μ_t is the multiplier on the budget constraint. The first order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial s_t} &= -\mu_t + \beta E_t \mu_{t+1}(1 + r_{t+1}^f) = 0 \\
\frac{\partial L}{\partial c_t} &= \beta^t\left(u_1(c_t, h_{t-1}) - \mu_t\right) = 0 \\
\frac{\partial L}{\partial h_t} &= \beta^t\left\{-\mu_t P_t + \beta E_t\left[\mu_{t+1}\left((1-\delta)P_{t+1} + u_2(c_{t+1}, h_t)\right)\right]\right\} = 0
\end{aligned}$$

For an equilibrium to exist, the absence of arbitrage is necessary. This in turn implies the existence of a strictly positive pricing kernel, which represents the present value of a unit of numeraire one period ahead:

$$M_{t+1} = \beta\frac{u_1(c_{t+1}, h_t)}{u_1(c_t, h_{t-1})} \tag{4}$$

After substituting the pricing kernel into first order conditions, we arrive at expressions for the risk-free interest rate and house price:

$$1 = E_t\left[(1 + r_{t+1}^f)M_{t+1}\right] \tag{5}$$

$$P_t = E_t\left[M_{t+1}\left(P_{t+1}(1-\delta) + \frac{u_2(c_{t+1}, h_t)}{u_1(c_{t+1}, h_t)}\right)\right] \tag{6}$$

Equation (5) defines the interest rate in equilibrium. Equation (6) is the central housing pricing formula. The price of housing asset, P_t, is the expected present value of future price, plus the marginal utility derived from housing consumption in the following period, discounted back by M_{t+1}. While the first term in the bracket represents the financial gains associated with the current home ownership, the second term can be interpreted as the rent savings from holding one unit of housing at time $t + 1$. To show this, assume there is a rental market in which the time-$(t + 1)$ rent for one unit of housing service is Q_{t+1}. Hypothetically, at time $t + 1$ the household rents housing service h_t for just one period. In this case, the household chooses the optimal level of

h_t to minimize expenditure $c_{t+1} + Q_{t+1}h_t$ to achieve a desired utility level, $u(c_{t+1}, h_t) = \bar{u}$. This yields the following first order condition:

$$Q_{t+1} = \frac{u_2(c_{t+1}, h_t)}{u_1(c_{t+1}, h_t)} \tag{7}$$

Thus, the pricing relation can be written as

$$P_t = E_t\left[M_{t+1}\left(P_{t+1}(1-\delta) + Q_{t+1}\right)\right] \tag{8}$$

Hence, the price P_t of the housing asset is the expected present value of its future sale price, adjusted for depreciation, plus one-period rent at time $t+1$. The housing pricing relation (8) represents a close parallel to the basic asset pricing relation in the standard CCAPM of Lucas (1978), in which the role of future dividends associated with financial securities is replaced by future rents associated with home ownership in our context. Defining the total housing return as $R_{t+1} = \frac{P_{t+1}(1-\delta)+Q_{t+1}}{P_t}$, Equation (8) can now be rewritten as

$$E_t(R_{t+1}M_{t+1}) = 1 \tag{9}$$

The model presented so far has been fairly unrestrictive. To solve the model analytically, I now make assumptions about the utility function and the labor income process. Following Chambers, Garriga and Schlagenhauf (2009), I assume that the preferences are given by

$$u(c_t, h_{t-1}) = \gamma \frac{c_t^{1-\tau_1}}{1-\tau_1}(1-\gamma)\frac{h_{t-1}^{1-\tau_2}}{1-\tau_2} \tag{10}$$

For convenience, I assume that the income growth follows a stochastic martingale process:

$$\ln \frac{y_{t+1}}{y_t} = \frac{1}{\tau_1}(\ln \beta + r_t^f) + \frac{1}{2\tau_1}\sigma_{y,t}^2 + \frac{1}{\tau_1}\epsilon_{y,t+1} \tag{11}$$

with $\epsilon_{y,t+1}|I_t \sim N(0, \sigma_{y,t}^2)$.[2] Solving the household maximization problem yields $M_{t+1} = \beta\left(\frac{y_{t+1}}{y_t}\right)^{-\tau_1}$. The separability of the utility function implies that $\sigma_{M,t}^2$ is driven purely by uncertainties in per capita labor income, that is, $\sigma_{M,t}^2 = \sigma_{y,t}^2$. It then follows that

$$M_{t+1} = exp\left(-r_t^f - \frac{1}{2}\sigma_{M,t}^2 + \epsilon_{M,t+1}\right) \tag{12}$$

where $\epsilon_{M,t+1}|I_t \sim N(0, \sigma_{M,t}^2)$.

Having obtained M_{t+1} as an equilibrium outcome, I now turn to the process of house prices and rents. Let $g_{t+1}^P = \ln \frac{P_{t+1}}{P_t}$ be the growth rate of the current house's price. Let $g_{t+1}^Q = \ln \frac{Q_{t+1}}{Q_t}$ be the growth rate of the future house's rent. Both follow a stochastic process:

$$g_{t+1}^j = \alpha_0^j + \alpha_1^j g_t^j + \epsilon_{j,t+1} \tag{13}$$

$$\epsilon_{j,t+1} \sim N(0, \sigma_{j,t}^2) \tag{14}$$

[2]The coefficients in the labor income process are restricted in a way so that the equilibrium can be solved analytically. This restriction can be relaxed without affecting the model's key implications.

where $0 < \alpha_1^j < 1$ and $j = \{P, Q\}$. It is well known that housing returns exhibit positive autocorrelation in many markets; see Case and Shiller (1990) for U.S. cities and Englund and Ioannides (1997) for international comparative data. This provides empirical justification for the AR(1) process specified in Equation (13).

To pin down the risk-return relationship, I make further assumptions about the joint distribution of house price, rent and pricing kernel. First, the shocks to the price growth and pricing kernel are negatively correlated. Specifically, $cov(\epsilon_{P,t+1}, \epsilon_{M,t+1}) = \rho_{PM}\sigma^2_{P,t}$, where $\rho_{PM} < 0$. Second, the shocks to the price growth and rental growth are positively correlated: $cov(\epsilon_{P,t+1}, \epsilon_{Q,t+1}) = \rho_{PQ}\sigma^2_{P,t}$, where $\rho_{PQ} > 0$. Section 3.1 provides justification for each of these two key assumptions. Finally, for simplicity, I assume $corr(\epsilon_{Q,t+1}, \epsilon_{M,t+1}) = 0$.

Proposition 1 *Define $p_t = \ln P_t$, $q_t = \ln Q_t$ and $r_{t+1} = \ln R_{t+1}$. Given the model specified by Equations (1), (2), (10)-(12), there exists a linear solution to the log price rent ratio*

$$p_t - q_t = c_0 + c_1 g_t^P + c_2 g_t^Q + c_3\sigma^2_{P,t} + c_4\sigma^2_{Q,t} \tag{15}$$

where the parameters are

$$\begin{aligned}
c_0 &= \frac{-r_t^f + k + \gamma\ln(1-\delta) + \gamma\alpha_0^P + (1-\gamma)\alpha_0^Q}{1-\gamma} \\
c_1 &= \frac{\gamma\alpha_1^P}{1-\gamma} \\
c_2 &= \alpha_1^Q \\
c_3 &= \frac{\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}}{1-\gamma} \\
c_4 &= \frac{1}{2}(1-\gamma)
\end{aligned}$$

Furthermore, there exists a linear solution to the log return process

$$\begin{aligned}
r_{t+1} &= r_t^f + \gamma\epsilon_{P,t+1} + (1-\gamma)\epsilon_{Q,t+1} - \left(\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}\right)\sigma^2_{P,t} \\
&\quad -\frac{1}{2}(1-\gamma)^2\sigma^2_{Q,t}
\end{aligned} \tag{16}$$

Proof: To solve the model, I apply Campbell and Shiller (1988)'s approximation. Let $p_t = \ln P_t$ and $q_t = \ln Q_t$.

$$\begin{aligned}
r_{t+1} &= \ln R_{t+1} \\
&= k + \gamma\ln(1-\delta) + \gamma g^P_{t+1} + (1-\gamma)g^Q_{t+1} - (1-\gamma)(p_t - q_t)
\end{aligned} \tag{17}$$

where parameter γ is the average ratio of the house price to the sum of the house price and the rent, a number slightly smaller than one, and k is a constant related to γ. Substituting equation (4) and equation (17) into equation (9), we obtain

$$E_t exp\left[-r_t^f - \frac{1}{2}\sigma_{M,t}^2 + \epsilon_{M,t+1} + k + \gamma\ln(1-\delta) + \gamma g_{t+1}^P + (1-\gamma)g_{t+1}^Q - (1-\gamma)(p_t - q_t)\right] = 1 \quad (18)$$

I first postulate a solution to the log price rent ratio in terms of the state variables. I then verify this solution and solve for the parameters of the solution. The linear solution takes the following form.

$$p_t - q_t = c_0 + c_1 g_t^P + c_2 g_t^Q + c_3\sigma_{P,t}^2 + c_4\sigma_{Q,t}^2 \quad (19)$$

Substituting this solution to equation (18), we get

$$1 = E_t exp[A(\cdot)] \quad (20)$$

where

$$\begin{aligned} A(\cdot) &= -r_t^f - \frac{1}{2}\sigma_{M,t}^2 + \epsilon_{M,t+1} + k + \gamma\ln(1-\delta) + \gamma g_{t+1}^P + (1-\gamma)g_{t+1}^Q \\ &\quad -(1-\gamma)(c_0 + c_1 g_t^P + c_2 g_t^Q + c_3\sigma_{P,t}^2 + c_4\sigma_{Q,t}^2) \\ &= \left(-r_t^f + k + \gamma\ln(1-\delta) - (1-\gamma)c_o + \gamma\alpha_0^P + (1-\gamma)\alpha_0^Q\right) - \frac{1}{2}\sigma_{M,t}^2 + \epsilon_{M,t+1} \\ &\quad + \left(\gamma\alpha_1^P + (1-\gamma)c_1\right) g_t^P + \Big((1-\gamma)(\alpha_1^Q - c_2)\Big) g_t^Q - (1-\gamma)c_3\sigma_{P,t}^2 - (1-\gamma)c_4\sigma_{Q,t}^2 \\ &\quad +\gamma\epsilon_{P,t+1} + (1-\gamma)\epsilon_{Q,t+1} \end{aligned} \quad (21)$$

This leads to

$$\begin{aligned} E_t A(\cdot) &= const. - \frac{1}{2}\sigma_{M,t}^2 + \left(\gamma\alpha_1^P + (1-\gamma)c_1\right) g_t^P + \Big((1-\gamma)(\alpha_1^Q - c_2)\Big) g_t^Q - (1-\gamma)c_3\sigma_{P,t}^2 - (1-\gamma)c_4\sigma_{Q,t}^2 \\ Var_t A(\cdot) &= \sigma_{m,t}^2 + \gamma^2\sigma_{P,t}^2 + (1-\gamma)^2\sigma_{Q,t}^2 + 2\gamma\rho_{PM}\sigma_{P,t}^2 + 2\gamma(1-\gamma)\rho_{PQ}\sigma_{P,t}^2 \end{aligned} \quad (22)$$

where $const. = -r_t^f + k + \gamma\ln(1-\delta) - (1-\gamma)c_o + \gamma\alpha_0^P + (1-\gamma)\alpha_0^Q$. Since $A(\cdot)$ is a normal random variable, we must have

$$E_t A(\cdot) + \frac{1}{2}Var_t A(\cdot) = 0 \quad (23)$$

Substituting equations (22) into equation (23), we obtain

$$\begin{aligned} 0 &= (const.) + \left(\gamma\alpha_1^P - (1-\gamma)c_1\right) g_t^P + \Big((1-\gamma)(\alpha_1^Q - c_2\Big) g_t^Q \\ &\quad + \left(-(1-\gamma)c_3 + \frac{1}{2}\gamma^2 + \gamma\rho_{PM} + (1-\gamma)\gamma\rho_{PQ}\right)\sigma_{P,t}^2 + \left(\frac{1}{2}(1-\gamma)^2 - (1-\gamma)c_4\right)\sigma_{Q,t}^2 \end{aligned} \quad (24)$$

Note that for this equation to hold, the terms in the five brackets must be equal to zero, since g_t^j and $\sigma_{j,t}^2$ are random variables. Solving the resulting equations produces the solutions below:

$$p_t - q_t = c_0 + c_1 g_t^P + c_2 g_t^Q + c_3\sigma_{P,t}^2 + c_4\sigma_{Q,t}^2 \quad (25)$$

where

$$\begin{aligned} c_0 &= \frac{-r_t^f + k + \gamma\ln(1-\delta) + \gamma\alpha_0^P + (1-\gamma)\alpha_0^Q}{1-\gamma} \\ c_1 &= \frac{\gamma\alpha_1^P}{1-\gamma} \\ c_2 &= \alpha_1^Q \\ c_3 &= \frac{\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}}{1-\gamma} \\ c_4 &= \frac{1}{2}(1-\gamma)\sigma_{Q,t}^2 \end{aligned}$$

Substituting equation (25) into equation (17), we then obtain

$$r_{t+1} = r_t^f + \gamma\epsilon_{P,t+1} + (1-\gamma)\epsilon_{Q,t+1} - \left(\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}\right)\sigma_{P,t}^2 - \frac{1}{2}(1-\gamma)^2\sigma_{Q,t}^2$$

■

The solution method is consistent with Campbell and Shiller (1988) and Wu (2001). The first part of Proposition 1 shows that the log price-rent ratio ($p_t - q_t$) depends on the risk-free rate, expected house price growth and rent growth, and their conditional volatilities. The results are fairly intuitive. For example, a higher risk-free rate represents a higher opportunity cost of housing investment, which makes holding current housing less attractive. In contrast, higher price growth and rent growth increase the value of housing, leading to a higher price-rent ratio. Finally, the parameter c_3 characterizes the net effect of price risk on the price-rent ratio. The economic implications of this term are the focus of the next proposition.

The second part of Proposition 1 shows that the log realized housing return r_{t+1} also follows a simple and intuitive pattern. It depends on the risk-free rate, shocks to price growth and rent growth, and their conditional volatilities. The positive dependence of r_{t+1} on $\epsilon_{P,t+1}$ and $\epsilon_{Q,t+1}$ confirms the conventional wisdom that housing is valuable not only as a financial asset but also as a consumption good (e.g., Henderson and Ioannides, 1983; Brueckner, 1997).

Proposition 2 *Assuming* $-1 < \alpha_1^P < 1$ *and* $-1 < \alpha_1^Q < 1$.

(i) In an economy with an exogenous housing supply, the equilibrium housing risk-return relationship for the log total return process is characterized by

$$\frac{\partial \mathrm{E}_t(r_{t+1})}{\partial \sigma_{r,t}^2} = -\frac{\frac{1}{2}\gamma + \rho_{PM} + (1-\gamma)\rho_{PQ}}{\gamma + 2(1-\gamma)\rho_{PQ}} \tag{26}$$

where $\sigma_{r,t}^2 = \mathrm{VAR}_t(r_{t+1}) = \left(\gamma^2 + 2\gamma(1-\gamma)\rho_{PQ}\right)\sigma_{P,t}^2 + (1-\gamma)^2\sigma_{Q,t}^2$.

(ii)

$$\begin{aligned} \frac{\partial \mathrm{E}_t(r_{t+1})}{\partial \sigma_{r,t}^2} &< 0 \ \textit{if}\ \rho_{PQ} > \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \\ \frac{\partial \mathrm{E}_t(r_{t+1})}{\partial \sigma_{r,t}^2} &> 0 \ \textit{if}\ \rho_{PQ} < \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \end{aligned} \tag{27}$$

(iii)

$$\frac{\partial^2 \mathrm{E}_t(r_{t+1})}{\partial \sigma_{r,t}^2 \partial \rho_{PQ}} < 0 \tag{28}$$

Proof: (i) It follows from Equation (16) that

$$E_t(r_{t+1}) = r_t^f - \left(\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}\right)\sigma_{P,t}^2 - \frac{1}{2}(1-\gamma)^2\sigma_{Q,t}^2 \tag{29}$$

$$\sigma_{r,t}^2 = \left(\gamma^2 + 2\gamma(1-\gamma)\rho_{PQ}\right)\sigma_{P,t}^2 + (1-\gamma)^2\sigma_{Q,t}^2 \tag{30}$$

$$\begin{aligned}
\frac{\partial E_t(r_{t+1})}{\partial \sigma^2_{r,t}} &= \frac{\partial E_t(r_{t+1})}{\partial \sigma^2_{P,t}} \frac{\partial \sigma^2_{P,t}}{\partial \sigma^2_{r,t}} \\
&= -\left(\frac{1}{2}\gamma^2 + \gamma\rho_{PM} + \gamma(1-\gamma)\rho_{PQ}\right) \frac{1}{\gamma^2 + 2\gamma(1-\gamma)\rho_{PQ}} \\
&= -\frac{\frac{1}{2}\gamma + \rho_{PM} + (1-\gamma)\rho_{PQ}}{\gamma + 2(1-\gamma)\rho_{PQ}} \qquad (31)
\end{aligned}$$

where the last step comes from Equations (29) and (30). It follows from Equation (31) that

$$\begin{aligned}
\frac{\partial E_t(r_{t+1})}{\partial \sigma^2_{r,t}} &< 0 \text{ if } \rho_{PQ} > \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \\
\frac{\partial E_t(r_{t+1})}{\partial \sigma^2_{r,t}} &> 0 \text{ if } \rho_{PQ} < \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \qquad \blacksquare
\end{aligned}$$

(ii) Proof: Following Equation (31), we have

$$\begin{aligned}
\frac{\partial^2 E_t(r_{t+1})}{\partial \sigma^2_{r,t} \partial \rho_{PQ}} &= -\frac{1}{(\gamma + 2(1-\gamma)\rho_{PQ})}(1-\gamma) + \frac{\frac{1}{2}\gamma + \rho_{PM} + (1-\gamma)\rho_{PQ}}{(\gamma + 2(1-\gamma)\rho_{PQ})^2} 2(1-\gamma) \\
&= \frac{2(1-\gamma)\rho_{PM}}{(\gamma + 2(1-\gamma)\rho_{PQ})^2} < 0
\end{aligned}$$

where the inequality follows from the assumption that $\rho_{PM} < 0$. $\blacksquare$

Proposition 2 (i) provides a clean characterization of the net risk-return relationship. Following this, Proposition 2 (ii) shows that the consumption hedge effect dominates in markets with sufficiently large ρ_{PQ}, whereas the financial risk effect dominates in other markets. In this sense, the model delivers cross-sectional variation in the *sign* of the risk-return relationship through the relative strength of ρ_{PQ} and ρ_{PM}. Proposition 2 (iii) further delivers cross-sectional variation in the *magnitude* of the risk-return relationship. That is, across markets, the marginal expected return required to compensate for the risk decreases with ρ_{PQ}. This provides the first testable implication for our empirical work.

The analysis above focuses on the log total return process, r_t, which includes the dividend yield. In principle, one can proxy the dividends with rents and combine rents with house prices to impute the total returns. However, doing so implicitly assumes the direct comparability of owned units to rental units and of owners to renters. In a recent work by Glaeser and Gyourko (2007), they document that rental units tend to be very different from owner-occupied units and that owners are different from owners in economically meaningful ways. They also argue that, while it is possible to construct comparable rent and house price data in one market, it is infeasible to do so for large scale statistical work that involves most of the key markets in the country. For this reason, the empirical analysis in this paper examines the relationship between the expected log capital return process $E_t g^P_{t+1} = E_t\left(ln(P_{t+1}) - ln(P_t)\right)$ and the expected price risk $\sigma^2_{P,t}$. In what follows, I shall show that the implications for the log total return process in Proposition 2 essentially carry over to the log capital return process, because the capital gain return tends to dominate the total return.

Proposition 3 *Assuming* $-1 < \alpha_1^P < 1$ *and* $-1 < \alpha_1^Q < 1$.

(i) In an economy with an exogenous housing supply, the equilibrium housing risk-return relationship for the log capital return process is characterized by

$$\frac{\partial \mathrm{E}_t(g_{t+1}^P)}{\partial \sigma_{P,t}^2} = -\frac{\frac{1}{2}\gamma + \rho_{PM} + (1-\gamma)\rho_{PQ}}{1-\gamma} \tag{32}$$

. *(ii)*

$$\begin{aligned} \frac{\partial \mathrm{E}_t(g_{t+1}^P)}{\partial \sigma_{P,t}^2} &< 0 \;\; \textit{if} \;\; \rho_{PQ} > \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \\ \frac{\partial \mathrm{E}_t(g_{t+1}^P)}{\partial \sigma_{P,t}^2} &> 0 \;\; \textit{if} \;\; \rho_{PQ} < \frac{-\rho_{PM} - \frac{1}{2}\gamma}{1-\gamma} \end{aligned} \tag{33}$$

(iii)

$$\frac{\partial^2 \mathrm{E}_t(g_{t+1}^P)}{\partial \sigma_{P,t}^2 \partial \rho_{PQ}} < 0 \tag{34}$$

Proof: (i) Combining Equation (16) and Equation (17), we obtain

$$\begin{aligned} \frac{\partial E_t(g_{t+1}^P)}{\partial \sigma_{P,t}^2} &= \frac{\partial E_t(g_{t+1}^P)}{\partial E_t r_{t+1}} \frac{\partial E_t(r_{t+1})}{\partial \sigma_{P,t}^2} + \frac{\partial E_t(g_{t+1}^P)}{\partial (p_t - q_t)} \frac{\partial (p_t - q_t)}{\partial \sigma_{P,t}^2} \\ &= \frac{1}{\gamma}(-c_3) + \frac{1-\gamma}{\gamma} c_3 \\ &= -\frac{\frac{1}{2}\gamma + \rho_{PM} + (1-\gamma)\rho_{PQ}}{1-\gamma} \end{aligned} \tag{35}$$

(ii) and (iii) from Equation (35) naturally. ■

To complete the solution of our model economy, we need to find housing and nondurable consumption in terms of exogenous forces. The results will of course depend on what the rest of the economy looks like. Since households are identical with respect to their preferences and endowments, it follows from the market clearing conditions that in equilibrium $h_t^* = \frac{H_t}{N_t}$ and $c_t^* = y_t$. Since housing endowment is exogenous, the house price adjusts so that an exogenous shock in price risk is fully translated into changes in house prices and returns. The equilibrium risk-return relationship is therefore fully characterized by Equation (32).

Appendix II. Population-Based Urban Growth Measure

The key parameter of interest in equation (4) is β_1, the coefficient on the interaction term $RISK_{it} \times HEDGE_{it} \times FAST_{it}$. A negative β_1 indicates that the consumption hedge effect is stronger in fast-growing markets. One econometric issue confronting this estimate is the possibility that the underlying urban growth indicator, $FAST_{it}$, is mis-measured, making it hard to interpret the resulting estimate. In measuring urban growth, Section 4.3 compares the overall population (and hence the number of households) with existing housing units. The comparison is intuitively appealing, as population is an obvious indicator of urban growth (Helsley, 2003, chapter 8). Given the extremely tight link between population and housing stock (Glaeser and Gyourko, 2005), the difference between the two would naturally reflect the imbalance between the aggregate housing demand and supply. However, what happens when we take into account the difference between the aggregate housing demand and the demand for owner-occupied housing, as well as changes in mortgage rates and expectations over future house price appreciation? Following DiPasquale and Wheaton (2000), I now present a simple model that captures these additional considerations, and show that the estimated urban growth effect remains robust after accounting for these factors.

Assume the demand for owner-occupied housing is proportional to the number of households (H_t), namely, $H_t(\alpha_0 - \alpha_1 UC_t)$. The parameter α_0 can be considered as the fraction of households who would own homes if the cost of owning a house is zero; α_1 as the responsiveness of this fraction to changes in the annual cost of owning, UC_t. A simple way to express UC_t is $P_t(M_t - I_t)$, where P_t indicates house price, M_t indicates the after-tax mortgage rate, and I_t indicates the expected rate of future house price appreciation. This specification is useful because it incorporates two important considerations. First, the proportionality factor, $\alpha_0 - \alpha_1 P_t(M_t - I_t)$, allows us to focus on the demand for owner-occupied housing, rather than the aggregate housing demand. Second, the current housing demand will be higher, all else being equal, when the after-tax mortgage rate is lower, or when expectations about future price appreciation are more optimistic.

In equilibrium, house price today adjusts so that the existing stock of housing units, S_t, equals the *ex ante* demand for owner-occupied housing units:

$$S_t = H_t(\alpha_0 - \alpha_1 P_t(M_t - I_t)) \tag{36}$$

Thus, a city exhibits *relatively* faster population growth and hence higher *owner-occupied* housing demand relative to supply if

$$\frac{H_t}{S_t} \geq \frac{1}{\alpha_0 - \alpha_1 P_t(M_t - I_t)} \tag{37}$$

Empirically, I do not observe α_0, α_1, M_t, and I_t. I therefore replace the cutoff point on the right-hand-side of equation (37) with 1. Given that $\frac{1}{\alpha_0 - \alpha_1 P_t(M_t - I_t)} \geq 1$, the fraction of fast-growing cities based on the owner-occupied housing demand (indicated by equation 37) is smaller than the fraction of fast-growing cities based on the aggregate demand – the latter being used to form the measure of urban growth in the main analysis. However, this should not affect the interpretation of our results for two reasons.

First, as discussed in Section 3.3, the urban growth measure is considered as a comparative static shift in hedging demand. In this sense, what we need here is a measure that can rank different cities based on their relative urban growth rates, rather than a measure of housing demand. I measure urban growth by comparing population with housing stock and by comparing average house price with costruct costs. This is consistent with the spirit of the urban literature (Helsley, 2003; Glaeser, Gyourko and Saks, 2006) and hence more suitable for the purpose here.

Second, to the extent that the owner-occupied housing demand matters for hedging demand in some unspecified way, equation (37) and the subsequent discussion indicate that we have a classical measurement error problem (e.g. Wooldridge, 2001, Chapter 4), which produces an attenuation bias on the parameter β_1 (i.e., a bias towards zero). In this case, a significantly negative estimate of β_1 should be considered as conservative evidence for the urban growth effect, providing reassuring support for the prediction that the consumption hedge effect is stronger in fast-growing cities.

Appendix III. Robustness Check for the Leverage Hypothesis

There is a long tradition in finance that attributes the negative risk-return relationship for stocks to the leverage effect. A leverage effect exists if a drop in the value of the stock (negative return) increases financial leverage, which could potentially make the stock riskier and increase its volatility. To the extent that this effect operates in the housing market through downpayment constraints (Stein, 1995), the negative risk-return relationship for housing could be induced by the leverage effect rather than by hedging incentives.

To address this leverage concern, I extend the GARCH(1,1)-in-mean specification presented above to an EGARCH(1,1)-in-mean specification. In particular, equation (9) in the main paper can be modified as:

$$\ln(\sigma_{i,t}^2) = b_{i,0} + b_{i,1} z_{i,t-1} + b_{i,2}(|z_{i,t-1}| - E|z_{i,t-1}|) + \lambda_i \ln(\sigma_{i,t-1}^2) \tag{38}$$

where $z_{i,t} \equiv u_{i,t}/\sigma_{i,t}$. This model is proposed by Nelson (1991) and allows an asymmetric effect of unanticipated changes in housing returns. Equation (38) specifies the log conditional variance as an asymmetric function of last period's standardized innovation, $z_{i,t-1}$, and last period's log conditional variance, $\ln(\sigma_{i,t-1}^2)$. The term $(|z_{i,t-1}| - E|z_{i,t-1}|)$ represents the size effect and the term $z_{i,t-1}$ represents the sign effect. The coefficient $b_{i,1}$ should be negative if leverage induces an inverse relationship between the housing return and risk. The estimate of θ_i therefore serves as a direct estimate of the risk-return relationship after controlling for the possible leverage effect.

Estimation of (7) and (38) converges in 51 MSAs in the FHFA sample. Table A4 reports the summary of estimates for the full sample. The coefficient, $b_{i,1}$, measuring the asymmetric impact of past innovation on current volatility (the leverage effect), is statistically significant at the 10% level for 27 MSAs. Among these MSAs, the sign of $b_{i,1}$ is significantly negative in 11 MSAs, indicating that in these areas negative innovations (price declines) increase volatility more than positive innovations (price increases). The sample presents rich cross-market variation in the strength of the leverage effect: the more negative the value of $b_{i,1}$, the stronger the leverage effect. Having controlled for the leverage effect, I find that the estimates of the risk-return relationship coefficient, θ_i, now range from -67.80 to 58.03 with a mean of -1.97 and a median of -2.91. In particular, the sign of θ_i is significantly negative in 15 MSAs, out of which only five MSAs are found to have leverage effects. Thus, a negative risk-return relationship in these markets cannot be attributed to leverage.

The small sample size prevents me from directly testing the relationship between θ_i and hedging incentives. As an alternative check, Column 6 presents a Spearman correlation between the estimates of θ_i and local out-migration rates. The positive correlation of 0.23 suggests that markets with lower out-migration rates are more likely to exhibit a negative risk-return relationship. This is consistent with the consumption hedge hypothesis explored in the main analysis. Moreover, the Spearman correlation between θ_i and the WRLURI is -0.53. This is consistent with the idea that the relative strength of the consumption hedge effect is larger in more supply-constrained markets.

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Table A1: Metropolitan Areas by Constraints and by Growth

Fast-Growing	Slow-Growing	Declining
	Less Constrained Markets	
Albuquerque, NM	Akron, OH	Detroit, MI
Allentown-Bethlehem-Easton, PA	Beaumont-Port Arthur, TX	New Orleans, LA
Atlanta, GA	Buffalo-Niagara Falls, NY	Wichita, KS
Ann Arbor, MI	Cleveland-Lorain-Elyria, OH	
Austin-San Marcos, TX	Columbus, OH	
Biloxi-Gulfport-Pascagoula, MS	Davenport-Moline-Rock Island, IA-IL	
Birmingham, AL	Dayton-Springfield, OH	
Charlotte-Gastonia-Rock Hill, NC-SC	Des Moines, IA	
Chattanooga, TN-GA	Duluth-Superior, MN-WI	
Cincinnati, OH-KY-IN	Erie, PA	
Columbia, SC	Evansville-Henderson, IN-KY	
Corpus Christi, TX	Flint, MI	
Dallas, TX	Fort Wayne, IN	
Denver, CO	Indianapolis, IN	
El Paso, TX	Gary, IN	
Fort Worth-Arlington, TX	Jacksonville, FL	
Grand Rapids-Muskegon-Holland, MI	Kalamazoo-Battle Creek, MI	
Greensboro-Winston-Salem-High Point, NC	Kansas City, MO-KS	
Houston, TX	Macon, GA	
Huntsville, AL	Omaha, NE-IA	
Jersey City, NJ	Peoria-Pekin, IL	
Knoxville, TN	Pittsburgh, PA	
Las Vegas, NV-AZ	Lansing-East Lansing, MI	
Little Rock-North Little Rock, AR	Rockford, IL	
Louisville, KY-IN	Scranton-Wilkes-Barre-Hazleton, PA	
Lubbock, TX	South Bend, IN	
Madison, WI	Springfield, MO	
Memphis, TN-AR-MS	St. Louis, MO-IL	
Minneapolis-St. Paul, MN-WI	Toledo, OH	
Mobile, AL	Topeka, KS	
Nashville, TN	Utica-Rome, NY	
Norfolk-Virginia Beach-Newport News, VA	Youngstown-Warren, OH	
Oklahoma City, OK		
Phoenix-Mesa, AZ		
Providence-Fall River-Warwick, RI-MA		
Raleigh-Durham-Chapel Hill, NC		

Table A1: Metropolitan Areas by Constraints and by Growth (cont'd)

Fast-Growing	Slow-Growing	Declining
Reno, NV		
Richmond-Petersburg, VA		
Roanoke, VA		
Rochester, NY		
Salt Lake City-Ogden, UT		
San Antonio, TX		
Springfield, MO		
Syracuse, NY		
Tampa-St. Petersburg-Clearwater, FL		
Tulsa, OK		
Washington, DC-MD-VA-WV		
More Constrained Markets		
Albany-Schenectady-Troy, NY	Milwaukee-Waukesha, WI	Philadelphia, PA-NJ
Allentown-Bethlehem-Easton, PA	Spokane, WA	
Anchorage, AK		
Baton Rouge, LA		
Bakersfield, CA		
Baltimore, MD		
Boston, MA-NH		
Bridgeport, CT		
Charleston-North Charleston, SC		
Chicago, IL		
Colorado Springs, CO		
Eugene-Springfield, OR		
Fort Lauderdale, FL		
Fresno, CA		
Hartford, CT		
Honolulu, HI		
Lawrence, MA-NH		
Lexington, KY		
Los Angeles-Long Beach, CA		
Lowell, MA-NH		
Miami, FL		
New York, NY		
Newark, NJ		
Orlando, FL		

Table A1: Metropolitan Areas by Constraints and by Growth (cont'd)

Fast-Growing	Slow-Growing	Declining
More Constrained Markets		
Phoenix-Mesa, AZ		
Portland-Vancouver, OR-WA		
Riverside-San Bernardino, CA		
Sacramento, CA		
San Diego, CA		
San Francisco, CA		
Santa Barbara-Santa Maria-Lompoc, CA		
Seattle-Bellevue-Everett, WA		
Springfield, MA		
Stockton-Lodi, CA		
Tacoma, WA		
Tucson, AZ		
Vallejo-Fairfield-Napa, CA		

Table A2: Summary Statistics of Fraction of Population Aged 20-45[a]

year	mean (%)	s.d. (%)	p50 (%)	min. (%)	max. (%)
1980	38.07	3.88	37.54	20.10	51.71
1981	38.67	3.88	38.02	21.20	52.44
1982	39.18	3.87	38.62	22.10	52.96
1983	39.61	3.85	39.05	22.90	53.37
1984	40.00	3.83	39.49	23.58	53.70
1985	40.25	3.80	39.78	24.10	54.26
1986	40.38	3.75	39.85	24.53	54.41
1987	40.39	3.71	39.91	24.87	54.59
1988	40.22	3.69	39.73	24.93	54.55
1989	40.14	3.68	39.68	25.08	54.64
1990	40.14	3.66	39.73	25.04	54.70
1991	40.28	3.66	39.97	25.30	55.55
1992	39.86	3.64	39.53	24.72	53.84
1993	39.47	3.59	39.06	24.31	53.59
1994	39.07	3.52	38.66	23.98	52.09
1995	38.71	3.49	38.27	23.75	51.35
1996	38.31	3.45	37.91	23.29	50.62
1997	37.51	3.43	37.52	22.78	49.55
1998	37.49	3.41	37.16	22.30	49.19
1999	37.10	3.40	36.76	22.16	48.79
2000	36.73	3.41	36.41	21.72	48.90
2001	36.51	3.39	36.11	22.16	49.10
2002	36.23	3.32	35.84	22.64	49.08
2003	35.87	3.22	35.49	23.01	48.83
2004	35.58	3.18	35.19	23.68	48.50
2005	35.23	3.12	34.85	23.72	48.14
2006	34.90	3.10	34.61	24.35	47.78
2007	34.57	3.05	34.18	25.13	47.35

[a]Data Source: U.S. Census. One unit of observation is an MSA in a given year.

Table A3: Regressions from Single-Family Housing Markets (1980-2010)[a]

Variable	1	2	3	4	5	6	7
risk	-0.19^{***}	0.10^{**}	0.15^{***}	0.17^{*}	0.12^{*}	0.14	0.13
	(2.85)	(2.01)	(3.04)	(1.68)	(1.72)	(0.89)	(0.67)
risk × hedge		-0.08^{***}	0.14	-0.09	-0.41^{*}	0.25	-0.28
		(-2.78)	(0.75)	(-1.08)	(-1.89)	(1.39)	(-1.36)
risk × hedge × constraint			-0.47^{***}	-0.36^{***}		-1.62^{**}	-0.54^{**}
			(-3.06)	(-4.12)		(-2.37)	(-1.98)
N	7638	6091	5320	5320	1147	1122	1122

[a]This table repeats the regressions in Table 11 based on the FHFA all-transaction house price indices between 1980-2010. Hedging incentives are based on the within-market measure in columns 2-4, and the cross-market measure in columns 5-7. Supply constraints are measured by the undevelopable land share in columns 3 and 6, and the WRLRUI in columns 4 and 7. The t-statistics are adjusted for the intra-MSA correlation and reported in parentheses. $***$, $**$, and $*$ denote statistically significant at the 1%, 5%, and 10% levels, respectively.

Table A4: EGARCH(1,1)-in-Mean Estimates[a]

	mean	se(mean)	median	max	min	$r_s(*, m_i)$	$r_s(*, WRLURI_i)$
θ_i	-1.97	3.80	-2.91	58.03	-67.80	0.23(0.15)	$-0.53^{*}(0.00)$
λ_i	0.93	0.01	0.95	0.99	0.49	$-0.21(0.20)$	$-0.16(0.33)$
$b_{i,1}$	-0.01	0.02	-0.02	0.18	-0.27	0.04(0.83)	0.29(0.08)
$b_{i,2}$	0.40	0.03	0.35	0.98	0.14	$0.36^{*}(0.03)$	$-0.03(0.87)$
retcon $a_{i,0}$	0.002	0.002	0.004	0.013	-0.016	$-0.11(0.51)$	$-0.46^{*}(0.00)$
$b_{i,0}$	-0.5	0.104	-0.36	0.11	-3.92	$-0.20(0.25)$	$-0.14(0.39)$

[a]a. se(mean) indicates standard errors of the means. b. $r_s(*, m_i)$ denotes the Spearman rank correlation coefficient of the metropolitan area out-migration rate and the model parameter, with its significance level in parentheses. c. The correlation coefficients significant at the 5% level or lower are marked with a star. d. The data are from the FHFA, the 2000 5% PUMS and WRLURI housing supply constraints indices. The EGARCH(1,1)-in-mean model converges in 51 MSAs. The number of MSAs is reduced to 38 after being merged with the WRLURI indices; and 37 after being merged with the out-migration data.