

# ON THE LAW OF THE $i^{th}$ WAITING TIME IN A BUSY PERIOD OF $G/M/c$ QUEUES

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## Abstract

We use induction to derive the distribution of the waiting time of the  $i^{th}$  waiting customer in a busy period for a  $G/M/1$  queue with a first come first serve service. A trivial implication gives the law for the  $i^{th}$  waiting time in a busy period for a  $G/M/c$  queue. Finally, we use the Lindley recursion to relate our results to the distribution of random walks.

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**Short Title:** LAW FOR THE  $i^{th}$  WAITING TIME IN A BUSY PERIOD  
OF  $G/M/c$  QUEUES

## 1 Introduction

Treatments of the  $G/M/1$  queue may be found in several books on queueing theory e.g., Asmussen [2], Cohen [3], and Prabhu [4]. There are alternative methods to explicitly express the steady state waiting times in a  $G/M/1$  queue (e.g., Prabhu [4] page 109, starting from the transient behavior of the system, or Asmussen [2], pages 228 and 238, based on the Wiener-Hopf factorization identity). The busy period in  $G/M/1$  queues was also characterized e.g., Cohen [3] page 225 and, recently, it has been studied by Adan, Boxma, and Perry [1] using sample path approach.

In this study, we characterize the conditional law of the waiting time of the  $i^{th}$  customer within a busy period of a  $G/M/1$  queue given that at least  $i$  customers were served within this busy period. As we expect this distribution is a normalized sum of  $Erlang(\mu, j)$ ,  $E_j^\mu$ , distributions with  $j = 1, \dots, i - 1$ . Therefore, our contribution is the explicit expressions for the probability that on arrival the  $i^{th}$  waiting customer in the busy period sees  $j$  customers in the system (or equivalently,  $j - 1$  customers in the queue).

We also state two straightforward applications for our results. The first is the law for the waiting time of the  $i^{th}$  customer in a busy period of a  $G/M/c$  queue. The second uses the equivalence of the Lindley recursion, describing the waiting time in a  $G/G/1$  queue with a First Come First Serve (FCFS) service discipline, to random walks to describe the distribution of random walks condition on that they do not cross a given threshold.

For completeness, we remind that the probability density function (PDF) and CDF of an  $Erlang(\mu, j)$  random variable, with  $j$  phases each with mean

$\mu^{-1}$ , are given by:

$$f_{E_j^\mu}(x) = e^{-\mu x} \frac{\mu^j x^{j-1}}{(j-1)!}, \quad \forall x \geq 0, \quad \text{and} \quad (1)$$

$$F_{E_j^\mu}(x) = 1 - e^{-\mu x} \sum_{k=0}^{j-1} \frac{(\mu x)^k}{k!}, \quad \forall x \geq 0, \quad (2)$$

respectively and the superscript,  $\mu$ , emphasizes the mean of each phase.

## 2 The Waiting Time of the $i^{\text{th}}$ Customer within a Busy Period

### 2.1 For the $G/M/1$ Queue

Consider a  $G/M/1$  queue with interarrival times  $Z_i$  for  $i \geq 1$  that are independent and identically distributed (*i.i.d.*) random variables with a cumulative distribution function (CDF)  $F_Z(z)$ , and service requirements  $S_i$  for  $i \geq 1$  that are *i.i.d.* and exponentially distributed with mean  $\mu^{-1}$ .

We let  $N+1$  be the number of customers served during a busy period. We number the arrivals during a busy period such that the customer that initiates the busy period (and does not wait) is the  $i=0$  customer in the busy period. Then, the first customer that waits, if such exists, is the  $i=1$  customer and so on until the  $i=N^{\text{th}}$  customer. Note that upon completion of service for the  $N^{\text{th}}$  customer the server is idle. Furthermore, because  $E(X) < 0$ , busy periods are *i.i.d.* and  $E(N) < \infty$  almost surely.

We let  $W_i$  be distributed as the waiting time of the  $i^{\text{th}}$  customer in a busy cycle on<sup>1</sup>

$$I\{N \geq i\} = 1. \quad (3)$$

Then, the CDF of  $W_i$  is defined as

$$F_{W_i}(x) = P(W_i \leq x) = E(I\{0 < W_i \leq x\}), \quad \forall x \geq 0. \quad (4)$$

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<sup>1</sup>We could define  $\tilde{W}_i$  such that  $P(\tilde{W}_i = 0 | N < i) = 1$  and  $\tilde{W}_i I\{N \geq i\} = W_i$ , but this makes our notation cumbersome.

We let

$$a_i^\mu = \int_0^\infty e^{-\mu t} \frac{(\mu t)^i}{i!} dF_Z(t)$$

be the probability that exactly  $i$  customers would be served within an interarrival time if there are infinite number of customers in the queue. The superscript,  $\mu$ , emphasizes the mean service time. Moreover, it is clear that  $\sum_{i=0}^\infty a_i^\mu = 1$ .

Using the memoryless of the service time,  $W_1$ , the waiting time of the first customer that waits, is exponentially distributed with mean  $\mu^{-1}$ . In the proof of Theorem 1 we use this fact to express the distribution of  $W_i$ , for each  $i \geq 1$ . Let  $\stackrel{d}{=}$  denote an equality in distribution, then:

**Theorem 1** *Let  $W_i$  be a random variable with a CDF defined by Eq. (4) for a G/M/1 queue with interarrival times  $Z$  with a CDF  $F_Z(z)$  and exponential service with mean  $\mu^{-1}$ . Then:*

$$W_i \stackrel{d}{=} \sum_{j=1}^i P_j^i E_j^\mu, \quad (5)$$

where  $E_j^\mu$  is an Erlang  $(\mu, j)$  distribution with a PDF and CDF given in Eq. (1),  $P_1^1 = 1$ ,  $P_0^i = 0$  for each  $i \geq 1$ , and  $P_j^i$  for  $i \geq 2$  is:

$$P_j^i = \sum_{k=j-1}^{i-1} P_k^{i-1} a_{k-j+1}^\mu, \quad \forall j = 1, \dots, i. \quad (6)$$

**Proof.** We argue by induction. For  $i = 1$  the distribution of  $W_1$  is exponential. Thus  $P_1^1 = 1$  and  $W_1 \stackrel{d}{=} E_1^\mu$ , so the claim holds for  $i = 1$ . Now, we assume that for  $i - 1 \geq 1$

$$W_{i-1} \stackrel{d}{=} \sum_{j=1}^{i-1} P_j^{i-1} E_j^\mu, \quad (7)$$

and show that the claim holds for  $i$ .

We prove the Theorem using the following observation: For each  $j = 1, \dots, i$  we have  $W_i \stackrel{d}{=} E_j^\mu$ , if two independent event happened. The first is that  $W_{i-1} \stackrel{d}{=} E_k^\mu$  for  $k = j - 1, \dots, i - 1$ ,<sup>2</sup> and the second event is that during the interarrival time of the  $i^{\text{th}}$  customer there were exactly  $l = k + 1 - j$  service completions. That is, upon arrival, the  $i - 1^{\text{th}}$  customer is the  $k + 1^{\text{th}}$  customer in the system (queue+service), then  $k + 1 - j$  service completions take place until the arrival of the  $i^{\text{th}}$  customer. These two independent events lead to that  $j$  customers are seen by the  $i^{\text{th}}$  arrival. Furthermore,  $P(W_{i-1} \stackrel{d}{=} E_j^\mu) = P_j^{i-1}$  for  $k = j - 1, \dots, i - 1$  and the probability that exactly  $k + 1 - j$  services are completed during an interarrival time is  $a_{k+1-j}^\mu$ . Thus, because the  $i^{\text{th}}$  customer can see  $j$  customers in the system only if  $W_{i-1}$  saw at least  $j - 1$ , we have

$$\begin{aligned} P(W_i \stackrel{d}{=} E_j^\mu) &= \sum_{k=j-1}^{i-1} P(W_{i-1} \stackrel{d}{=} E_k^\mu) a_{k+1-j}^\mu \\ &= \sum_{k=j-1}^{i-1} P_k^{i-1} a_{k+1-j}^\mu, \end{aligned} \quad (8)$$

where the last equality follows from the induction assumption in Eq. (7). Observing that the  $i^{\text{th}}$  customer can see 1 to  $i$  customers in the system upon arrival completes the proof. ■

We conclude this subsection by establishing that the distribution given for  $W_i$  in Theorem 1 is proper. This is equivalent to proving that

$$\sum_{j=1}^i P_j^i = \sum_{j=1}^i \sum_{k=j-1}^{i-1} P_k^{i-1} a_{k-j+1}^\mu = E(I\{N \geq i\}). \quad (9)$$

To establish Eq. (9) we observe that given  $W_{i-1} \stackrel{d}{=} E_k^\mu$  the busy cycle ends if

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<sup>2</sup>Of course with  $j = 1$ ,  $k = 1, \dots, i - 1$ , so we could define in general  $k = \max\{j - 1, 1\}, \dots, i - 1$ . But, we preferred to define  $P_0^i = 0$  for each  $i$ . Both definitions are equivalent as is evident in Eq. (8).

there were  $k+1$  or potentially more service completions during the interarrival time of the  $i^{\text{th}}$  customer. Thus,

$$P\left(N \geq i | W_{i-1} \stackrel{d}{=} E_k^\mu\right) = E\left(I\left\{N \geq i | W_{i-1} \stackrel{d}{=} E_k^\mu\right\}\right) = \sum_{l=0}^k a_l^\mu \quad (10)$$

and using Eq. (10)

$$\begin{aligned} \sum_{j=1}^i \sum_{k=j-1}^{i-1} P_k^{i-1} a_{k-j+1}^\mu &= \sum_{k=0}^{i-1} P_k^{i-1} \sum_{j=1}^{k+1} a_{k-j+1}^\mu = \sum_{k=1}^{i-1} P_k^{i-1} \sum_{l=0}^k a_l^\mu \\ &= \sum_{k=1}^{i-1} P_k^{i-1} P\left(N \geq i | W_{i-1} \stackrel{d}{=} E_k^\mu\right) = E\left(I\{N \geq i\}\right), \end{aligned} \quad (11)$$

where the last from the Total Expectation Theorem. This establishes Eq. (9).

## 2.2 For $G/M/c$ Queues

Consider a  $G/M/c$  queue, where each server has an exponential service time with mean  $\mu^{-1}$ , as was investigated by Asmussen [2] and Wolff [6]. The busy period in this queue is the time from an arrival that makes all  $c$  servers busy to the first time when only  $c-1$  servers are busy. Denoting the waiting time of the  $i^{\text{th}}$  customer in a busy period of a  $G/M/c$  queue by  $W_i^c$ , similar proof to the one of Theorem 1 establishes:

**Corollary 1** *For a  $G/M/c$  queue with interarrival times  $Z$  with a CDF  $F_Z(z)$  and exponential service with mean  $\mu^{-1}$  we have*

$$W_i^c \stackrel{d}{=} \sum_{j=1}^i P_j^i E_j^{c\mu}, \quad (12)$$

where  $E_j^{c\mu}$  has an Erlang  $(c\mu, j)$  distribution with a PDF and CDF given in

Eq. (1),  $P_1^1 = 1$ ,  $P_0^i = 0$  for each  $i \geq 1$ , and  $P_j^i$  for  $i \geq 2$  is:

$$P_j^i = \sum_{k=j-1}^{i-1} P_k^{i-1} a_{k-j+1}^{c\mu}, \quad \forall j = 1, \dots, i. \quad (13)$$

### 2.3 The Busy Period in $G/M/1$ Queue and Random Walks

For the  $G/M/1$  queue with a FCFS service discipline let

$$X_i = S_i - Z_{i-1}, \quad \forall i \geq 1, \quad (14)$$

with  $Z_0 = 0$ . Thus,  $\{X_i\}_{i=2}^{\infty}$  is a sequence of *i.i.d.* random variables. Let  $X$  be the generic random variable of this sequence and assume that  $E(X) < 0$  (i.e.,  $E(Z) > 1/\mu$ ).

Consider the *random walk*  $\{V_i\}_{i=1}^{\infty}$  given by  $V_n = \sum_{i=1}^n X_i$  (with  $V_0 \equiv 0$ ) and observe that because  $E(X) < 0$ ,  $V$  tends to  $-\infty$  almost surely, e.g., Ross [5]. A *one-sided regulated random walk*  $\{Y_i\}_{i=1}^{\infty}$  that is regulated at 0 is given by:

$$Y_0 = 0 \text{ and } Y_{i+1} = \max\{0, Y_i + X_{i+1}\} \quad \forall i \geq 0. \quad (15)$$

Then, from the equivalence of the waiting time in a  $G/G/1$  queue with a FCFS service discipline to Eq. (15), known also as the Lindley recursion (e.g., Cohen [3]), and from Theorem 1 it follows that:

**Corollary 2** *For random walks with steps defined by Eq. (14) with  $S_i$  that follow an exponential distribution with mean  $\mu^{-1}$ , the PDF's of  $\{Y_i | \min_{0 < j \leq i} \{Y_j\} > 0\}$  and  $\{V_i | \min_{0 < j \leq i} \{V_j\} > 0\}$  are identical to this of  $W_i$  given in Eq. (5) of Theorem 1.*

### References

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