Technical Note No. 1*
Options, Futures, and Other Derivatives, Eighth Edition
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Convexity Adjustments to Eurodollar Futures

In the Ho-Lee model the risk-neutral process for the short rate in teh traditional risk-neutral world is

$$dr = \theta(t)dt + \sigma\, dz$$

where r is the instantaneous short rate, θ is a function of time, a and σ are constants, and dz is a Wiener process. Define $P(t,T)$ as the price of a bond paying \$1 at time T as seen at time t. As explained in Chapter 30 the bond price has the form $P(t,T) = A(t,T)e^{-r(T-t)}$. From Itô's lemma the process for the bond price in a traditional risk-neutral world is

$$dP(t,T) = r(t)P(t,T)dt - (T-t)\sigma P(t,T)\, dz$$

Define $f(t,T_1,T_2)$ as the forward rate (continuously compounded) at time t for the period between T_1 and T_2.

$$f(t,T_1,T_2) = \frac{\ln[P(t,T_1)] - \ln[P(t,T_2)]}{T_2 - T_1}$$

From Ito's lemma the process followed by $f(t,T_1,T_2)$ is

$$df(t,T_1,T_2) = \frac{\sigma^2(T_2-t)^2 - \sigma^2(T_1-t)^2}{2(T_2-T_1)}dt + \sigma\, dz$$

The expected change in the forward rate between time zero and time T_1 is determined by integrating the coefficient of dt between 0 and T_1. It is $\sigma^2 T_1 T_2/2$.

The forward rate equals the spot rate at time T_1. The expected value of the forward rate at time T_1 is therefore the expected value of the spot rate at time T_1. It follows that the forward rate at time zero equal the expected spot rate minus $\sigma^2 T_1 T_2/2$.

Because we are in the traditional risk-neutral world the expected value of the spot rate is the same as the futures rate. The futures rate is therefore greater than the forward rate by $\sigma^2 T_1 T_2/2$ when both are expressed with continuous compounding. (There is a small approximation here in that we know that the futures rate with quarterly compounding equals the expected future spot rate with quarterly compounding. We assume that the same is true when both both rates are converted continuous compounding.)

When converting the futures rate to the forward rate we should therefore subtract $\sigma^2 T_1 T_2/2$ from the futures rate. This is known as a convexity adjustment. As explained in Chapter 6, there are actually two parts to the convexity adjustment:

1. The difference between a futures contract that is settled daily and a similar contract that is settled entirely at time T_1
2. The difference between the contract that is settled at time T_1 and a similar contract that is settled at time T_2

This convexity adjustment is also discussed in Problem 30.13 and its answer.

To prove a corresponding result for the Hull-White model

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz$$

we proceed similarly. The process for the bond price is

$$dP(t,T) = r(t)P(t,T)dt - B(t,T)\sigma P(t,T)\,dz$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

The result after crunching through similar but more involved math is that the convexity adjustment is

$$\frac{B(T_1,T_2)}{T_2 - T_1}[B(T_1,T_2)(1 - e^{-2aT_1}) + 2aB(0,T_1)^2]\frac{\sigma^2}{4a}$$

This reduces to the earlier result when we take the limit as a tends to zero. See also Problem 30.14.