

Components of bull and bear markets: bull corrections and bear rallies

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This Draft: July 2011

Abstract

Existing methods of partitioning the market index into bull and bear regimes do not identify market corrections or bear market rallies. In contrast, our probabilistic model of the return distribution allows for rich and heterogeneous intra-regime dynamics. We focus on the characteristics and dynamics of bear market rallies and bull market corrections, including, for example, the probability of transition from a bear market rally into a bull market versus back to the primary bear state. A Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. We show how to compute the predictive density of long-horizon returns and discuss the improvements our model provides over benchmarks.

Key Words: predictive density, long-horizon returns, Markov switching

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1 Introduction

There is a widespread belief both by investors, policy makers and academics that low frequency trends do exist in the stock market. Traditionally these positive and negative low frequency trends have been labelled as bull and bear markets respectively. We propose a model that provides answers to typical questions such as, 'Are we in a bull market or a bear market rally?' or 'Will this bull market correction become a bear market?'. We propose a latent 4-state Markov-switching model for weekly stock returns. Our focus is on modeling the component states of bull and bear market regimes in order to identify and forecast bull, bull correction, bear and bear rally states.

Traditional methods of identifying bull and bear markets are based on an ex post assessment of the peaks and troughs of the price index. Formal dating algorithms based on a set of rules for classification are found, for example, in Gonzalez, Powell, Shi, and Wilson (2005), Lunde and Timmermann (2004) and Pagan and Sossounov (2003). Some of this work is related to the dating methods used to identify turning points in the business cycle (Bry and Boschan (1971)). A drawback is that a turning point can only be identified several observations after it occurs. Ex post dating methods cannot be used for statistical inference on returns or for investment decisions which require more information from the return distribution, such as changing risk assessments. For adequate risk management and investment decisions, we need a probability model for returns and one for which the distribution of returns changes over time.

Stock markets are perceived to have a cyclical pattern which can be captured with regime-switching models. For example, Hamilton and Lin (1996) relate business cycles and stock market regimes, Chauvet and Potter (2000) and Maheu and McCurdy (2000a) use a duration-dependent Markov-switching parameterization to analyze properties of bull and bear market regimes extracted from aggregate stock market returns. Lunde and Timmermann (2004) study duration dependence after sorting stock returns into either a bull or bear market using their dating algorithm. Ntantamis (2009) explores potential explanatory variables for stock market regimes' duration. Applications that explore the implications of nonlinearities due to regimes switches for asset allocation and/or predictability of returns include Turner, Startz, and Nelson (1989), van Norden and Schaller (1997), Maheu and McCurdy (2000b), Perez-Quiros and Timmermann (2001), Ang and Bekaert (2002) and Guidolin and Timmermann (2007).

In a related literature that investigates cyclical patterns in a broader class of assets, Guidolin and Timmermann (2005) use a 3-state regime-switching model to identify bull and bear markets in monthly UK stock and bond returns and analyze implications for predictability and optimal asset allocation. Guidolin and Timmermann (2006) add an additional state in order to model the nonlinear joint dynamics of monthly returns associated with small and large cap stocks and long-term bonds. Kim, Nelson, and Startz (1998) also find 3 states in the variance of monthly returns provides a good fit.

Moving these models to higher data frequencies tends to obscure any structure in the conditional mean in favor of the conditional variance dynamics. Our objective is to use higher-frequency weekly data and to provide a real-time approach to identifying phases of the market that relate to investors' perceptions of primary and secondary trends in aggregate stock returns. Existing approaches do not explicitly model bull market corrections and bear market rallies.

The bear and bear rally states govern the bear regime; the bull correction and bull states govern the bull regime. Therefore each regime consists of two primitive states. The model can accommodate short-term reversals (secondary trends) within each regime of the market. For example, in the bull regime it is possible to have a series of persistent negative returns (a bull correction), despite the fact that the expected long-run return (primary trend) is positive in that regime. Analogously, bear markets often exhibit persistent rallies which are subsequently reversed as investors take the opportunity to sell with the result that the average return in that regime is still negative.

It is important to note that our additional states allow for both intra and inter-regime transitions. A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. A bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state, exactly as many investors feel we observe in the data. Because, the realization of states in a regime will differ over time, bull and bear regimes can be heterogeneous over time. These important intra and inter-regime dynamics are absent in the existing literature.

Our Bayesian estimation approach accounts for parameter and regime uncertainty and provides probability statements regarding future regimes and returns. An important contribution of this paper is to show how to compute the predictive density of long-horizon returns. Model comparison based on the density of long-horizon returns is very informative in differentiating models and shows the importance of a richer specification of bull and bear phases.

Applied to 125 years of data our model provides superior identification of trends in stock prices. One important difference with our specification is that the richer dynamics in each regime, facilitated by our 4-state model, allow us to extract bull and bear markets in higher frequency data. A two-state Markov-switching model applied to higher frequency data results in too many switches between the high and low return states. It is incapable of extracting the low frequency trends in the market. In high frequency data it is important to allow for short-term reversals in the regime of the market. Relative to a two-state model we find that market regimes are more persistent and there is less erratic switching. According to Bayes factors, our 4-state model of bull and bear markets is strongly favored over several alternatives.

Bull regimes have an average duration of just under 5 years, while the duration of a bull correction is 4 months on average and a bear rally is just over half a year. The cumulative return mean of the bull market state is 7.88% but bull corrections offset this by 2.13% on average. Average cumulative return in the bear market state is -12.4% but bear market rallies counteract that steep decline by yielding a cumulative return of 7.1% on average. Note that these states are combined into bull and bear market regimes in heterogeneous patterns over time yielding an average cumulative return in the bull market regime of 33% while that for the bear market regime is about -10%. Although the average cumulative return in the bear rally state is not much less than in the bull market state, the latter state's standard deviation is 50% less. This result highlights the importance of also considering assessments of volatility associated with the alternative states, for example, when identifying bear market rallies versus bull markets.

The model identifies in real time a transition from a bull market correction to a bear market in early October 2008. The bear rally and bull correction states are critical to modeling turning points between regimes; our results show that most transitions between bull and bear regimes occur through these states. This is consistent with investors' perceptions. Further, we find asymmetries in intra-regime dynamics, for example, a bull market correction returns to the bull market state more often than a bear market rally reverts to the bear state. These are important features that the existing literature on bull and bear markets ignores.

Compared to a two state model and a GARCH(1,1) model our 4 state model provides superior density forecasts of long-horizon returns. The dynamics of our model become important at 3 month returns and remain important to annual returns. Thus, bear rallies and bull corrections have important implications for cumulative returns. We also show the importance of the model in a value-at-risk application.

This paper is organized as follows. The next section describes the data, Section 3 discusses two alternative ex post market regime dating algorithms. Section 4 summarizes the benchmark 2-state model and develops our proposed 4-state specification. Estimation and model comparison are discussed in Section 5. Section 6 presents results including: parameter estimates; probabilistic identification of the market states and regimes including real time bear market forecasts; Value-at-Risk forecasts; and out-of-sample density forecasts. Section 7 concludes.

2 Data

We begin with 125 years of daily capital gain returns on a broad market equity index. Our source for the period 1926-2008 inclusive is the value-weighted return excluding dividends associated with the CRSP S&P 500 index. The 1885:02-1925 daily capital gain returns are courtesy of Bill Schwert (see Schwert (1990)). For 2009-2010, we use

the daily rates of change of the S&P 500 index level (SPX) obtained from Reuters.

Returns are converted to daily continuously compounded returns from which we construct weekly continuously compounded returns by summing daily returns from Wednesday close to Wednesday close of the following week. If a Wednesday is missing, we use Tuesday close. If the Tuesday is also missing, we use Thursday. Weekly realized variance (RV) is computed as the sum of daily (intra-week) squared returns.

Weekly returns are scaled by 100 so they are percentage returns. Unless otherwise indicated, henceforth returns refer to weekly continuously compounded returns expressed as a percentage. We have 6498 weekly observations covering the period February 25, 1885 to January 20, 2010. Summary statistics are shown in Table 1.

3 Bull and Bear Dating Algorithms

Ex post sorting methods for classification of stock returns into bull and bear phases are called dating algorithms. Such algorithms attempt to use a sequence of rules to isolate patterns in the data. A popular algorithm is that used by Bry and Boschan (1971) to identify turning points of business cycles. Pagan and Sossounov (2003) adapted this algorithm to study the characteristics of bull/bear regimes in monthly stock prices. First a criterion for identifying potential peaks and troughs is applied; then censoring rules are used to impose minimum duration constraints on both phases and complete cycles. Finally, an exception to the rule for the minimum length of a phase is allowed to accommodate 'sharp movements' in stock prices.

The Pagan and Sossounov (2003) BB algorithm is summarized in the appendix. There are alternative dating algorithms or filters for identifying turning points. For example, the Lunde and Timmermann (2004) (LT) algorithm identifies bull and bear markets using a cumulative return threshold of 20% to locate peaks and troughs moving forward. They define a binary market indicator variable I_t which takes the value 1 if the stock market is identified by their algorithm to be in a bull state at time t and 0 if it is in a bear state. Our application of this LT dating algorithm is also summarized in the appendix.

The classification of our data into bull and bear regimes using these two filters is found in Figure 1. The shaded portions under the cumulative return denote bull markets while the white portions of the figure are the bear markets. The exact dates for the bull and bear regimes can be found in Table 2 of the working paper version. There are several features to note. First, the sorting of the data is broadly similar but with important differences. For example, during the 1930s the BB approach finds many more switches between market phases than does the LT algorithm. More recently, both identify 1987-12 as a trough but the subsequent bull phase ends in 1990-06 for LT but 2000-03 for BB. The average bear duration is similar (66 weeks) while the average bull duration is quite

different, 117.0 weeks (BB) versus 166.7 (LT). In other words, the different parameters and assumptions in the filtering methods can result in a different classification of market phases.

Although the ex post dating algorithms can filter the data to locate different regimes, they cannot be used for forecasting or inference. In addition, since the sorting rule focuses on the first moment, it does not characterize the full distribution of returns. The latter is required if we wish to derive features of the regimes that are useful for measuring and forecasting risk. Also, as noted above, ex post dating algorithms sort returns into a particular regime with probability zero or one. However, the data provides more information allowing one to estimate probabilities associated with particular states.

Nevertheless, the dating algorithms are still very useful. For example, we use the LT algorithm to sort data simulated from our candidate parametric models in order to determine whether the latter can match commonly perceived features of bull and bear markets.

4 Models

In this section, we briefly review a benchmark two-state model, our proposed 4-state model, and some alternative specifications of the latter used to evaluate robustness of our best model.

4.1 Two-State Markov-Switching Model

The concept of bull and bear markets suggests cycles or trends that get reversed. Since those regimes are not observable, as discussed in Section 1, two-state latent-variable MS models have been applied to stock market data. A two-state 1st-order Markov model can be written

$$r_t | s_t \sim N(\mu_{s_t}, \sigma_{s_t}^2) \quad (4.1)$$

$$p_{ij} = p(s_t = j | s_{t-1} = i) \quad (4.2)$$

$i = 1, 2, j = 1, 2$. We impose $\mu_1 < 0$ and $\mu_2 > 0$ so that $s_t = 1$ is the bear market and $s_t = 2$ is the bull market.

Modeling of the latent regimes, regime probabilities, and state transition probabilities, allows explicit model estimation and inference. In addition, in contrast to dating algorithms or filters, forecasts are possible. Investors can base their investment decisions on the posterior states or the whole forecast density.

4.2 MS-4 to allow Bull Corrections and Bear Rallies

Consider the following general K -state first-order Markov-switching model for returns

$$r_t | s_t \sim N(\mu_{s_t}, \sigma_{s_t}^2) \quad (4.3)$$

$$p_{ij} = p(s_t = j | s_{t-1} = i) \quad (4.4)$$

$i = 1, \dots, K, j = 1, \dots, K$. We explore a 4-state model, $K = 4$, in order to focus on modeling potential phases of the aggregate stock market. Without any additional restrictions we cannot identify the model and relate it to market phases. Therefore, we consider the following restrictions. First, the states $s_t = 1, 2$ are assumed to govern the bear market; *we label these two states as the bear regime*. The states $s_t = 3, 4$ are assumed to govern the bull market; *these two states are labelled the bull regime*. Each regime has 2 states which allows for positive and negative periods of price growth within each regime. In particular we impose

$$\begin{array}{l} \text{Bear regime} \\ \text{Bull regime} \end{array} \quad \begin{cases} \mu_1 < 0 \text{ (bear market state),} \\ \mu_2 > 0 \text{ (bear market rally),} \\ \mu_3 < 0 \text{ (bull market correction),} \\ \mu_4 > 0 \text{ (bull market state).} \end{cases} \quad (4.5)$$

This structure can capture short-term reversals in market trends. Each state can have a different variance and can accommodate autoregressive heteroskedasticity in returns. In addition, conditional heteroskedasticity within each regime can be captured.

Consistent with the 2 states in each regime the full transition matrix is

$$P = \begin{pmatrix} p_{11} & p_{12} & 0 & p_{14} \\ p_{21} & p_{22} & 0 & p_{24} \\ p_{31} & 0 & p_{33} & p_{34} \\ p_{41} & 0 & p_{43} & p_{44} \end{pmatrix}. \quad (4.6)$$

This structure allows for several important features that are excluded in the smaller Markov-switching models in the literature. First, a bear regime can feature several episodes of the bear state and bear rally state, exactly as many investors feel we observe in the data. Similarly, the bull regime can be characterized by a combination of bull states and bull corrections. Because, the realization of states in a regime will differ over time, both bull and bear regime will tend to look heterogeneous to some extent. For instance, based on returns, a bear regime lasting 5 periods made of the states

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2, s_{t+4} = 2, s_{t+5} = 2, s_{t+6} = 4$$

will tend to look very different than

$$s_t = 1, s_{t+1} = 1, s_{t+2} = 1, s_{t+3} = 2, s_{t+4} = 1, s_{t+5} = 1, s_{t+6} = 4.$$

A second important contribution is that a bear rally is allowed to move either into the bull state or back to the bear state; analogously, a bull correction can move to a bear state or back to the bull state. These important inter and intra-regime dynamics are absent in the existing literature.

The unconditional probabilities associated with P can be solved (Hamilton (1994))

$$\pi = (A'A)^{-1}A'e \quad (4.7)$$

where $A' = [P' - I, \iota]$ and $e' = [0, 0, 0, 0, 1]$ and $\iota = [1, 1, 1, 1]'$.

Using the matrix of unconditional state probabilities given by (4.7), we impose the following conditions on long-run returns in the bear and bull regimes respectively

$$E[r_t | \text{bear regime}, s_t = 1, 2] = \frac{\pi_1}{\pi_1 + \pi_2} \mu_1 + \frac{\pi_2}{\pi_1 + \pi_2} \mu_2 < 0 \quad (4.8)$$

$$E[r_t | \text{bull regime}, s_t = 3, 4] = \frac{\pi_3}{\pi_3 + \pi_4} \mu_3 + \frac{\pi_4}{\pi_3 + \pi_4} \mu_4 > 0. \quad (4.9)$$

We do not impose any constraint on the variances.

The equations (4.5) and (4.6), along with equations (4.8) and (4.9), serve to identify bull and bear regimes in the MS-4 model. The bull (bear) regime has a long-run positive (negative) return. Each market regime can display short-term reversals that differ from their long-run mean. For example, a bear regime can display a bear market rally (temporary period of positive returns), even though its long-run return is negative. Similarly for the bull market.

4.3 Other Models for Robustness Checks

Besides the 2-state and 4-state model we consider several other specifications and provide model comparisons among them. The dependencies in the variance of returns are the most dominate feature of the data. This structure may adversely dominate dynamics of the conditional mean. The following specifications are included to investigate this issue.

1. Restricted 4-state model. This is identical to the 4-state model in Section 4.2 except that inside a regime the return innovations are homoskedastic. That is, $\sigma_1^2 = \sigma_2^2$ and $\sigma_3^2 = \sigma_4^2$. In this case, the variance within each regime is restricted to be constant although the overall variance of returns can change over time due to switches between regimes.
2. Markov-Switching with Decoupled Conditional Mean and Variance. In this model,

the mean and variance dynamics are decoupled and directed by independent latent Markov chains. This is a robustness check to determine to what extent the variance dynamics might be driving the regime transitions. This specification has the following structure

$$r_t|s_t, w_t \sim N(\mu_{s_t}, \sigma_{w_t}^2) \quad (4.10)$$

$$p_{ij} = p(s_t = j|s_{t-1} = i), \quad i, j = 1, \dots, K \quad (4.11)$$

$$q_{ij} = p(w_t = j|w_{t-1} = i), \quad i, j = 1, \dots, L. \quad (4.12)$$

We focus on the case $K = 4$ and $L = 4$, again to allow us to capture at least four phases of cycles for aggregate stock returns. The conditional means have the same restriction imposed as the MS-4 while the conditional variances have no prior restriction.

3. A GARCH(1,1) model

$$r_t|I_{t-1} \sim N(\mu, \sigma_t^2) \quad (4.13)$$

$$\sigma_t^2 = \omega + \alpha(r_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2 \quad (4.14)$$

where $I_{t-1} = \{r_1, \dots, r_{t-1}\}$. This model captures volatility dynamics but does not model changes in the conditional mean.

4. MS-4 with $p_{24} = p_{31} = 0$ which restricts the moves between bull and bear regime to be from state 1 to 4 and state 4 to 1.
5. MS-4 model with unrestricted transition matrix. In this case all 16 elements of P are estimated.

5 Estimation and Model Comparison

5.1 Estimation

In this section we discuss Bayesian estimation for the most general model introduced in Section 4.2 assuming there are K states, $k = 1, \dots, K$. The other models are estimated in a similar way with minor modifications.

There are 3 groups of parameters $M = \{\mu_1, \dots, \mu_K\}$, $\Sigma = \{\sigma_1^2, \dots, \sigma_K^2\}$, and the elements of the transition matrix P . Let $\theta = \{M, \Sigma, P\}$ and given data $I_T = \{r_1, \dots, r_T\}$ we augment the parameter space to include the states $S = \{s_1, \dots, s_T\}$ so that we sample from the full posterior $p(\theta, S|I_T)$. Assuming conditionally conjugate priors $\mu_i \sim N(m_i, n_i^2)$, $\sigma_i^{-2} \sim G(v_i/2, w_i/2)$ and each row of P following a Dirichlet distribution, allows for a Gibbs sampling approach following Chib (1996). Gibbs sampling

iterates on sampling from the following conditional densities given startup parameter values for M , Σ and P :

$$\begin{aligned} S|M, \Sigma, P \\ M|\Sigma, P, S \\ \Sigma|M, P, S \\ P|M, \Sigma, S \end{aligned}$$

Sequentially sampling from each of these conditional densities results in one iteration of the Gibbs sampler. Dropping an initial set of draws to remove any dependence from startup values, the remaining draws $\{S^{(j)}, M^{(j)}, \Sigma^{(j)}, P^{(j)}\}_{j=1}^N$ are collected to estimate features of the posterior density. Simulation consistent estimates can be obtained as sample averages of the draws. For example, the posterior mean of the state dependent mean and standard deviation of returns are estimated as

$$\frac{1}{N} \sum_{j=1}^N \mu_k^{(j)}, \quad \frac{1}{N} \sum_{j=1}^N \sigma_k^{(j)}, \quad (5.1)$$

for $k = 1, \dots, K$ and are simulation consistent estimates of $E[\mu_k|I_T]$ and $E[\sigma_k|I_T]$ respectively.

The first sampling step of $S|M, \Sigma, P$ involves a joint draw of all the states. Chib (1996) shows that this can be done by a so-called forward and backward smoother. The second and third sampling steps are straightforward and use results from the linear regression model. Given the conjugate Dirichlet prior on each row of P , the final step is to sample $P|M, \Sigma, S$ from the Dirichlet distribution (Geweke (2005)).

An important byproduct of Gibbs sampling is an estimate of the smoothed state probabilities $p(s_t|I_T)$ which can be estimated as

$$p(\widehat{s_t = i}|I_T) = \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{s_t=i}(S^{(j)}) \quad (5.2)$$

for $i = 1, \dots, K$.

At each step, if a parameter draw violates any of the prior restrictions in (4.5), (4.6), (4.8) and (4.9), then it is discarded. For the 4-state model we set the independent priors as

$$\mu_1 \sim \mathbf{N}(-0.7, 1), \mu_2 \sim \mathbf{N}(0.2, 1), \mu_3 \sim \mathbf{N}(-0.2, 1), \mu_4 \sim \mathbf{N}(0.3, 1) \quad (5.3)$$

$$\sigma_i^{-2} \sim G(0.5, 0.05) \text{ for } i = 1, 2, 3, 4 \quad (5.4)$$

$$\{p_{11}, p_{12}, p_{14}\} \sim Dir(8, 1.5, 0.5), \{p_{21}, p_{22}, p_{24}\} \sim Dir(1.5, 8, 0.5) \quad (5.5)$$

$$\{p_{31}, p_{33}, p_{34}\} \sim Dir(0.5, 8, 1.5), \{p_{41}, p_{43}, p_{44}\} \sim Dir(0.5, 1.5, 8). \quad (5.6)$$

These priors are informative but cover a wide range of empirically relevant parameter

values.

5.2 Model Comparison

If the marginal likelihood can be computed for a model it is possible to compare models based on Bayes factors. Non-nested models can be compared as well as specifications with a different number of states. Note that the Bayes factor compares the out-of-sample prediction record of models and penalizes over-parameterized models that do not deliver improved predictions. This is referred to as an Ockham's razor effect. See Kass and Raftery (1995) for a discussion on the benefits of Bayes factors. For the general Markov-switching model with K states, the marginal likelihood for model \mathcal{M}_i is defined as

$$p(r|\mathcal{M}_i) = \int p(r|\mathcal{M}_i, \theta)p(\theta|\mathcal{M}_i)d\theta \quad (5.7)$$

which integrates out parameter uncertainty. $p(\theta|\mathcal{M}_i)$ is the prior and

$$p(r|\mathcal{M}_i, \theta) = \prod_{t=1}^T f(r_t|I_{t-1}, \theta) \quad (5.8)$$

is the likelihood which has S integrated out according to

$$f(r_t|I_{t-1}, \theta) = \sum_{k=1}^K f(r_t|I_{t-1}, \theta, s_t = k)p(s_t = k|\theta, I_{t-1}). \quad (5.9)$$

The term $p(s_t = k|\theta, I_{t-1})$ is available from the Hamilton filter. Chib (1995) shows how to estimate the marginal likelihood for MS models. His estimate is based on re-arranging Bayes' theorem as

$$\widehat{p(r|\mathcal{M}_i)} = \frac{p(r|\mathcal{M}_i, \theta^*)p(\theta^*|\mathcal{M}_i)}{p(\theta^*|r, \mathcal{M}_i)} \quad (5.10)$$

where θ^* is a point of high mass in the posterior pdf. The terms in the numerator are directly available above while the denominator can be estimated using additional Gibbs sampling runs. The integrating constant in the prior pdf is estimated by simulation.

A log-Bayes factor between model \mathcal{M}_i and \mathcal{M}_j is defined as

$$\log(BF_{ij}) = \log(p(r|\mathcal{M}_i)) - \log(p(r|\mathcal{M}_j)). \quad (5.11)$$

Kass and Raftery (1995) suggest interpreting the evidence for \mathcal{M}_i versus \mathcal{M}_j as: not worth more than a bare mention for $0 \leq \log(BF_{ij}) < 1$; positive for $1 \leq \log(BF_{ij}) < 3$; strong for $3 \leq \log(BF_{ij}) < 5$; and very strong for $\log(BF_{ij}) \geq 5$.

5.3 Predictive Density

An important feature of our probabilistic approach is that a predictive density of future returns can be computed that integrates out all uncertainty regarding states and parameters. We first discuss the predictive density of the one period ahead return. Then we show how to compute the predictive density of multiperiod returns (cumulative log-returns). Model comparison based on the density of multiperiod returns is a new result and as we show very informative in differentiating model.

The predictive density for future returns based on current information at time t is computed as

$$p(r_{t+1}|I_t) = \int f(r_{t+1}|I_t, \theta)p(\theta|I_t)d\theta \quad (5.12)$$

which involved integrating out both state and parameter uncertainty using the posterior distribution $p(\theta|I_t)$. From the Gibbs sampling draws $\{S^{(j)}, M^{(j)}, \Sigma^{(j)}, P^{(j)}\}_{j=1}^N$ based on data I_t we approximate the predictive density as

$$p(\widehat{r_{t+1}}|I_t) = \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^K f(r_{t+1}|\theta^{(i)}, s_{t+1} = k)p(s_{t+1} = k|s_t^{(i)}, \theta^{(i)}) \quad (5.13)$$

where $f(r_{t+1}|\theta^{(i)}, s_t = k)$ follows $N(\mu_k^{(i)}, \sigma_k^{2(i)})$ and $p(s_{t+1} = k|s_t^{(i)}, \theta^{(i)})$ is the transition probability.

The predictive mean of a future state s_{t+1} can also be easily estimated by simulating from the distribution $p(s_{t+1} = k|s_t^{(i)}, \theta^{(i)})$ a state $s_{t+1}^{(j)}$ for each state and parameter draw $s_t^{(i)}, \theta^{(i)}$. The average of these draws, $\{s_{t+1}^{(j)}\}_{j=1}^N$ is an estimate of $E[s_{t+1}|I_t]$.

The predictive density can also be computed for multiperiod returns. Define the h -period return as $r_{t,h} = \sum_{i=1}^h r_{t+i}$. Then the predictive density for the multiperiod return is

$$p(r_{t,h}|I_t) = \int \left[\prod_{j=1}^{h-1} f(r_{t+j}|\theta, s_{t+j}) \right] f\left(r_{t,h} - \sum_{j=1}^{h-1} r_{t+j} \middle| \theta, s_{t+h}\right) \\ \times \left[\prod_{j=2}^h p(s_{t+j}|s_{t+j-1}, \theta) \right] p(s_{t+1}|I_t, \theta)p(\theta|I_t)d\theta ds_{t+1} \dots ds_{t+h} dr_{t+1} \dots dr_{t+h-1}. \quad (5.14)$$

This integrates out all parameter uncertainty and future state uncertainty as well as all possible sample paths of returns $\{r_{t+1}, \dots, r_{t+h-1}\}$ such that $r_{t,h} = \sum_{i=1}^h r_{t+i}$. The predictive density can be approximated as

$$p(\widehat{r_{t,h}}|I_t) = \frac{1}{N} \sum_{i=1}^N f\left(r_{t,h} - \sum_{j=1}^{h-1} r_{t+j}^{(i)} \middle| \theta^{(i)}, s_{t+h}^{(i)}\right)$$

Given $\theta^{(i)}, s_t^{(i)}$ from the posterior we simulate out $\{s_{t+j}^{(i)}, r_{t+j}^{(i)}\}_{j=1}^{h-1}$ and $s_{t+h}^{(i)}$ from the MS-4 and evaluate the sum above at the data $r_{t,h}$. Similar calculations are used for the other models we compare to.

Evaluating the predictive density at the data $r_{t,h}$ gives the predictive likelihood. Models can be compared on their out-of-sample performance for observations τ_1 to τ_2 , $\tau_1 < \tau_2$ based on the cumulative log predictive likelihood,

$$\sum_{t=\tau_1}^{\tau_2} \log(\widehat{p}(r_{t,h}|I_t)). \quad (5.15)$$

To compute this the model must be re-estimated at each point in the out-of-sample period. As with the marginal likelihood, better models will have a larger cumulative log predictive likelihood value. A model with a larger value is better able to account for the data.

6 Results

6.1 Parameter Estimates and Implied Distributions

Model estimates for the 2-state Markov-switching (MS-2) model are found in Table 2. State 1 has a negative conditional mean along with a high conditional variance whereas state 2 displays a high conditional mean with a low conditional variance. Both regimes are very persistent. These results are consistent with the sorting of bull and bear regimes in Maheu and McCurdy (2000a) and Guidolin and Timmermann (2005).

Estimates for our proposed 4-state model (MS-4) are found in Table 3. All parameters are precisely estimated indicating that the data are quite informative. Recall that states $s_t = 1, 2$ capture the bear regime while states $s_t = 3, 4$ capture the bull regime. Each regime contains a state with a positive and a negative conditional mean. We label states 1 and 2 the bear and bear rally states respectively; states 3 and 4 are the bull correction and bull states.

Consistent with the MS-2 model, volatility is highest in the bear regime. In particular, the highest volatility occurs in the bear regime in state 1. This state also delivers the lowest average return. The highest average return and lowest volatility is in state 4 which is part of the bull regime. The bear rally state ($s_t = 2$) delivers a conditional mean of 0.23 and conditional standard deviation of 2.63. However, this mean is lower and the volatility higher than the bull positive growth state ($s_t = 4$). Analogously, the bull correction state ($s_t = 3$) has a larger conditional mean ($-0.13 > -0.94$) and smaller volatility ($2.18 < 6.01$) than the bear state 1.

State 4 provides the most favorable risk-return tradeoff followed by state 2, 3 and 1. Note that the bull state 4 delivers a larger expected return with a standard deviation

that is 50% smaller than the bear rally (state 2). Even though the bear rally delivers a positive expected return, that return is much more variable than in the bull state.

All states display high persistence (p_{ii} is high for all i). However, the transition probabilities display some asymmetries. For example, the probability of a bear rally moving back to the bear state 1 ($p_{21} = 0.015$) is a little lower than changing regime to a bull market ($p_{24} = 0.019$). On the other hand, the probability of a bull correction returning to a bull market ($p_{34} = 0.051$) is considerably higher than changing regime to the bear state ($p_{31} = 0.010$).

Figure 2 displays the density of each of the 4 states. The differences in the illustrated densities are in accord with the parameter estimates in Table 3. Differences in the spreads of the densities are most apparent but the locations are also different. There is no suggestion from these plots that states 1 and 2 are the same or that states 3 and 4 are the same, as a two-state Markov-switching model would assume.

What do the dynamics of the MS-4 model imply for investors? First, the probability of a positive return in any state is high with $P(r_t > 0|s_t) = 0.44, 0.53, 0.48,$ and 0.59 respectively for states $s_t = 1, 2, 3, 4$. Second, investing in any state eventually results in future returns being positive. For instance, the $H = \arg \min\{E(r_{t+H}|s_t) > 0\}$ is 37 ($s_t = 1$), 1 ($s_t = 2$), 15 ($s_t = 3$) and 1 ($s_t = 4$). Therefore, investors concerned with long investment horizons or obtaining a high probability of a positive return may find investing in states 1 and 3 desirable.

Integrating state 1 and 2 gives the bear regime and doing the same for states 3 and 4 produces the bull regime. These densities are shown in Figure 3. The bear regime has a mean slightly below 0 but with a much larger variance than the bull regime. The implied unconditional density of returns is a mixture of these two regimes and displayed in the middle of the figure. Table 4 reports the unconditional probabilities for the states. On average the market spends 0.157 of time in a bear rally while 0.304 in a bull correction. The most time is spent in the bull growth state 4. The unconditional probability of the bull regime is 0.773.

A comparison of the regime statistics implied by the parameter estimates for the MS-2 and MS-4 models is found in Table 5. The expected duration of regimes is much longer in the 4-state model. That is, by allowing heterogeneity within a regime in our 4-state model, we switch between bull and bear markets less frequently. For instance, in a MS-2 parameterization the bull market has a duration of only 82.6 weeks, about 18 months, while the richer MS-4 model has a bull duration of just under 5 years. As we will see below, there is much more switching between regimes in the MS-2 model.

In the 2-state model, the expected return and variance are fixed within a regime. For the bear regime in the MS-2 model, the expected variance is $E[\text{Var}(r_t|s_t = 1)] = 19.6$. In contrast, the average variance for each regime in the 4-state model can be attributed to changes in the conditional mean as well as to the average conditional variance of the

return innovations. For instance, the average variance of returns in the bear regime can be decomposed as $\text{Var}(r_t|s_t = 1, 2) = \text{Var}(E[r_t|s_t]|s_t = 1, 2) + E[\text{Var}(r_t|s_t)|s_t = 1, 2] = 0.31 + 16.1$, with a similar result for the bull regime. For the bull and bear phase, the mean dynamics account for a small share, 2% of the total variance.

The MS-2 model assumes normality in both market regimes while the MS-4 shows that the data is at odds with this assumption. Skewness is present in bear markets while excess kurtosis is found in both bull and bear regimes. Overall the bear market deviates more from a normal distribution; it has thicker tails and captures more extreme events.

Table 6 summarizes features of the MS-4 parameterization for both the regimes and their component states derived from the posterior parameter estimates. The bear regime duration is 77.8 weeks, much shorter than the bull regime duration of 256.0 weeks. The average cumulative return in the bear (bull) regime is -9.94 (33.0). The volatility in the bear market is more than twice that in the bull market. The third panel provides a breakdown of cumulative return means in each of the component states of the market regimes. The bear rally yields a cumulative return of 7.10 on average which partially offsets the average decline of -12.4 in state 1. On the other hand the bull correction has a cumulative return mean of -2.13 which diminishes the average cumulative return of 7.88 in state 4. Note that these states are combined into bull and bear market regimes in heterogeneous patterns over time yielding the statistics for regimes summarized in the first two panels of Table 6.

Although the stock market spends most of the time in the bull regime (states 3 and 4), in terms of individual states it is state 2 that has the longest duration while the shortest is state 1.

6.2 Model Comparisons

We conduct formal model comparisons based on the marginal likelihoods reported in Table 7. The constant mean and variance model performs the worst (has the lowest marginal likelihood). The next model is the MS-2 followed by the restricted MS-4, $\sigma_1^2 = \sigma_2$, $\sigma_3^2 = \sigma_4^2$ (model 1 in Section 4.3). Next is the decoupled model with 4 states in the conditional mean and an independent 4 states in the conditional variance (model 2 in Section 4.3). Following this is a GARCH(1,1) model (model 3 in Section 4.3) and an MS-4 with $p_{24} = p_{31} = 0$ (model 4 in Section 4.3). Following this is an MS-4 with an unrestricted transition matrix (model 5 in Section 4.3). The last specification is our preferred MS-4 model and the final column of the Table presents log-Bayes factor for this model against the alternatives.

The log-Bayes factor between the 2-state MS and the 4-state MS in the conditional mean restricted to have only a 2-state conditional variance is large at $53.4 = -13849.9 - (-13903.3)$. This improved fit comes when additional conditional mean dynamics (going from 2 to 4 states) are added to the basic 2-state MS model. The best model is the 4-

state Markov-switching model. The log-Bayes factor in support of the 4-state versus the 2-state model is $162.9 = -13740.4 - (-13903.3)$. The zero restrictions in the transition matrix (4.6) are also strongly supported by the data. For instance, the log-Bayes factor is $6.9 = -13740.4 - (-13747.3)$ in support of the MS-4 model with P matrix (4.6) as compared to a 4 state model with an unrestricted transition matrix (all 16 elements of P are estimated). The evidence is also against additional restrictions on P such as $p_{24} = p_{31} = 0$. The data favor the more flexible transitions of the MS-4 between bull and bear regimes.

Our 4-state model also dominates the more flexible model with 4 states for conditional mean and 4 independent states for the conditional variance (decoupled 4 state model). Similarly the MS-4 model improves upon the GARCH(1,1) model.

Overall, there is very strong evidence that the 4-state specification of Section 4.2 provides the best fit to weekly returns. The comparisons also show that this improved fit comes from improved fit to both the conditional mean and variance. Not only does our MS-4 model provide a better economic characterization of differences in stock market cycles but the model statistically dominates other alternatives.

The Markov-switching models specify a latent variable that directs low frequency trends in the data. As such, the regime characteristics from the population model are not directly comparable to the dating algorithms of Section 3. Instead, we consider the dating algorithm as a lens to view both the S&P500 data and data simulated from our preferred MS-4 model. Using parameter draws from the Gibbs sampler, we simulate return data from the model and then apply the LT dating algorithm to those simulated returns. This is done many times and the average and 0.70 density intervals of these statistics are reported in Table 8 along with the statistics from the S&P500 data. Although our model provides a richer 4 state description of bull and bear markets it does account for all of the data statistics associated with a simpler 2 state view of the market using the LT dating algorithm.

6.3 Identification of Historical Turning Points in the Market

The dating of the market regimes using the LT dating algorithm are found in the top panel of Figure 4. Below this panel is the smoothed probability of a bull market, $p(s_t = 3|I_T) + p(s_t = 4|I_T)$ for the 4-state model. The final plot in Figure 4 is the smoothed probability of a bull market, $p(s_t = 2|I_T)$ from the 2-state model. The 4-state model produces less erratic shifts between market regimes, closely matches the trends in prices, and generally corresponds to the dating algorithm. The 2-state model is less able to extract the low frequency trends in the market. In high frequency data it is important to allow intra-regime dynamics, such as short-term reversals.

Note that the success of our model should not be based on how well it matches the results from dating algorithms. Rather this comparison is done to show that the

latent-state MS models can identify bull and bear markets with similar features to those identified by conventional dating algorithms. Beyond that, the Markov-switching models presented in this paper provide a superior approach to modeling stock market trends as they deliver a full specification of the distribution of returns along with latent market dynamics. Such an approach permits out-of-sample forecasting which we turn to in Section 6.4.

The following subsections discuss how our model identifies sub-regime dynamics using examples from various subperiods. There are several important points revealed by this discussion. First, bear (bull) markets are persistent but are made of many regular transitions between states 1 and 2 (3 and 4). Second, in each of the examples the move between regimes occurs through either the bear rally or the bull correction state. In other words, these additional dynamics are critical to fully capturing turning points in stock market cycles. This is also borne out by our model estimates. The most likely route for a bear market to go to a bull market is through the bear rally state. Given that a bull market has just started, the probability is 0.9342 that the previous state was a bear rally and only 0.0658 that it was a bear state (i.e. $p(s_t = 2 | s_{t+1} = 4, s_t = 1 \text{ or } 2) \propto \frac{p_{24}\pi_2}{\pi_1 + \pi_2}$). Similarly, given that a bear market has just started, the probability is 0.8663 that the previous state was a bull correction, and only 0.1337 that it was a bull state. The following subperiod descriptions provide examples of this richer specification of turning points plus frequent reversals within a regime.

6.3.1 1927-1939

Figure 5 displays the log-price and the realized volatility (square root of realized variance) in the top panel, the smoothed states of the MS-4 model in the second panel, and the posterior probability of the bull market, $p(s_t = 3 | I_T) + p(s_t = 4 | I_T)$, in the last panel.

Just before the crash of 1929 the model identifies a bull correction state. The transition from a bull to bear market occurs as a move from a bull market state to a bull correction state and then into the bear regime. For the week ending October 16 1929, there was a return of -3.348 and the market transitioned from the bull correction state into the bear market state with $p(s_t = 1 | I_T) = 0.63$. This is further reinforced so that the next 5 weeks have essentially probability 1 for state 1.

As this figure shows, the remainder of this subperiod is decisively a bear market, but displays considerable heterogeneity in that there are several short-lived bear rallies. The high levels of realized volatility coincide with the high volatility in the bear market states. Periods of somewhat lower volatility are associated with the bear rally states.

In Figure 5, a strong bear rally begins in late November 1933 and lasts until August 25, 1937, at which time there is a move back into the bear market state. Realized volatility increases with this move into state 1.

6.3.2 1980-1985

In Figure 6, the market displays several moves between the bull market state and the bull correction state before a short-term move into a bear market in August of 1982. Once again the transition from a bull to bear market is through a bull correction state. However, the bear market that emerges has state 1 that lasts only about 4 weeks. This is followed by a bear rally that results in increased prices accompanied with substantial volatility. The bear rally turns into a bull market in late April of 1983, thereafter are periods of the bull market state and bull corrections.

6.3.3 1987 crash

Prior to the 1987 crash there is a dramatic run-up in stock prices with generally low volatility, as illustrated in the top panel of Figure 7. It is interesting to note that the model shows a great deal of uncertainty about the state of the market well before the crash. In the first week of October, just before the crash, the most likely state is the bull correction with $p(s_t = 3|I_T) = 0.37$. The bear state which starts the following week lasts for about 5 weeks after which a strong bear rally quickly emerges as of the week ending November 18, 1987. It is the bear rally state that exits into a bull market during the week of August 17, 1988. Prices resume their strong increase until they plateau with a bull correction beginning the week of October 4, 1989.

6.3.4 2006-2010

We conclude with an analysis of recent market activity in Figure 8. The bull market state turned into a bull correction in mid-July 2007, which persisted until an abrupt move into the bear market state in early September 2008. This transition was accompanied by a dramatic increase in realized volatility. According to our model, the bear market became a bear market rally in the third week of March 2009 where it stayed until mid-November 2009 when it moved into the bull market state. As noted earlier, the positive trend in returns during a bear market rally do not get interpreted as a bull market until the market volatility declines to levels more typical of bull markets.

6.4 Density Forecasts

The different phases of the market according to the MS-4 model should have important implications for the forecast density of returns. In this section we discuss two illustrations.

An industry standard measure of potential portfolio loss is the Value-at-Risk (VaR) which measures the tail area of the return distribution. $\text{VaR}_{(\alpha),t}$ is defined as the 100α percent quantile of the portfolio value or return distribution given information at time

$t - 1$. We compute $\text{VaR}_{(\alpha),t}$ from the predictive density of the MS-4 model as

$$p(r_t < \text{VaR}_{(\alpha),t} | I_{t-1}) = \alpha. \quad (6.1)$$

Given a correctly specified model, the probability of a return of $\text{VaR}_{(\alpha),t}$ or less is α .

To compute the Value-at-Risk from the MS-4 model we do the following. First, N draws from the predictive density are taken as follows: draw θ and s_{t-1} from the Gibbs sampler, a future state \tilde{s}_t is simulated based on P and $\tilde{r}_t | \tilde{s}_t \sim N(\mu_{\tilde{s}_t}, \sigma_{\tilde{s}_t}^2)$. The details are discussed in Section 5.3. From the resulting draws, the \tilde{r}_t with rank $[N\alpha]$ is an estimate of $\text{VaR}_{(\alpha),t}$.

Figure 9 displays the conditional VaR from January 3, 2007 to January 20, 2010 predicted by the MS-4 model, as well as that implied by the normal benchmark for $\alpha = 0.05$. At each point the model is estimated based on information up to $t - 1$. Similarly, the benchmark, $N(0, \sigma^2)$, sets σ^2 to the sample variance using I_{t-1} .

The normal benchmark overestimates the VaR for the early part of this subsample but starts to understate it at times, beginning in mid-2007, and then severely underestimates in the last few months of 2008. The MS-4 model provides a very different $\text{VaR}_{(0.05),t}$ over time because it takes into account the predicted regime, as indicated by the middle and bottom panels of Figure 9 which show *forecasts* of the states and regimes respectively. Note that the potential losses, shown in the top panel, increase considerably in September and October 2008 as the model identifies a move from a bull to a bear market.

The density for long horizon returns will be sensitive to the different states in the MS-4 model and if they are empirically important should result in better density forecasts. To investigate this we compute the predictive likelihood according to Section 5.3 for a range of investment horizons for the time period 1913/11/12 – 2010/01/20. Along with the MS-4 model the MS-2 and a GARCH(1,1) specifications are included in Table 9. We can compare the models based on their relative log-probabilities for long horizon returns just as we would normally compare models using the log-Bayes factor in Section 5.2.

The GARCH(1,1) model provides the most accurate density forecasts for 1 month ahead returns while for all longer horizon returns the MS-4 dominates. From 3 months onward the difference in the log-predictive likelihoods is very large at 60 (3 months), 117 (6 months) and 210 (12 months) and is strong support for the 4 phases of the stock market that the MS-4 captures. On the other hand, except for the 12 month density forecasts the MS-2 is worse than the GARCH model. In summary, our model accurately identifies the phases of bull and bear markets which lead to superior long horizon out-of-sample density forecast.

6.4.1 Real-time Identification of the Bear Market

This out-of-sample application also gives us an opportunity to assess in real time when our model identified a move into the bear regime. In Section 6.3.4 this was discussed in the context of the full sample smoothed estimates. We now consider the identification process that would have been historically available to investors using the model forecasts. This will differ from the previous results as we are using a smaller sample and updating estimates as new data arrives.

The second and third panel of Figure 9 report the predictive mean of the states and regimes. Prior to 2008, forecasts of the bull states occur the most, including some short episodes of bull corrections. In the first week of October 2008, the probability of a bull regime drops from 0.85 to essentially zero and remains there for some time. In other words, the model in real time detects a turning point in the first week of October 2008 from the bull to the bear regime. The first half of the bear regime that follows is characterized by the bear state while the second half is largely classified as a bear rally.

Toward the end of our sample there is a move from the bear market rally state to a bull market. In real time, in early December 2009 the model forecasts a move from the bear rally to the bull market state. For the week ending December 9, we have $p(s_t = 1|I_{t-1}) = 0.02$, $p(s_t = 2|I_{t-1}) = 0.17$, $p(s_t = 3|I_{t-1}) = 0.14$ and $p(s_t = 4|I_{t-1}) = 0.67$. The evidence for a bull market regime gradually strengthens; the last observation in our sample, January 20, 2010, has probabilities 0.01, 0.11, 0.07 and 0.81 for states 1,2,3 and 4, with the bull market state being the most likely.

7 Conclusion

This paper proposes a new 4-state Markov-switching model to identify the components of bull and bear market regimes in weekly stock market data. Bull correction and bull states govern the bull regime; bear rally and bear states govern the bear regime. Our probability model fully describes the return distribution while treating bull and bear regimes and their component states as unobservable.

A bear rally is allowed to move back to the bear state or to exit the bear regime by moving to a bull state. Likewise, a bull correction can move back to the bull state or exit the bull regime by transitioning to a bear state. This implies that regimes can feature several episodes of their component states. For example, a bull regime can be characterized by a combination of bull states and bull corrections. Similarly, a bear regime can consist of several episodes of the bear state and the bear rally state. Because the realization of states in a regime will differ over time, bull and bear regimes can be heterogeneous over time. This richer structure, including both intra-regime and inter-regime dynamics, results in a richer characterization of market cycles.

Probability statements on regimes and future returns are available. Our model

strongly dominates other alternatives. Model comparisons show that the 4-state specification of bull and bear markets is strongly favored over several alternatives including a two-state model, as well as various alternative specifications for variance dynamics. For example, relative to a two-state model, there is less erratic switching so that market regimes are more persistent.

We find that bull corrections and bear rallies are empirically important for out-of-sample forecasts of turning points and VaR predictions. Our model provides superior density forecasts of long-horizon returns.

8 Acknowledgements

The authors are grateful for comments from the Editor, Associate editor and two anonymous referees, Christos Ntantamis, Hao Zhou, participants at the 6th World Congress of the Bachelier Finance Society, the Bayesian Econometrics workshop at RCEA, the Canadian Econometrics Study Group, the Third Risk Management Conference, Mont Tremblant and seminar participants from University of Colorado, Brock University and McMaster University. We thank the Social Sciences and Humanities Research Council of Canada for financial support.

9 Appendix: Ex Post Dating Algorithms

The Pagan and Sossounov (2003) adaptation of the Bry-Boschan (BB) algorithm can be summarized as follows:

1. Identify the peaks and troughs by using a window of 8 months.
2. Enforce alternation of phases by deleting the lower of adjacent peaks and the higher of adjacent troughs.
3. Eliminate phases less than 4 months unless changes exceed 20%.
4. Eliminate cycles less than 16 months.

Window width and phase duration constraints will depend on the particular series and will obviously be different for smoothed business cycle data than for stock prices. Pagan and Sossounov (2003) provide a detailed discussion of their choices for these constraints.

The Lunde and Timmermann (2004) dating algorithm defines a binary market indicator variable I_t which takes the value 1 if the stock market is in a bull state at time t and 0 if it is in a bear state. The stock price at the end of period t is labelled P_t . Our application of their dating algorithm can be summarized as: use a 6-month window to locate the initial local maximum or minimum.

Suppose we have a local maximum at time t_0 , in which case we set $P_{t_0}^{\max} = P_{t_0}$.

1. Define stopping-time variables associated with a bull market as

$$\begin{aligned}\tau_{\max}(P_{t_0}^{\max}, t_0 \mid I_{t_0} = 1) &= \inf\{t_0 + \tau : P_{t_0+\tau} \geq P_{t_0}^{\max}\} \\ \tau_{\min}(P_{t_0}^{\max}, t_0 \mid I_{t_0} = 1) &= \inf\{t_0 + \tau : P_{t_0+\tau} \leq 0.8P_{t_0}^{\max}\}\end{aligned}$$

2. One of the following happens:

- If $\tau_{\max} < \tau_{\min}$, bull market continues, update the new peak value $P_{t_0+\tau_{\max}}^{\max} = P_{t_0+\tau_{\max}}$ and set $I_{t_0+1} = \dots = I_{t_0+\tau_{\max}} = 1$. Update $t_0 = t_0 + \tau_{\max}$ still as local maximum and continue with step 1 above.
- If $\tau_{\max} > \tau_{\min}$, we find a trough at time $t_0 + \tau_{\min}$ and we have been in a bear market from $t_0 + 1$ to $t_0 + \tau_{\min}$. Set $I_{t_0+1} = \dots = I_{t_0+\tau_{\min}} = 0$. Record the value $P_{t_0+\tau_{\min}}^{\min} = P_{t_0+\tau_{\min}}$ and update $t_0 = t_0 + \tau_{\min}$ as local minimum. Go to step 3 below since t_0 is a local minimum now.

When t_0 is a local minimum:

3 Bear market stopping times are

$$\tau_{\min}(P_{t_0}^{\min}, t_0 \mid I_{t_0} = 0) = \inf\{t_0 + \tau : P_{t_0+\tau} \leq P_{t_0}^{\min}\}$$

$$\tau_{\max}(P_{t_0}^{\min}, t_0 \mid I_{t_0} = 0) = \inf\{t_0 + \tau : P_{t_0+\tau} \geq 1.2P_{t_0}^{\min}\}$$

4 One of the following happens:

- If $\tau_{\min} < \tau_{\max}$, bear market continues, update the new trough value, $P_{t_0+\tau_{\min}}^{\min} = P_{t_0+\tau_{\min}}$ and set $I_{t_0+1} = \dots = I_{t_0+\tau_{\min}} = 0$. Update $t_0 = t_0 + \tau_{\min}$ and continue with step 3.
- If $\tau_{\min} > \tau_{\max}$ we find a peak at time $t_0 + \tau_{\max}$ and we have been in a bull market from $t_0 + 1$ to $t_0 + \tau_{\max}$. Set $I_{t_0+1} = \dots = I_{t_0+\tau_{\max}} = 1$. Record the value $P_{t_0+\tau_{\max}}^{\max} = P_{t_0+\tau_{\max}}$ and update $t_0 = t_0 + \tau_{\max}$ as a local maximum. Go to 1 above since t_0 is a local maximum now.

This process is repeated until the last data point. All periods with $I_t = 1$ are in bull regime and $I_t = 0$ are in bear regime.

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Table 1: Weekly Return Statistics (1885-2010)^a

| N | Mean | standard deviation | Skewness | Kurtosis | J-B ^b |
|------|-------|--------------------|----------|----------|------------------|
| 6498 | 0.085 | 2.40 | -0.49 | 11.2 | 18475.5* |

^a Continuously compounded returns

^b Jarque-Bera normality test: p-value = 0.00000

Table 2: MS-2-State Model Estimates

| | mean | median | std | 0.95 DI |
|------------|-------|--------|-------|----------------|
| μ_1 | -0.46 | -0.46 | 0.14 | (-0.73, -0.20) |
| μ_2 | 0.20 | 0.20 | 0.02 | (0.16, 0.25) |
| σ_1 | 4.42 | 4.42 | 0.13 | (4.18, 4.69) |
| σ_2 | 1.64 | 1.64 | 0.02 | (1.59, 1.69) |
| p_{11} | 0.94 | 0.94 | 0.01 | (0.92, 0.96) |
| p_{22} | 0.99 | 0.99 | 0.002 | (0.98, 0.99) |

This table reports the posterior mean, median, standard deviation and 0.95 density intervals for model parameters.

Table 3: MS-4-State Model Estimates

| | mean | median | std | 95% DI |
|------------|-------|--------|--------|----------------|
| μ_1 | -0.94 | -0.92 | 0.27 | (-1.50, -0.45) |
| μ_2 | 0.23 | 0.23 | 0.10 | (0.04, 0.43) |
| μ_3 | -0.13 | 0.12 | 0.08 | (-0.31, -0.01) |
| μ_4 | 0.30 | 0.29 | 0.04 | (0.22, 0.38) |
| σ_1 | 6.01 | 5.98 | 0.35 | (5.41, 6.77) |
| σ_2 | 2.63 | 2.61 | 0.18 | (2.36, 3.08) |
| σ_3 | 2.18 | 2.19 | 0.12 | (1.94, 2.39) |
| σ_4 | 1.30 | 1.30 | 0.04 | (1.20, 1.37) |
| p_{11} | 0.921 | 0.923 | 0.020 | (0.877, 0.955) |
| p_{12} | 0.076 | 0.074 | 0.020 | (0.042, 0.120) |
| p_{14} | 0.003 | 0.001 | 0.004 | (3e-6, 0.013) |
| p_{21} | 0.015 | 0.014 | 0.007 | (0.005, 0.031) |
| p_{22} | 0.966 | 0.967 | 0.009 | (0.945, 0.980) |
| p_{24} | 0.019 | 0.018 | 0.006 | (0.009, 0.034) |
| p_{31} | 0.010 | 0.009 | 0.003 | (0.004, 0.017) |
| p_{33} | 0.939 | 0.943 | 0.018 | (0.899, 0.965) |
| p_{34} | 0.051 | 0.048 | 0.017 | (0.027, 0.088) |
| p_{41} | 0.001 | 0.0003 | 0.0007 | (6e-7, 0.002) |
| p_{43} | 0.039 | 0.037 | 0.012 | (0.024, 0.067) |
| p_{44} | 0.960 | 0.963 | 0.012 | (0.933, 0.976) |

The posterior mean, median, standard deviation and 0.95 density intervals for model parameters.

Table 4: Unconditional State Probabilities

| | mean | 0.95 DI |
|---------|-------|----------------|
| π_1 | 0.070 | (0.035, 0.117) |
| π_2 | 0.157 | (0.073, 0.270) |
| π_3 | 0.304 | (0.216, 0.397) |
| π_4 | 0.469 | (0.346, 0.579) |

The posterior mean and 0.95 density intervals associated with the posterior distribution for π from Equation (4.7).

Table 5: Posterior Regime Statistics for MS-2 and MS-4 Models

| | MS-2 | MS-4 |
|--|-------------------------|---------------------------|
| bear mean | -0.46 (-0.73, -0.20) | -0.13 (-0.367, -0.005) |
| bear duration | 18.2 (13.2, 25.0) | 77.8 (44.4, 134.6) |
| bear standard deviation | 4.42 (4.18, 4.69) | 4.04 (3.51, 4.73) |
| bear variance from $\text{Var}(E[r_t s_t] s_t = 1, 2)$ | 0.00 | 0.31 (0.07, 0.68) |
| bear variance from $E[\text{Var}(r_t s_t) s_t = 1, 2]$ | 19.6 (17.5, 22.0) | 16.1 (12.1, 22.0) |
| bear skewness | 0 | -0.42 (-0.68, -0.20) |
| bear kurtosis | 3 | 5.12 (4.37, 5.93) |
| bull mean | 0.20 (0.16, 0.25) | 0.13 (0.07, 0.18) |
| bull duration | 82.6 (59.1, 115.9) | 256.0 (123.5, 509.6) |
| bull standard deviation | 1.64 (1.59, 1.69) | 1.71 (1.59, 1.83) |
| bull variance from $\text{Var}(E[r_t s_t] s_t = 3, 4)$ | 0.00 | 0.04 (0.02, 0.09) |
| bull variance from $E[\text{Var}(r_t s_t) s_t = 3, 4]$ | 2.69 (2.54, 2.85) | 2.89 (2.47, 3.30) |
| bull skewness | 0 | 0.04 (-0.11, 0.16) |
| bull kurtosis | 3 | 3.77 (3.51, 4.03) |

The posterior mean and 0.95 density interval for regime statistics.

Table 6: Posterior State Statistics for the MS-4 Model

| | mean | median | std | 95% DI |
|------------------------|-------|--------|-------|------------------|
| Bear mean | -0.13 | -0.11 | 0.10 | (-0.367, -0.005) |
| Bear duration | 77.8 | 74.0 | 23.1 | (44.4, 134.6) |
| Bear cumulative return | -9.94 | -8.28 | 7.89 | (-29.6, -0.41) |
| Bear std | 4.04 | 4.01 | 0.31 | (3.51, 4.73) |
| Bull mean | 0.13 | 0.13 | 0.03 | (0.07, 0.18) |
| Bull duration | 256.0 | 235.6 | 100.9 | (123.5, 509.6) |
| Bull cumulative return | 33.0 | 30.0 | 14.9 | (12.9, 70.3) |
| Bull std | 1.71 | 1.71 | 0.06 | (1.59, 1.83) |
| s=1: cumulative return | -12.4 | -11.8 | 4.49 | (-23.0, -5.45) |
| s=2: cumulative return | 7.10 | 6.97 | 3.10 | (1.47, 13.8) |
| s=3: cumulative return | -2.13 | -2.07 | 1.09 | (-4.46, -0.27) |
| s=4: cumulative return | 7.88 | 7.75 | 1.67 | (5.02, 11.6) |
| s=1: duration | 13.5 | 13.0 | 3.63 | (8.13, 22.2) |
| s=2: duration | 31.2 | 30.1 | 8.39 | (18.3, 51.0) |
| s=3: duration | 17.9 | 17.4 | 4.80 | (9.91, 28.8) |
| s=4: duration | 27.2 | 26.9 | 6.75 | (14.9, 41.4) |

This table report posterior statistics for various population moments.

Table 7: Log Marginal Likelihoods: Alternative Models

| Model | $\log f(Y \text{Model})$ | log-Bayes Factor ^a |
|--|----------------------------|-------------------------------|
| Constant mean with constant variance | -14924.1 | 1183.7 |
| MS-2 | -13903.3 | 162.9 |
| MS-4 with restricted variances ($\sigma_0^2 = \sigma_1^2$, and $\sigma_2^2 = \sigma_3^2$) | -13849.9 | 109.5 |
| MS-4, 4 state mean and decoupled 4 state variance | -13754.4 | 14.0 |
| GARCH(1,1) | -13809.1 | 68.7 |
| MS-4 with $p_{24} = p_{31} = 0$ | -13766.9 | 26.5 |
| MS-4 with unrestricted P | -13747.3 | 6.9 |
| MS-4 | -13740.4 | |

^a log-Bayes Factor for MS-4 vs each model

Table 8: Dating-algorithm filtering of data and simulated data

| | S&P | MS-4 |
|----------------------------------|-------|-------------------------------|
| Avg. number of bears | 29 | 31.7 (22, 42) ^a |
| Avg. bear duration | 63.1 | 55.9 (40.5, 74.7) |
| Avg. bear amplitude ^b | -45.0 | -43.4 (-52.7, -35.8) |
| Avg. bear return | -0.71 | -0.80 (-1.08, -0.57) |
| Avg. bear std | 3.16 | 3.15 (2.60, 3.73) |
| Avg. number of bulls | 28 | 31.4 (22, 42) |
| Avg. bull duration | 166.7 | 158.5 (103.0, 235.3) |
| Avg. bull amplitude | 66.4 | 60.2 (46.3, 80.0) |
| Avg. bull return | 0.40 | 0.39 (0.31, 0.48) |
| Avg. bull std | 2.53 | 2.42 (1.97, 2.91) |

^a 70% density interval

^b Aggregate return over one regime

Table 9: Long Horizon Return Density Forecasts

| | h=weeks | MS-4 | MS-2 | GARCH(1,1) |
|-----------|---------|------------------|-----------|------------------|
| 1 month | 4 | -14428.80 | -14575.67 | -14341.60 |
| 3 months | 13 | -17610.50 | -17796.85 | -17670.56 |
| 6 months | 26 | -19534.98 | -19725.67 | -19652.16 |
| 12 months | 52 | -21652.68 | -22454.47 | -21862.74 |

This table reports the log-predictive density of h-period returns: $\sum_{t=T_1}^{T_2} \log(p(r_{t,h}|I_t, \text{Model}))$ where $r_{t,h} = \sum_{i=1}^h r_{t+i}$. Out-of-sample data: 1913/11/12 – 2010/01/20. This gives 4997 (1 month), 4988 (3 months), 4975 (6 months) and 4949 (12 months) observations respectively.

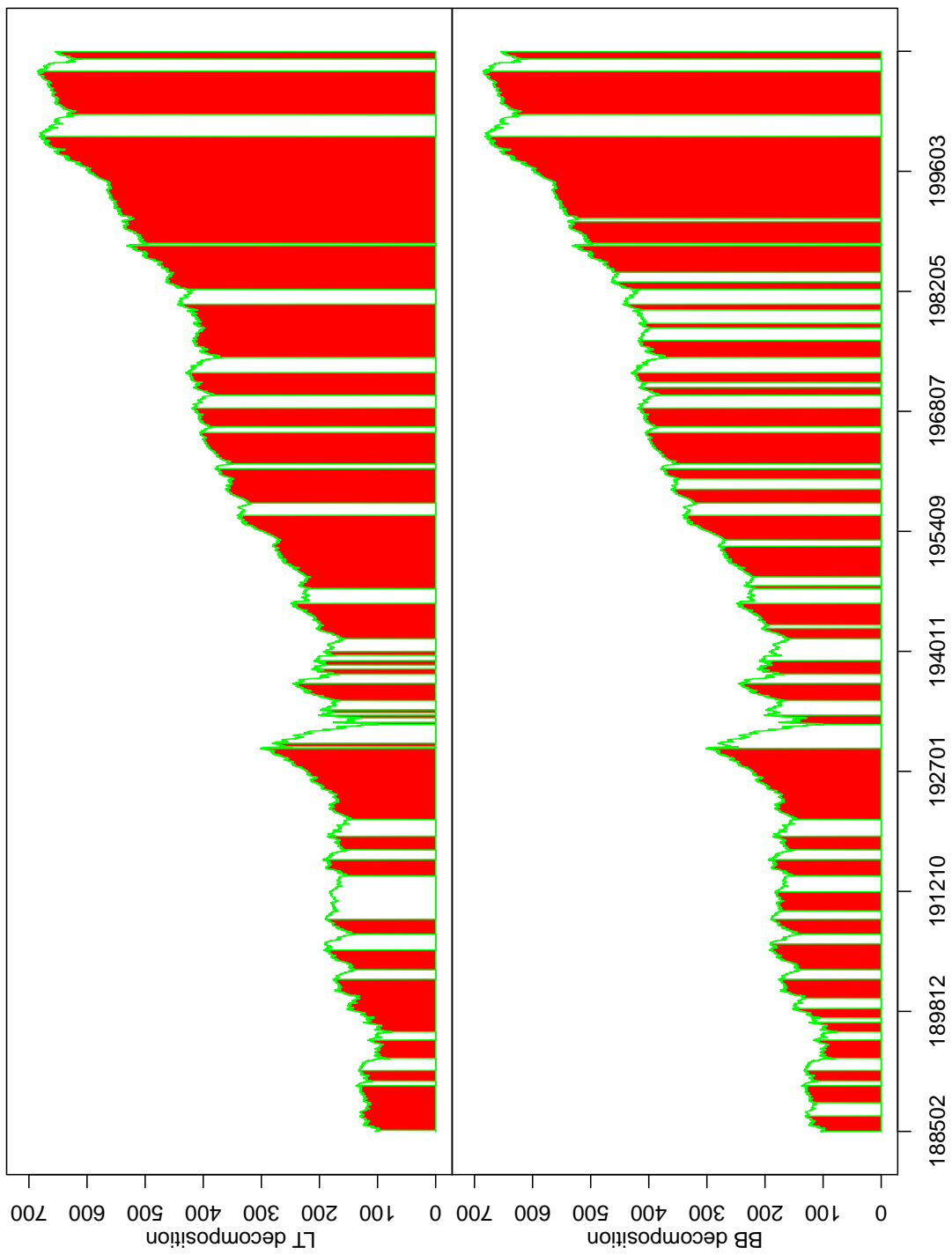


Figure 1: LT and BB dating algorithms

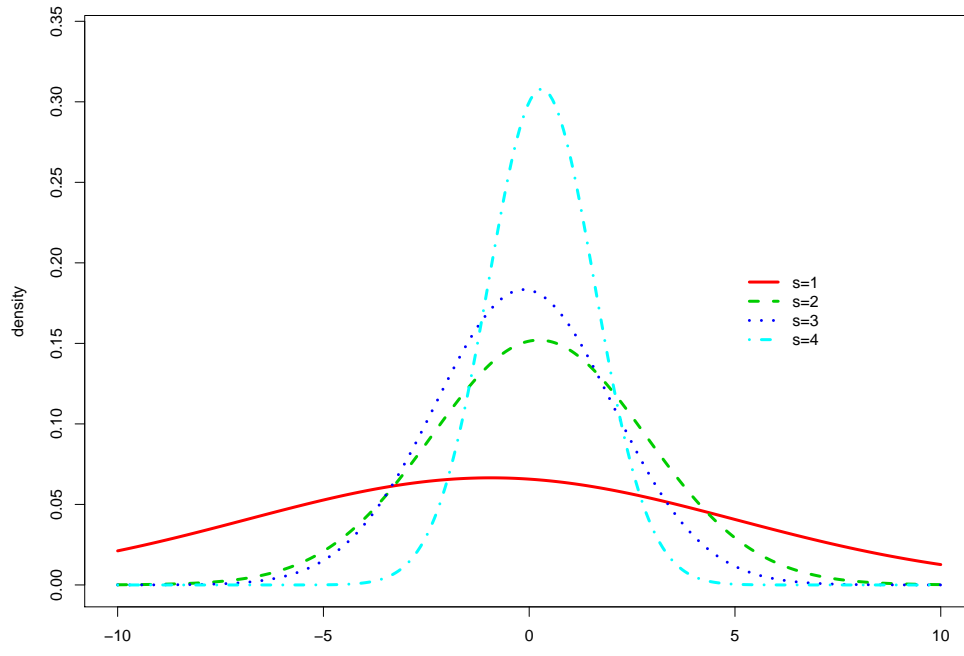


Figure 2: MS-4-States, State Densities

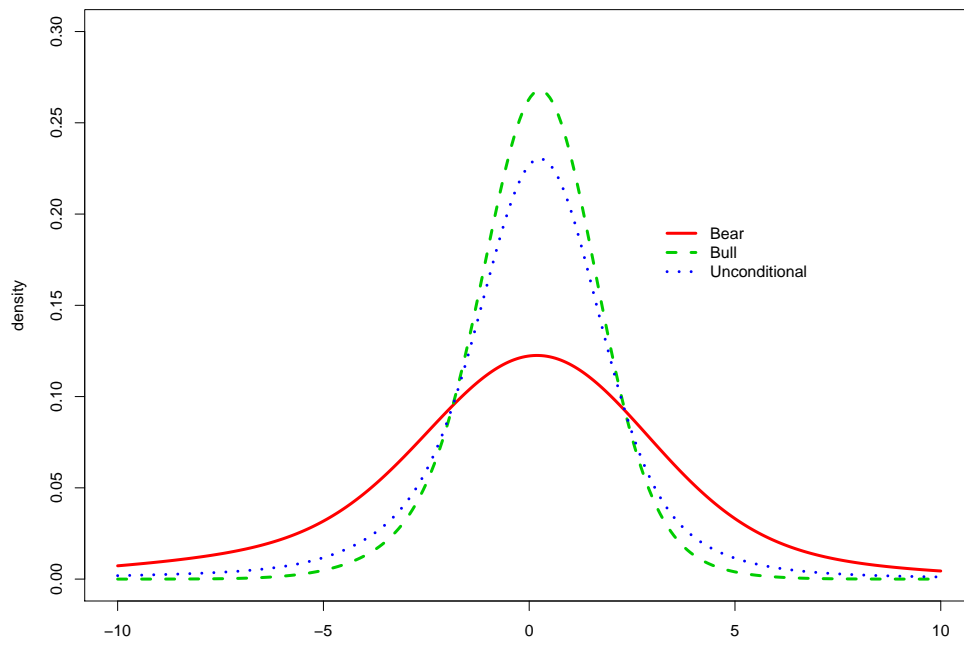


Figure 3: MS-4-States, Regime Densities

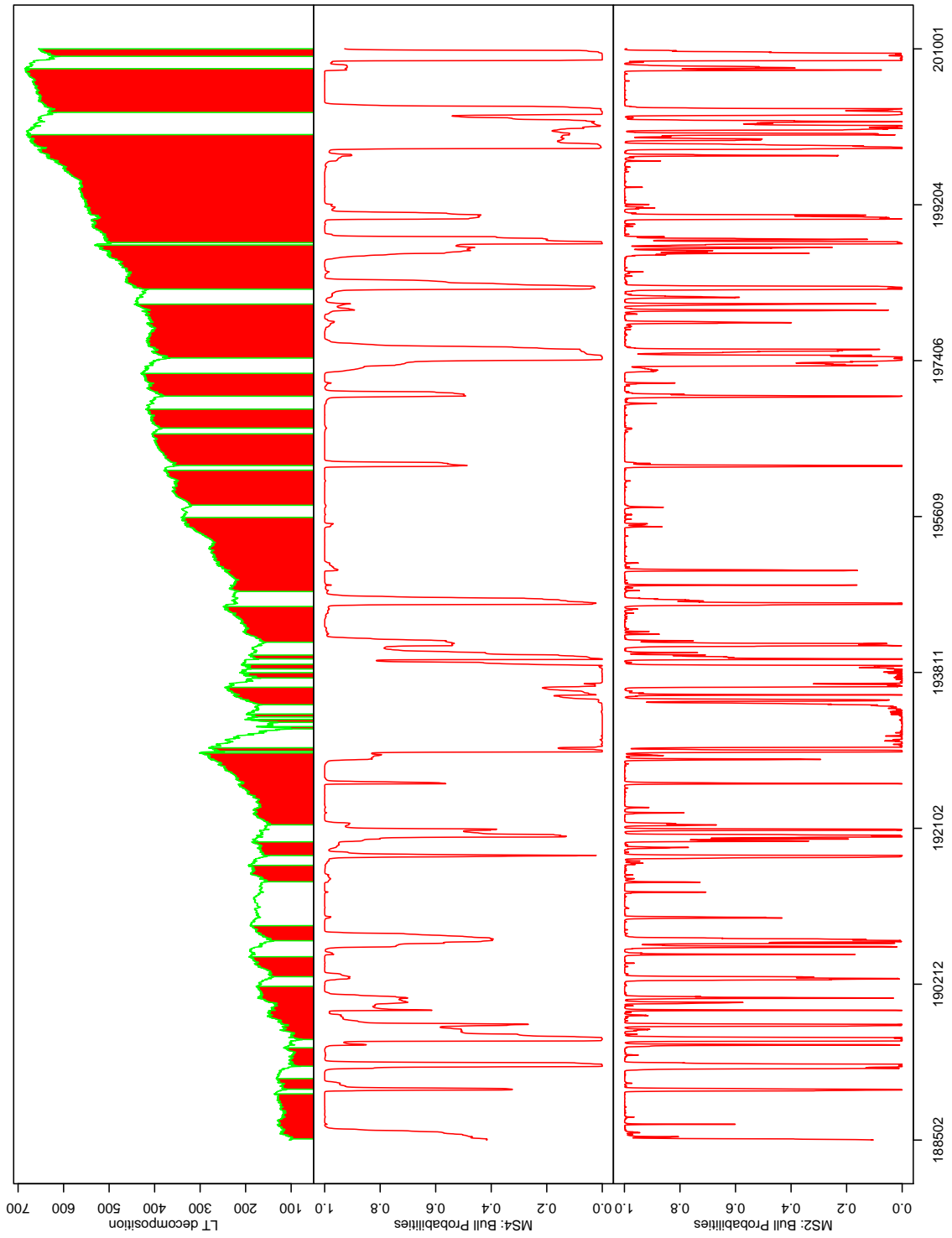


Figure 4: LT algorithm, MS-4 and MS-2

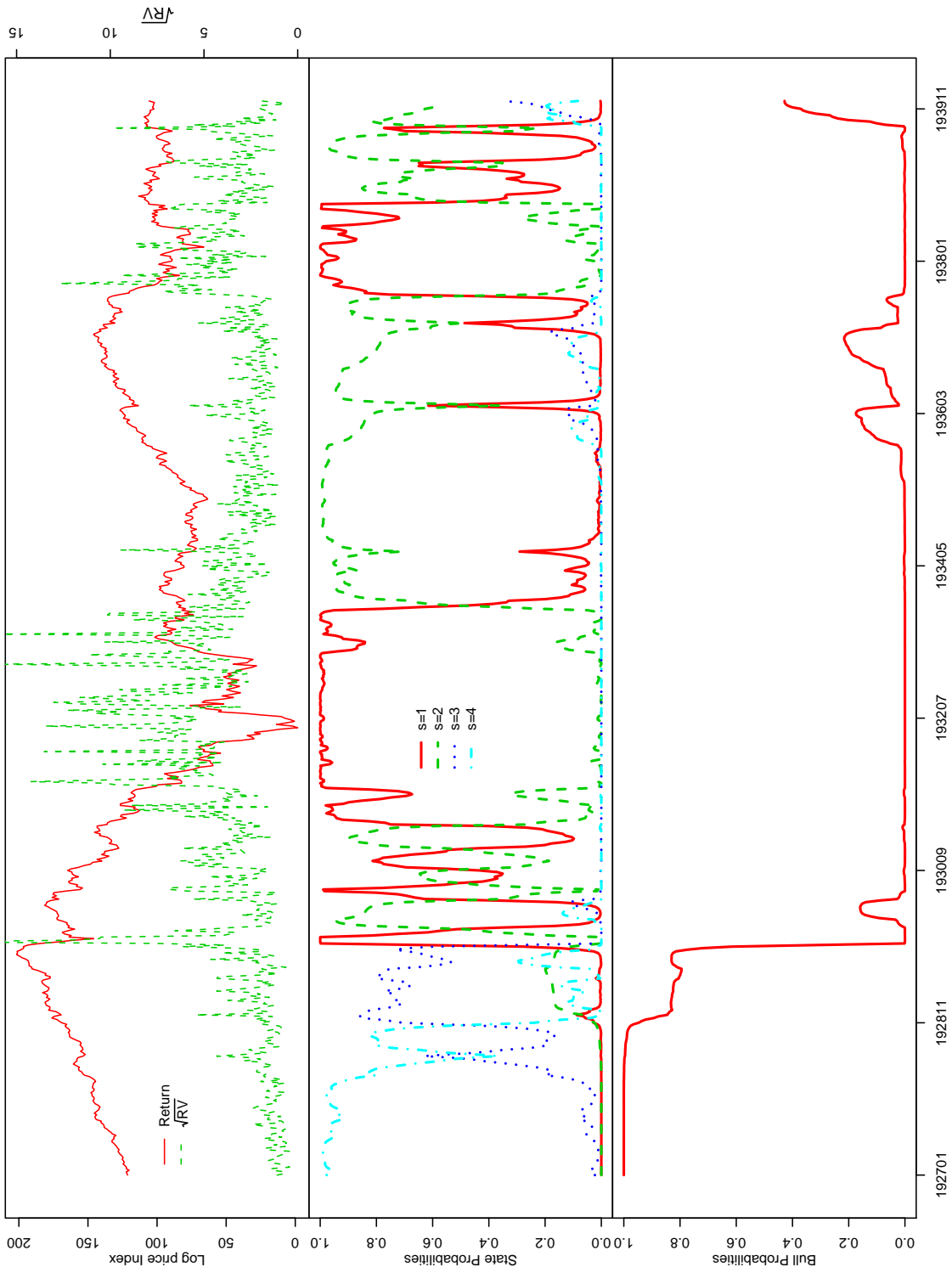


Figure 5: MS-4, 1927-1939

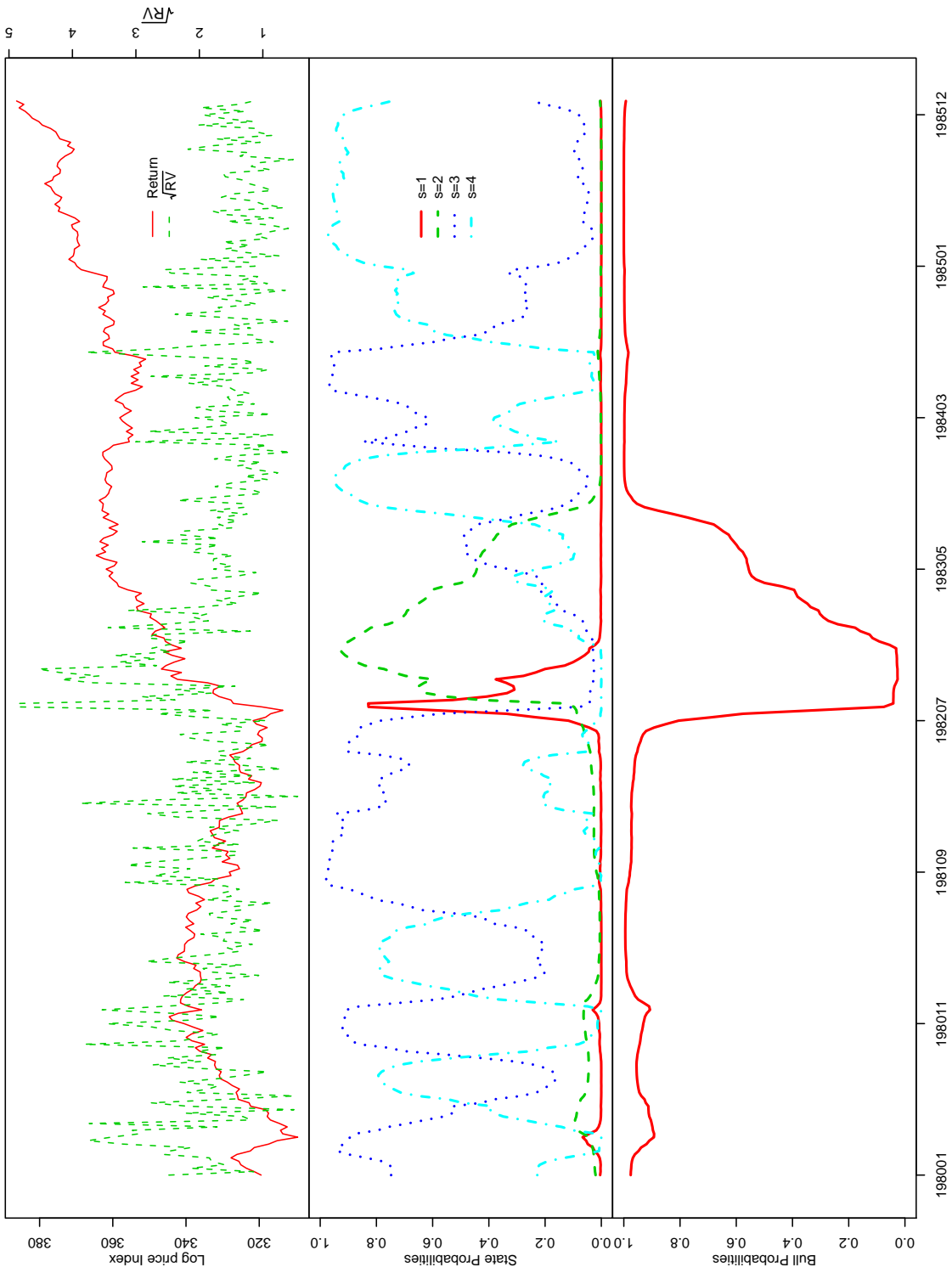


Figure 6: MS-4, 1980-1985

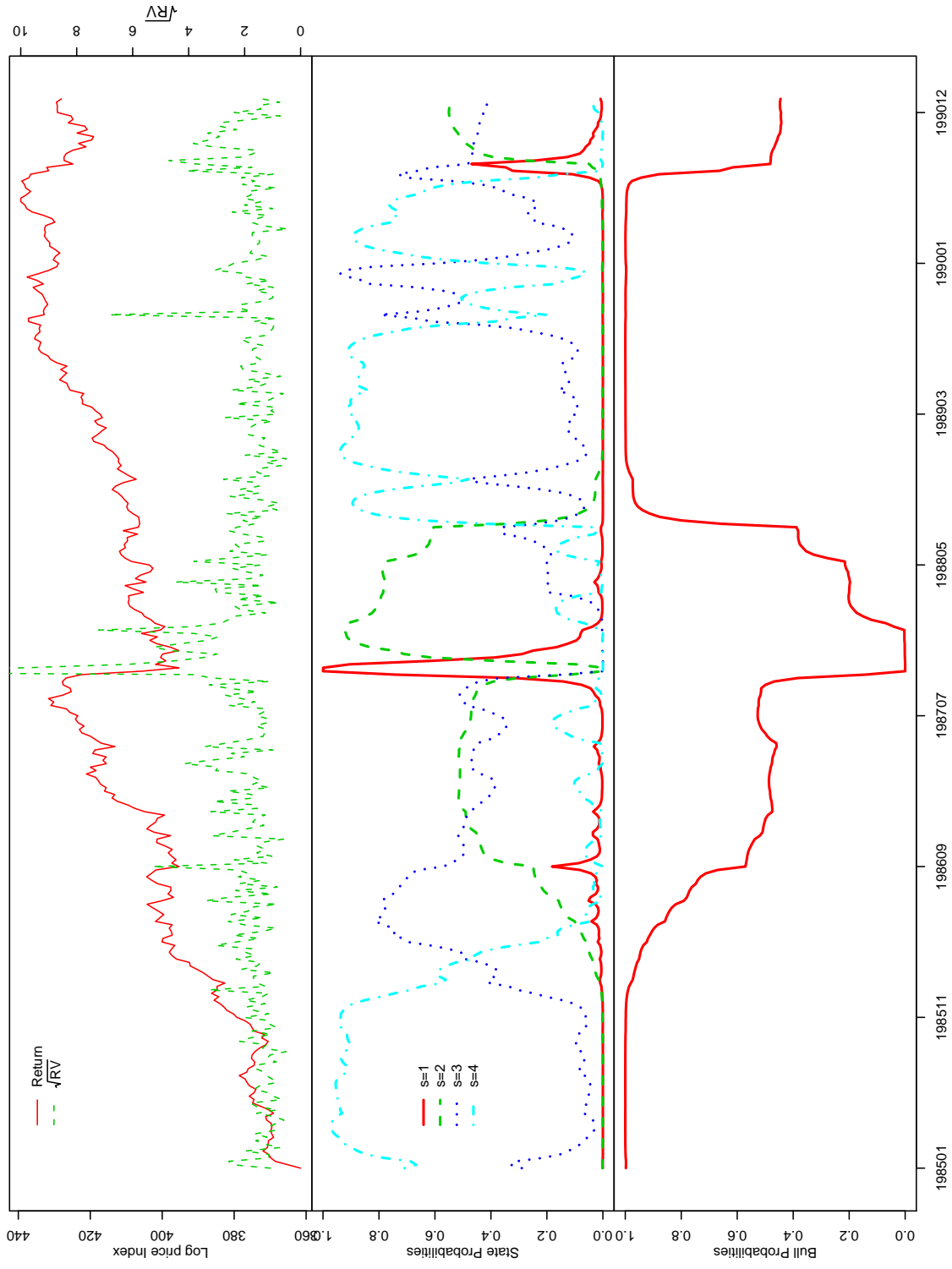


Figure 7: MS-4, 1985-1990

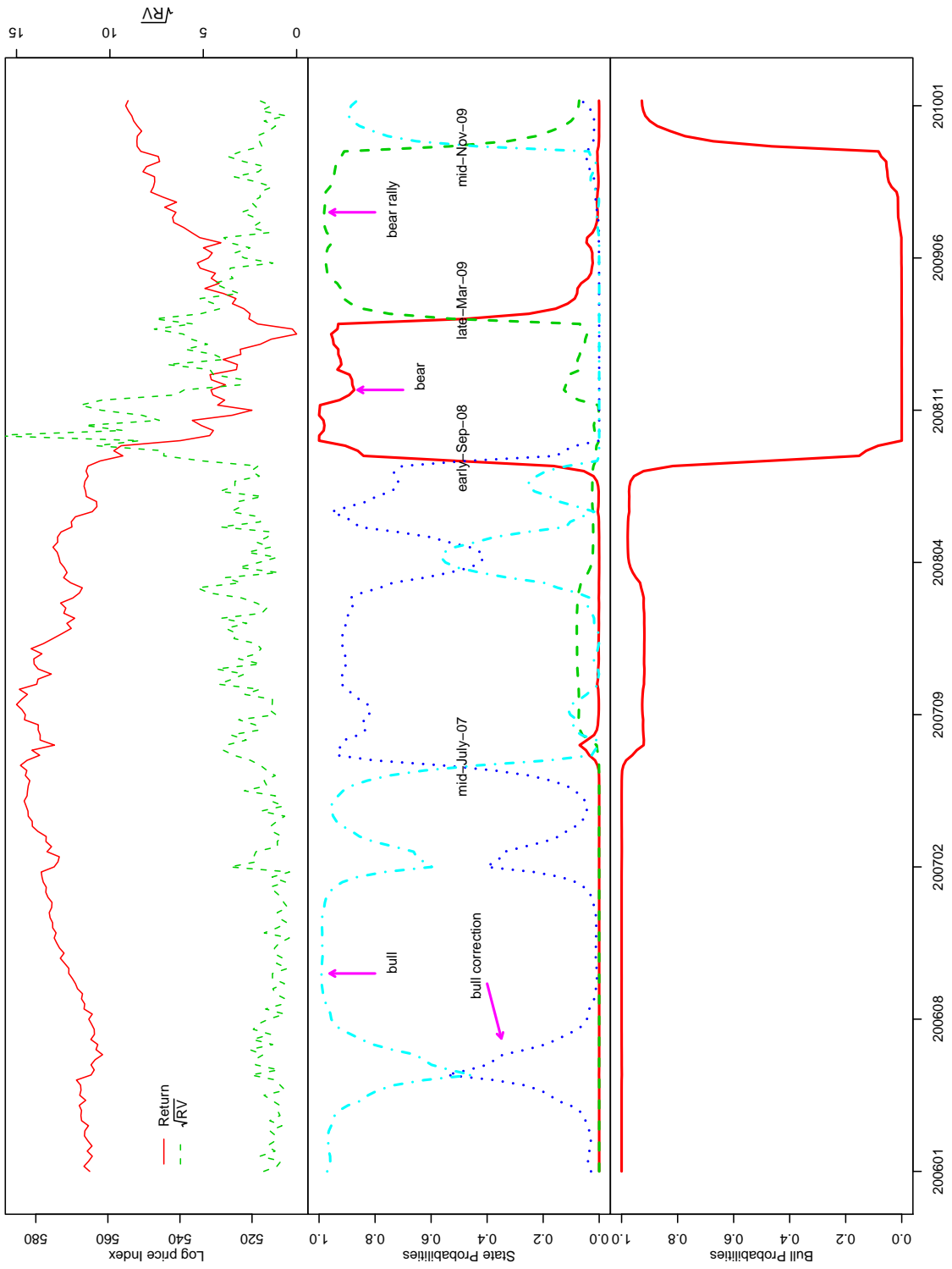


Figure 8: MS-4, 2006-2010

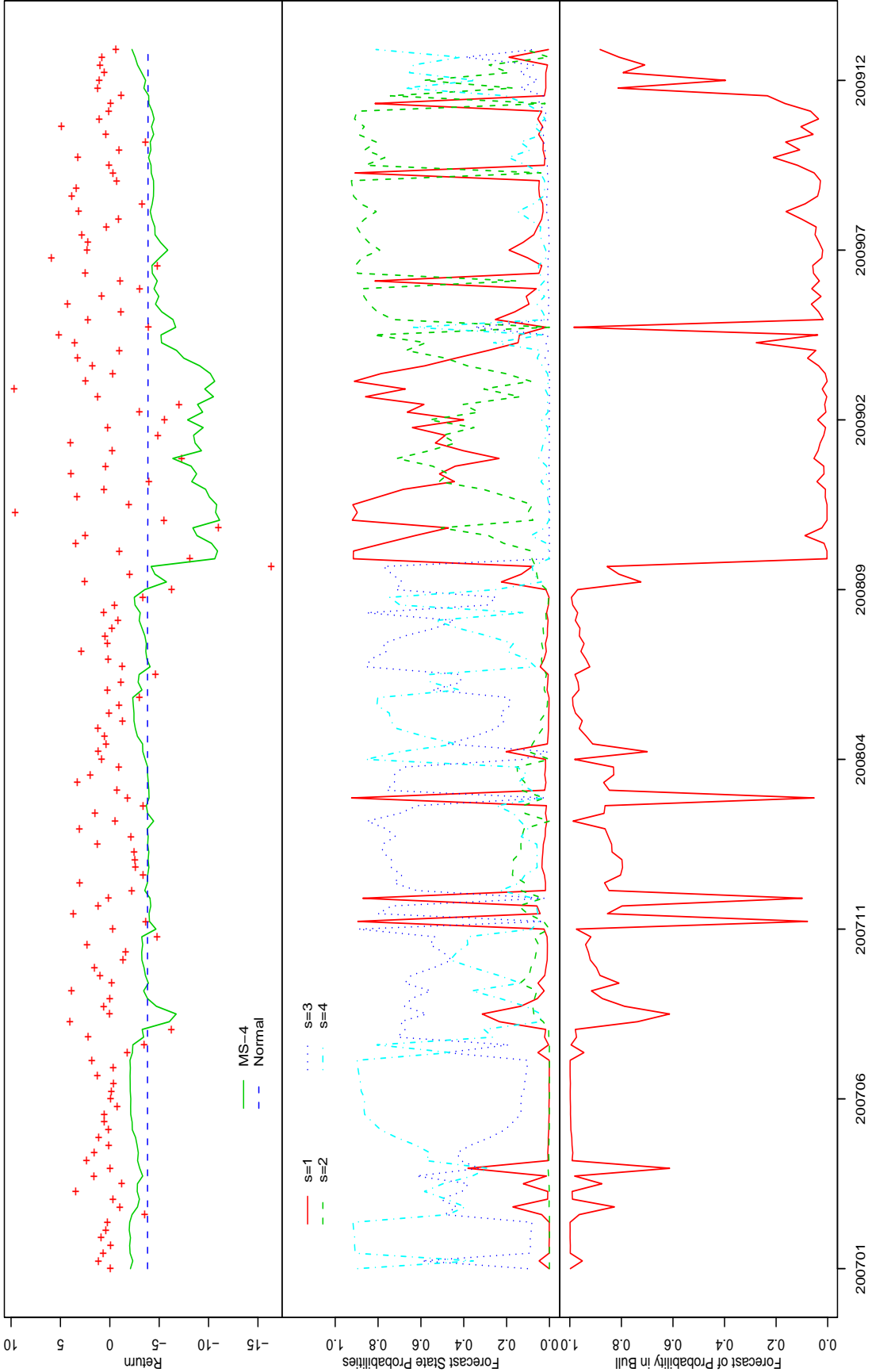


Figure 9: Value-at-Risk from MS-4 and Benchmark Normal distribution