Price Hedging with Local and Aggregate Quantity Risk

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The authors present a method to minimize Value-at-Risk where price, local quantity, and aggregate quantity are all stochastic and correlated. The framework is quite general in that it accommodates both local and aggregate quantity, and the quantity variable may be for an asset that is subject to a stochastic convenience yield. The framework is more general than that of Ahn et al. [1999]. The solution identifies an optimal strike price for a quantity-triggered put option used to minimize Value-at-Risk. The authors identify situations where this put option is more effective than its plain-vanilla counterpart in reducing both price and quantity risks.

I. INTRODUCTION

A rich literature has emerged that explores the various channels through which hedging can contribute to greater firm value (cf. Froot, Scharfstein, and Stein [1993], DeMarzo and Duffie [1995], Smith and Stulz [1985], and Smithson [1998]). While these studies have enhanced our understanding of what motivates corporations to manage risk, less attention has been directed toward understanding precisely how corporations should hedge.

Using hedging instruments with nonlinear payoffs, Ahn et al. [1999] and Brown and Toft [2002] demonstrate how corporations should hedge to achieve their unique objectives. Ahn et al. investigate the optimal hedging strategy of minimizing Value-at-Risk (VaR) for firms facing price risk only. They solve for the strike price of a plain-vanilla put option that minimizes the hedger's VaR. Taking into consideration both price risk and quantity risk (at the firm level), Brown and Toft also derive optimal hedging positions (maximizing firm cash flow) using a theoretical hedging instrument dubbed a custom exotic derivative. This product, selected as a result of an optimization procedure, has a quadratic state contingent payoff. By construction, the exotic derivative is superior to either forwards or options in terms of hedging efficiency.

Related to the issue of how firms should hedge, several articles in the agricultural risk management literature provide important early insight into the choice of linear and non-linear instruments (cf. Lence, Sakong, and Hayes [1994] and Sakong, Hayes, and Hallam [1993]). Also, Gay, Nam, and Turac [2002a, 2003] study the optimal mix of linear and non-linear derivative instruments for corporations facing both price and quantity risk (at the firm level). While these studies have tremendously increased our understanding as to how corporations should hedge, none of them takes into consideration the quantity risk at the aggregate level, the correlation between the firm quantity and the aggregate quantity. Moreover, these studies do not consider the effect of aggregate quantity risk on a firm's optimal hedging strategy.

The purpose of this research is to provide an optimal hedging technique for entities that face both price and quantity risk, where price, quantity at the local level, and quantity at the aggregate level are all stochastic and correlated.

Specifically, this article provides a method to minimize VaR with stochastic price, local and aggregate quantities using a real and recently introduced derivative product known as a Quantity-Triggered Option (QTO). Our approach is quite general because 1) we accommodate both local (firm level) and aggregate quantity; 2) the quantity variable could reflect any asset including various commodities, various weather measures, and the like; and 3) where appropriate, the quantity variable can be subject to a stochastic convenience yield. Consequently, our optimal hedging framework is more general than that of Ahn et al. [1999].[1]

A QTO is a relatively new product that was introduced in 2002 by major derivative dealers in the U.S., and it is becoming an increasingly important hedging instrument. One example of such a product is a put option on West Texas Crude whose payoff is triggered by whether the reported production output of OPEC for a particular month exceeds a prespecified output level. Another example is a put option on wheat whose payoff is triggered by whether the USDA's reported figure for the U.S. seasonal wheat harvest exceeds a prespecified harvest figure. Yet another example is a weather-triggered commodity option such as a call on the regional electricity price whose payoff is triggered if the regional daily average temperature for the month exceeds a prespecified level.

Producers and consumers of these and other commodities are attracted to the new products presumably because they represent less expensive alternatives to plain-vanilla commodity options for hedging quantity shocks and the consequences of said shocks on revenues, input costs of production, and earnings. For instance, envision a commodity producer that specializes in soybeans whose price is commonly determined by the aggregate quantity to be harvested. The supply curve for soybeans may shift rightward or leftward depending on whether mother nature is kind or unkind. When the aggregate harvest is higher than anticipated, the supply curve will shift rightward and soybean prices will tend to be lower—a classic inverse relationship between quantity and price. Here, to protect revenues against a positive quantity shock, the soybean producer would look to purchase a put option on the price of soybeans whose payoff is triggered by the aggregate quantity exceeding some threshold level. This level will most often represent a consensus forecast of harvest. The cost of this quantity-triggered soybean put will be less than that of a plain-vanilla counterpart.[2] Although all agents face a single market price for the same commodity, individual hedgers bear different quantity risks depending on where their production is located. Therefore, when analyzing the hedging effectiveness of QTOs, it is imperative to consider the correlation between firm-level quantity and aggregate-level quantity.[3]

Our article contributes to the literature in two important ways. First, our article presents a method to minimize Value-at-Risk where price, firm-level quantity, and aggregate-level quantity are all stochastic and correlated. This framework is quite general in that it accommodates potential effects of various correlations among asset price, firm-level, and aggregate-level quantities on a firm's optimal hedging strategy. Second, our article provides additional insights as to the design and effectiveness of a firm's optimal hedging strategy using the new type of hedging instrument. Specifically, we investigate whether quantity-triggered options are more effective than their plain-vanilla counterparts in managing both the price risk and the quantity risk. To provide a backdrop for investigating the hedging effectiveness of QTOs, we first derive the valuation formulas for the European-style, quantity-triggered commodity options. Our theoretical setup consists of correlated geometric Brownian motions for the asset price and the aggregate quantity. In addition, following Gibson and Schwartz [1990], Schwartz [1997], Hilliard and Reis [1998], and Schwartz [1998], we permit the commodity's convenience yield to follow an arithmetic Brownian motion.

Defining VaR over the total revenue (i.e., the product of price and firm-level quantity) and using a similar technique as in Ahn et al. [1999], we identify the optimal strike price for a quantity-triggered put option that allows the hedger to minimize VaR. Under the same hedging budget, we then compare the effectiveness of the quantity-triggered put option with the corresponding plain-vanilla put option. The QTO is more effective than its plain-vanilla counterpart in reducing VaR when any of the following situations prevails: 1) the asset price volatility is low, 2) the correlation between the asset price and the aggregate quantity is low, and 3) the correlation between the firm quantity and the aggregate quantity is high, coupled with high volatilities of the two quantity variables. If all the three conditions are met at the same time, then the quantity-trigger put option is the most effective in reducing VaR. The conditions discussed earlier also imply that the quantity-triggered option is more effective than its plain-vanilla counterpart when the price risk is low and the quantity risk is high.

The article proceeds as follows. The general valuation equations for the four types of quantity-triggered commodity options are derived in Section II. Section III specializes the model by specifying the convenience yield process and the volatility components of the stochastic processes. Comparative statics and related simulations are presented in Section IV. Section V contains our principal results: the VaR minimization procedure and the investigation of the hedging effectiveness of quantity-triggered put options. Section VI offers a brief conclusion.

II. GENERAL PRICING FORMULAS FOR QUANTITY-TRIGGERED OPTIONS

Following Schwartz [1997, 1998], we provide a two-factor model to describe commodity prices with the convenience yield being stochastic. Specifically, we assume the following processes for the commodity spot price S, the convenience yield δ, and the aggregate quantity Q:

$$dS = [\mu_S(S,t) - \delta]Sdt + \sigma_S(t)Sdz_S \tag{1}$$

$$d\delta = \mu_\delta(\delta,t)dt + \sigma_\delta(t)dz_\delta \tag{2}$$

$$dQ = \mu_Q(t)Qdt + \sigma_Q(t)Qdz_Q \tag{3}$$

where $E[dz_S, dz_\delta] = \rho_{S\delta}dt$, $E[dz_S, dz_Q] = \rho_{SQ}dt$, and $E[dz_\delta, dz_Q] = \rho_{\delta Q}dt$.

These processes imply that the future spot price, S_T, and the aggregate quantity, Q_T, are lognormally distributed, and the future convenience yield, δ_T, is normally distributed. The drifts and diffusion coefficients are left unspecified at this point, so that our pricing results are general. It is quite likely that both the drift and the diffusion coefficient for the quantity variable are time dependent. This is especially so for agricultural commodities such as wheat and soybeans.

In a general equilibrium framework, price is jointly determined by demand and supply. When the demand curve is fixed, supply shocks lead to price adjustments. Specifically, a positive shock in supply will lead to a price decline, and vice versa. The slope of the demand curve determines the sensitivity of price to quantity. When the demand curve has uncertainties of its own, the relationship between price and quantity is no longer deterministic. In our partial equilibrium framework characterized by Equations (1) and (3), the correlation coefficient ρ_{SQ} captures the inverse relationship between the aggregate quantity (or supply) and price. When $\rho_{SQ} = -1$, the demand curve is fixed, and as a result, price risk is identical to quantity risk. As long as ρ_{SQ} is not equal to ± 1, price risk and quantity risk will coexist.

For our problem at hand, the concern is how price and quantity/supply are related, not how demand and supply jointly determine the price. Therefore, a partial equilibrium framework is sufficient to derive hedging implications. In addition, although we only focus on quantity or supply, the magnitude of the correlation coefficient ρ_{SQ} determines the joint uncertainties in both demand and supply. In fact, as a general setup, we do not even require that the quantity variable always refers to the supply of the commodity in question. For instance, the setup described earlier can be applied to a weather-triggered commodity option where the underlying commodity could be natural gas, yet the quantity variable could be the average temperature over, say, a month. In this case, the temperature variable will be highly correlated with the demand of natural gas and actually becomes a proxy for the quantity of demand. Again, the key issue is how the price variable S and the quantity variable Q are correlated. Once the quantity variable is decided on, all we need to take care of is the ensuing correlations.

Note that we model only the aggregate quantity at this point. Most of the quantity-triggered options are based on publicly available aggregate quantity measures. Hence, the hedging effectiveness of this type of options will depend on, among other things, the correlation between the firm-level quantity and the aggregate-level quantity. We will defer this investigation to Section V.

Regardless of what the triggering quantity variable is and how it is measured, as long as the commodity in question is traded, then in the risk-neutral world, the processes discussed earlier become

$$dS = [r - \delta]Sdt + \sigma_S(t)Sdz_S^* \tag{4}$$

$$d\delta = [\mu_\delta(\delta,t) - \lambda(t)]dt + \sigma_\delta(t)dz_\delta^* \tag{5}$$

$$dQ = \mu_Q(t)Qdt + \sigma_Q(t)Qdz_Q^* \tag{6}$$

where r is the riskfree rate—an assumed constant—and $\lambda(t)$ is the risk premium for the convenience yield. Notice that the quantity variable is assumed to be invariant to the change of pricing measures. Inserting a risk premium term in the drift would not cause any pricing difficulty.

There are four types of quantity-triggered options and their values can be expressed as follows:

$$\begin{aligned} QTC_H &= e^{-r(T-t)}E[\max(0,S_T - X)\,|\,Q_T \geq Q^\star] \\ QTC_L &= e^{-r(T-t)}E[\max(0,S_T - X)\,|\,Q_T \leq Q^\star] \\ QTP_H &= e^{-r(T-t)}E[\max(0,X - S_T)\,|\,Q_T \geq Q^\star] \\ QTP_L &= e^{-r(T-t)}E[\max(0,X - S_T)\,|\,Q_T \leq Q^\star] \end{aligned} \tag{7}$$

where E is the expectation operator under the risk-neutral measure, X is the exercise price, QTC_H is a quantity-triggered call when the quantity needs to be higher than the threshold level $Q^\star$, QTC_L is a quantity-triggered call when the quantity needs to be lower than the threshold level $Q^\star$, and so on. By definition, $QTC_H + QTC_L = C$ and $QTP_H + QTP_L = P$, where C and P are the price of a plain-vanilla call and put, respectively. Hereinafter, we will call a quantity-triggered option a "high-quantity triggered option" when the quantity variable needs to be above a critical level and a "low-quantity triggered option" when the quantity variable needs to be below a critical level.

Since the options are European style, we can express the terminal values of the risk-neutral price and aggregate quantity, S_T and Q_T, in the following form:

$$\begin{aligned} S_T &= Se^{\int_t^T(r-\delta(\tau))d\tau - \frac{1}{2}\int_t^T\sigma_S^2(\tau)d\tau + \int_t^T\sigma_S(\tau)dz_S^\star(\tau)} \\ &\equiv Se^{r(T-t) - E(\int_t^T\delta(\tau)d\tau) - \frac{1}{2}\int_t^T\sigma_S^2(\tau)d\tau + \Sigma_S\varepsilon_S} \end{aligned} \tag{8}$$

and

$$\begin{aligned} Q_T &= Qe^{\int_t^T\mu_Q(\tau)d\tau - \frac{1}{2}\int_t^T\sigma_Q^2(\tau)d\tau + \int_t^T\sigma_Q(\tau)dz_Q^\star(\tau)} \\ &\equiv Qe^{\int_t^T\mu_Q(\tau)d\tau - \frac{1}{2}\int_t^T\sigma_Q^2(\tau)d\tau + \Sigma_Q\varepsilon_Q} \end{aligned} \tag{9}$$

where ε_S and ε_Q are standard normal variables with correlation coefficient $\bar{\rho}_{SQ}$, and

$$\begin{aligned} \Sigma_S^2 &= \mathrm{var}\left[-\int_t^T\delta(\tau)d\tau + \int_t^T\sigma_S(\tau)dz_S^\star(\tau)\right] \\ &= \mathrm{var}\left[\int_t^T\delta(\tau)d\tau\right] + \mathrm{var}\left[\int_t^T\sigma_S(\tau)dz_S^\star(\tau)\right] \\ &\quad -2\mathrm{cov}\left[\int_t^T\delta(\tau)d\tau, \int_t^T\sigma_S(\tau)dz_S^\star(\tau)\right] \\ \Sigma_Q^2 &= \mathrm{var}\left[\int_t^T\sigma_Q(\tau)dz_Q^\star(\tau)\right] \end{aligned}$$

$$\bar{\rho}_{SQ} = \frac{-\mathrm{cov}\left[\int_t^T\delta(\tau)d\tau, \int_t^T\sigma_Q(\tau)dz_Q^\star(\tau)\right] + \mathrm{cov}\left[\int_t^T\sigma_S(\tau)dz_S^\star(\tau), \int_t^T\sigma_Q(\tau)dz_Q^\star(\tau)\right]}{\Sigma_S\Sigma_Q}$$

with var[·] standing for variance and cov[·] standing for covariance.

In this framework, the futures price and the forward price are identical since the interest rate is constant. The forward price is

$$\begin{aligned} F = E(S_T) &= E[Se^{\int_t^T(r-\delta(\tau))d\tau - \frac{1}{2}\int_t^T\sigma_S^2(\tau)d\tau + \int_t^T\sigma_S(\tau)dz_S^\star(\tau)}] \\ &= Se^{r(T-t) - E(\int_t^T\delta(\tau)d\tau) - \frac{1}{2}\int_t^T\sigma_S^2(\tau)d\tau + \frac{1}{2}\Sigma_S^2} \equiv Se^H \end{aligned} \tag{10}$$

Using Equations (8) and (9) to evaluate the option payoffs in (7), we are able to obtain general expressions for the values of the quantity-triggered options. For instance,

$$\begin{aligned} QTC_H &= e^{-r(T-t)}E[\max(0, S_T - X)\,|\,Q_T \geq Q^\star], \\ &= e^{-r(T-t)}E[S_T|_{Q_T\geq Q^\star, S_T\geq X}] - e^{-r(T-t)}XE[\mathbf{1}\,|_{Q_T\geq Q^\star, S_T\geq X}] \end{aligned} \tag{11}$$

Tedious algebra leads to the following analytical solutions:

$$\begin{aligned} QTC_H = e^{-r(T-t)}[&F \cdot N(d_1, d_2, \bar{\rho}_{SQ}) \\ &- X \cdot N(d_1 - \Sigma_S\bar{\rho}_{SQ}, d_2 - \Sigma_S, \bar{\rho}_{SQ})] \end{aligned} \tag{12}$$

$$\begin{aligned} QTC_L = e^{-r(T-t)}[&F \cdot N(-d_1, d_2, -\bar{\rho}_{SQ}) \\ &- X \cdot N(-d_1 + \Sigma_S\bar{\rho}_{SQ}, d_2 - \Sigma_S, -\bar{\rho}_{SQ})] \end{aligned} \tag{13}$$

$$\begin{aligned} QTP_H = e^{-r(T-t)}[&X \cdot N(d_1 - \Sigma_S\bar{\rho}_{SQ}, -d_2 + \Sigma_S, -\bar{\rho}_{SQ}) \\ &- F \cdot N(d_1, -d_2, -\bar{\rho}_{SQ})] \end{aligned} \tag{14}$$

$$\begin{aligned} QTP_L = e^{-r(T-t)}[&X \cdot N(-d_1 + \Sigma_S\bar{\rho}_{SQ}, -d_2 + \Sigma_S, \bar{\rho}_{SQ}) \\ &- F \cdot N(-d_1, -d_2, \bar{\rho}_{SQ})] \end{aligned} \tag{15}$$

where

$$\begin{aligned} d_1 &= \frac{\ln Q/Q^\star + \int_t^T\mu_Q(\tau)d\tau}{\Sigma_Q} + \Sigma_S\bar{\rho}_{SQ} - \frac{1}{2}\Sigma_Q \\ d_2 &= \frac{\ln F/X + \frac{1}{2}\Sigma_S^2}{\Sigma_S} \end{aligned} \tag{16}$$

where $N(\cdot, \cdot, \rho)$ is the cumulative joint probability distribution for two standard normal variables with a correlation coefficient, ρ, and all other quantities are defined as before. Using the above formulas, we can easily verify the following:

$$C = QTC_H + QTC_L$$
$$= e^{-r(T-t)} [F \cdot N(d_2) - X \cdot N(d_2 - \Sigma_S)] \quad (17)$$

$$P = QTP_H + QTP_L = e^{-r(T-t)}$$
$$\times [X \cdot N(-d_2 + \Sigma_S) - F \cdot N(-d_2)] \quad (18)$$

where $N(\cdot)$ is the cumulative probability distribution for a standard normal variable.

Although the above pricing formulas are general in that the drifts and diffusion coefficients are left unspecified, the distributions are inherently restricted to normal for the convenience yield and lognormal for the spot price and the aggregate quantity. Pricing results will differ when alternative specifications are used. Non-normality, jump risk, and liquidity risk can all lead to different prices. In contrast to the literature on valuation of equity or interest rate derivatives, the literature on commodity derivatives pricing is scanty with respect to alternative model specifications. Hilliard and Reis [1998] consider stochastic interest rates and jumps in the spot price besides stochastic convenience yields. Thanks to the jump component, their setup permits different skewness than the lognormal diffusion for the spot price. They find that jumps do not affect forward or futures prices but do affect option prices. Instead of directly specifying a stochastic convenience yield as commonly done, Schwartz and Smith [2000] develop a two-factor model featuring mean-reversion in the short-term spot price and uncertainty in the reversion target. They show that, although equivalent to the stochastic convenience yield model of Gibson and Schwartz [1990], their model has several advantages. Indeed, they find a better fit of their model to two data sets on oil prices.

It remains an open question as to which specification is the "best" for commodity derivatives pricing. In the end, it will boil down to the task at hand and the trade-off between practicality and complexity. As Schwartz and Smith [2000] put it, "we must balance our desire for fidelity in the price models with the need for parsimony in models used to evaluate complex real or financial options." Our model choice is based on this philosophy. It is also worth noting that our task is to study the hedging efficacy of quantity-triggered options relative to the plain-vanilla options. Insofar as precise pricing is not the focus, any effects of mispricing will be significantly dampened when we only examine the relative hedging performance.

III. A SPECIALIZED MODEL: SPECIFYING THE CONVENIENCE YIELD PROCESS AND DIFFUSION COEFFICIENTS FOR THE ASSET PRICE AND QUANTITY PROCESSES

To begin with, following Gibson and Schwartz [1990], Schwartz [1997], Hilliard and Reis [1998], and Schwartz [1998], we specify the convenience yield process to be a mean-reverting process with the reversion speed (denoted by k), the reversion target (denoted by η), the diffusion coefficient, $\sigma_\delta(t)$, and the market price of risk, $\lambda(t)$, all being constant. Next, for the asset price process and the aggregate quantity process, we assume that the diffusion coefficients $\sigma_S(t)$ and $\sigma_Q(t)$ are all constant. Then the specialized processes in the risk-neutral world are

$$dS = [r - \delta]Sdt + \sigma_S S dz_S^* \quad (19)$$

$$d\delta = k(\overline{\eta} - \delta)dt + \sigma_\delta dz_\delta^* \quad (20)$$

$$dQ = \mu_Q(t)Qdt + \sigma_Q Q dz_Q^* \quad (21)$$

where $\overline{\eta} = \eta - \lambda/k$. Under these assumptions, the relevant quantities in the valuation equations (12)-(15) become

$$H = \left(\overline{\eta} - \delta + \frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k} + \frac{\sigma_\delta^2(e^{-k(T-t)} - 3)}{4k^2}\right)\frac{1 - e^{-k(T-t)}}{k}$$
$$+ \left(r - \overline{\eta} + \frac{\sigma_\delta^2}{2k^2} - \frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k}\right)(T - t)$$

$$\Sigma_S^2 = \left(\frac{\sigma_\delta^2}{k^2} - \frac{2\sigma_S\sigma_\delta\rho_{S\delta}}{k} + \sigma_S^2\right)(T - t) - \frac{2(1 - e^{-k(T-t)})}{k^2}$$
$$\times \left(\frac{\sigma_\delta^2}{k} - \sigma_S\sigma_\delta\rho_{S\delta}\right) + \frac{\sigma_\delta^2}{2k^3}(1 - e^{-2k(T-t)})$$

$$\Sigma_Q^2 = \sigma_Q^2(T - t)$$

$$\overline{\rho}_{SQ} = \frac{\sigma_\delta\rho_{\delta Q}/(k\sqrt{T-t})(\frac{1 - e^{-k(T-t)}}{k} - (T-t)) + \sigma_S\rho_{SQ}\sqrt{T-t}}{\Sigma_S}$$

and $\mu_Q(t)$ can be specified according to the specific commodity in question. It can incorporate seasonality or any other time-dependent features that may be pertinent to the commodity.

IV. COMPARATIVE STATICS AND NUMERICAL ANALYSIS FOR QTOs

Some comparative static results are obvious and a precise expression is not needed to sign them. For instance, for the aggregate quantity variable, we know that $\partial(QTC_H)/\partial Q > 0$, $\partial(QTC_L)/\partial Q < 0$, $\partial(QTP_H)/\partial Q > 0$, $\partial(QTP_L)/\partial Q < 0$, $\partial(QTC_H)/\partial Q^\star < 0$, $\partial(QTC_L)/\partial Q^\star > 0$, $\partial(QTP_H)/\partial Q^\star < 0$, and $\partial(QTP_L)/\partial Q^\star > 0$. Similarly, for the price and exercise price, we have $\partial(QTC_H)/\partial S > 0$, $\partial(QTC_L)/\partial S > 0$, $\partial(QTP_H)/\partial S < 0$, $\partial(QTP_L)/\partial S < 0$, $\partial(QTC_H)/\partial X < 0$, $\partial(QTC_L)/\partial X < 0$, $\partial(QTP_H)/\partial X > 0$, and $\partial(QTP_L)/\partial X > 0$. Moreover, insofar as the convenience yield can be likened to a dividend yield, we could easily infer that $\partial(QTC_H)/\partial\delta < 0$, $\partial(QTC_L)/\partial\delta < 0$, $\partial(QTP_H)/\partial\delta > 0$, and $\partial(QTP_L)/\partial\delta > 0$. Other comparative statics are mathematically cumbersome and we resort to numerical simulations to sign them.

For numerical analyses in this section and the next sections, we focus on crude oil as the underlying asset. We set the base value of the parameters with reference to the empirical estimates of Schwartz [1997]. Specifically, for the oil price process, we set $\sigma_S = 0.30$; for the convenience yield process, we set $\delta = 0.10$, $\eta = 0.15$, $k = 1.5$, $\sigma_\delta = 0.40$, and $\lambda = 0.2$; for the aggregate quantity process, we normalize the current level to $Q = 100$ and set $\mu_Q = 0.20$ (the drift is assumed to be constant for simplicity), and $\sigma_Q = 0.08$; the correlations between the three processes are set at $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, and $\rho_{\delta Q} = -0.60$;[4] the riskfree interest rate is set at $r = 0.06$ p.a.; the time to maturity of the option is one year; and the current oil price is set at $S = \$30$. The critical value of the quantity variable is set at the forward level, i.e., $Q^\star = Qe^{\mu_Q(T-t)} = 122.14$.

The examined parameters are ρ_{SQ}, $\rho_{S\delta}$, $\rho_{\delta Q}$, σ_Q, σ_δ, and σ_S. When varying each parameter, all other parameters take their base values as shown above. We do the calculations for three levels of the exercise price: $X = \$25$, \$30, and \$35. Exhibit 1 contains the results.

The correlation between the asset price and the aggregate quantity, ρ_{SQ}. To begin with, the value of a plain-vanilla option is independent of this correlation, as expected. For high-quantity (low-quantity) triggered call options, a higher negative correlation leads to a lower (higher) option value; the opposite is true for quantity-triggered put options. This observation applies to all moneyness situations. Intuitively, a negative correlation between the asset price and the quantity means that when the asset price is high, the quantity tends to be low; this hurts a high-quantity triggered call and helps a low-quantity triggered call, since when the option is in-the-money, the low quantity level may nullify the high-quantity triggered call while enhancing the survival of the low-quantity triggered call. A similar intuition can be obtained for quantity-triggered put options.

The correlation between the asset price and the convenience yield, $\rho_{S\delta}$. Here, a higher correlation reduces the value of every option for all moneyness situations. The relationship can be intuitively understood by considering the convenience yield as a dividend yield. It is well known that a higher dividend yield reduces a call option's value and increases a put option's value. When the correlation $\rho_{S\delta}$ is high, a higher stock price tends to be accompanied by a higher convenience yield, which hurts call options; and a lower stock price tends to be accompanied by a lower convenience yield, but in this case, the lower yield does not really help since the option is out-of-the-money anyway. A similar line of reasoning can be used to understand the results for put options.

The correlation between the convenience yield and the aggregate quantity, $\rho_{\delta Q}$. Understandably, this correlation does not affect the value of plain-vanilla options. For quantity-triggered call options, a higher negative correlation leads to a higher value of a high-quantity triggered call, and a lower value of a low-quantity triggered call. The opposite is true for quantity-triggered put options. The patterns apply to all moneyness situations. The intuitive understanding lies in the way the convenience yield and the quantity variable affect the option values. For instance, a higher negative correlation, $\rho_{\delta Q}$, means that a higher quantity level is most likely accompanied by a lower convenience yield, and this will help a high-quantity triggered call, but hurt a high-quantity triggered put (due to the role of the convenience yield); the opposite is true for low-quantity triggered call and put options.

The volatility of the aggregate quantity, σ_Q. Again, the value of a plain-vanilla option is not affected by this parameter, as we would expect. Otherwise, a higher σ_Q leads to a lower value of a high-quantity triggered call or put, and a higher value of a low-quantity triggered call or put. This is true for all moneyness situations. To understand the results, we need to closely examine the pricing formulas in (12)-(15) with the specializations following (19)-(21). It is seen that σ_Q enters the formulas only through d_1. Our choice of the critical quantity level $Q^\star = Qe^{\mu_Q(T-t)}$ means that $d_1 = \Sigma_S\bar{\rho}_{SQ} - \frac{1}{2}\sigma_Q\sqrt{T-t}$. It becomes obvious that a higher σ_Q will decrease (increase) the risk-neutral probability for the quantity variable to be above (below) the critical level, and hence the observed price patterns.[5]

The volatility of the convenience yield, σ_δ. To begin with, this variable affects both the quantity-triggered options and the plain-vanilla options. It appears that the impacts are parameter- and option specific. The complexities arise from the fact that this volatility affects just about every element of the pricing formula: the forward price of the asset, F, the composite correlation, $\bar{\rho}_{SQ}$, and the arguments for the bivariate normal distribution, d_1 and d_2.

The volatility of the underlying asset, σ_S. It is seen that for almost all options, a higher σ_S leads to a higher option value. This is largely expected. However, for an in-the-money, high-quantity triggered call option, a higher asset volatility is not always desirable. This has to do with the fact that σ_S affects both the forward price F (negatively) and the asset return distribution, and the fact that the correlation ρ_{SQ} is highly negative (-0.5). In fact, when ρ_{SQ} is, e.g., -0.8, then a higher σ_S will lead to a lower value for the high-quantity triggered option; when ρ_{SQ} is, e.g., -0.1, then a higher σ_S will lead to a higher value for the high-quantity triggered option. This could be understood in conjunction with the results for the case of ρ_{SQ}.

All told, the impacts of the three correlation coefficients, ρ_{SQ}, $\rho_{S\delta}$, and $\rho_{\delta Q}$, and the quantity variable's volatility, σ_Q can all be determined unambiguously directional-wise; those of the convenience yield's volatility, σ_δ, and the asset price volatility, σ_S can be option specific. This is especially so for the convenience yield's volatility, σ_δ.

EXHIBIT 1
Quantity-Triggered and Vanilla Option Prices

Panel A: Exercise Price X = $25

		QTC_H	QTC_L	C	QTP_H	QTP_L	P
ρ_{SQ}	−0.25	2.328	2.442	4.770	0.331	0.362	0.693
	−0.50	1.663	3.107	4.770	0.490	0.203	0.693
	−0.75	0.990	3.780	4.770	0.634	0.059	0.693
$\rho_{S\delta}$	0.45	2.085	3.532	5.617	0.716	0.395	1.111
	0.70	1.830	3.269	5.099	0.584	0.278	0.862
	0.95	1.544	2.997	4.541	0.420	0.151	0.571
$\rho_{\delta Q}$	−0.30	1.320	3.450	4.770	0.568	0.125	0.693
	−0.60	1.663	3.107	4.770	0.490	0.203	0.693
	−0.90	2.005	2.765	4.770	0.409	0.284	0.693
σ_Q	0.05	1.691	3.079	4.770	0.494	0.199	0.693
	0.15	1.600	3.170	4.770	0.482	0.212	0.693
	0.25	1.512	3.258	4.770	0.469	0.225	0.693
σ_d	0.20	1.579	3.692	5.270	0.718	0.228	0.946
	0.40	1.663	3.107	4.770	0.490	0.203	0.693
	0.60	1.889	2.649	4.538	0.344	0.225	0.569
σ_S	0.15	2.235	2.335	4.571	0.017	0.020	0.037
	0.30	1.663	3.107	4.770	0.490	0.203	0.693
	0.45	1.594	4.052	5.646	1.429	0.589	2.018

Panel B: Exercise Price X = $30

		QTC_H	QTC_L	C	QTP_H	QTP_L	P
ρ_{SQ}	−0.25	1.000	1.042	2.042	1.283	1.391	2.674
	−0.50	0.594	1.448	2.042	1.700	0.974	2.674
	−0.75	0.218	1.824	2.042	2.142	0.532	2.674
ρ_{Sd}	0.45	0.999	1.972	2.971	1.910	1.264	3.174
	0.70	0.753	1.656	2.409	1.786	1.095	2.881
	0.95	0.482	1.297	1.779	1.636	0.881	2.517
ρ_{dQ}	−0.30	0.395	1.647	2.042	1.922	0.752	2.674
	−0.60	0.594	1.448	2.042	1.700	0.974	2.674
	−0.90	0.801	1.241	2.042	1.484	1.190	2.674
σ_Q	0.05	0.605	1.437	2.042	1.716	0.958	2.674
	0.15	0.569	1.473	2.042	1.664	1.010	2.674
	0.25	0.534	1.508	2.042	1.611	1.063	2.674
σ_d	0.20	0.618	1.978	2.597	2.037	0.944	2.982
	0.40	0.594	1.448	2.042	1.700	0.974	2.674
	0.60	0.679	1.097	1.775	1.413	1.102	2.515
σ_S	0.15	0.506	0.515	1.020	0.567	0.629	1.196
	0.30	0.594	1.448	2.042	1.700	0.974	2.674
	0.45	0.807	2.552	3.358	2.921	1.518	4.439

V. MINIMIZING VaR WITH QUANTITY-TRIGGERED OPTIONS

The Preliminaries

Now we address the matter of the optimal use of quantity-triggered options in the context of VaR management. VaR is typically defined as the dollar loss at a given percentile on the distribution of an institution's portfolio over a certain period of time. In statistical terms, the phrasing of VaR usually reads like, "With a $(1 - \alpha)$ confidence level, the dollar loss on the portfolio or asset will not exceed \$VaR within the next τ period," where α represents the percentile of the lower tail of the distribution, e.g., $\alpha = 0.05$. As seen, there are three key elements in a VaR statement: the confidence level $(1 - \alpha)$, the length of the time interval τ, and the dollar amount of loss VaR. Keeping the time interval fixed, the higher the confidence level, the bigger the dollar loss; keeping

EXHIBIT 1
Continued

	Panel C: Exercise Price X = $35						
		QTC_H	QTC_L	C	QTP_H	QTP_L	P
ρ_{SQ}	20.25	0.350	0.361	0.711	2.911	3.141	6.052
	20.50	0.166	0.545	0.711	3.552	2.500	6.052
	20.75	0.030	0.681	0.711	4.233	1.819	6.052
$\rho_{S\delta}$	0.45	0.431	0.996	1.427	3.621	2.717	6.339
	0.70	0.261	0.719	0.980	3.574	2.587	6.161
	0.95	0.109	0.425	0.534	3.543	2.438	5.981
$\rho_{\delta Q}$	20.30	0.087	0.624	0.711	3.894	2.158	6.052
	20.60	0.166	0.545	0.711	3.552	2.500	6.052
	20.90	0.258	0.453	0.711	3.220	2.832	6.052
σ_Q	0.05	0.170	0.541	0.711	3.588	2.464	6.052
	0.15	0.158	0.553	0.711	3.467	2.585	6.052
	0.25	0.148	0.564	0.711	3.345	2.707	6.052
σ_δ	0.20	0.205	0.921	1.125	3.903	2.316	6.219
	0.40	0.166	0.545	0.711	3.552	2.500	6.052
	0.60	0.184	0.348	0.531	3.197	2.783	5.980
σ_S	0.15	0.033	0.032	0.065	2.373	2.576	4.949
	0.30	0.166	0.545	0.711	3.552	2.500	6.052
	0.45	0.388	1.529	1.917	4.782	2.925	7.707

Note:
1. QTC_H: quantity-triggered call option when the quantity needs to be above a critical level, QTC_L: quantity-triggered call option when the quantity needs to be below a critical level, QTP_H: quantity-triggered put option when the quantity needs to be above a critical level, QTP_L: quantity-triggered put option when the quantity needs to be below a critical level, C: plain-vanilla call option, and P: plain-vanilla put option.
2. Base parameter values: $S = \$30$, $Q = 100$, $Q^* = 122.14$ (the forward level), $T - t = 1$, $r = 0.06$, $\sigma_S = 0.30$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$.

the confidence level fixed, the longer the time interval, the bigger the dollar loss. Therefore, a primary task in VaR management is to seek for the hedging scheme that will minimize VaR subject to a fixed hedging cost, a given confidence level, and a specified time interval.

In this study, we adopt a setup similar to that in Ahn et al. [1999]. Specifically, we assume that the firm uses only put options to hedge, and, in the end, as demonstrated by Ahn et al. [1999], the firm needs only one put option (as opposed to a combination of options with different exercise prices) to minimize VaR. Ahn et al. [1999] use plain-vanilla options to conduct the hedge. They show that, for a given hedging cost, a prespecified confidence level and a time period, there indeed exists an optimal exercise price for the single put option that will minimize VaR. In other words, optimal hedging can be achieved in their framework. For commodities, we could usually identify a relationship between the price and the total supply or aggregate quantity. This suggests that, for more effective hedging, firms could utilize quantity-triggered put options. Then, the questions at hand are What is the optimal exercise price of the quantity-triggered put that will minimize Value-at-Risk? Given the same hedging cost, confidence level, and the hedging horizon, which type of option is more effective in reducing VaR—the quantity-triggered option or the plain-vanilla option? What is the role of the correlation between the firm-level quantity and the aggregate-level quantity given that only the aggregate quantity affects the option's payoff?

To address the above questions, we need to properly quantify VaR. In defining VaR, Ahn et al. [1999] use the riskfree investment as the benchmark. However, in our case, because the convenience yield is also stochastic, it is appropriate to use the forward price as the benchmark.[6] Moreover, unlike Ahn et al. [1999], we consider both the price risk and the quantity risk. Although we will use QTOs based on the aggregate quantity, our VaR should be based on the firm's quantity To this end, we assume that the firm's quantity Q^i also follows a lognormal process. Assuming a constant volatility, we can write the risk-neutral process as

$$dQ^i = \mu_{Q^i}(t)Q^i dt + \sigma_{Q^i}Q^i dz^{\star}_{Q^i} \qquad (22)$$

The correlations with previously defined variables are $E[dz_S, dz_{Q^i}] = \rho_{SQ^i}dt$, $E[dz_\delta, dz_{Q^i}] = \rho_{\delta Q^i}dt$, and $E[dz_Q, dz_{Q^i}] = \rho_{QQ^i}dt$.[7] We can then define our risk variable as $Y = SQ^i$, which is simply the total revenue of the firm. Obviously, the variable Y encompasses both the price risk and the quantity risk.

The above elements enable us to define VaR in our framework as

$$VaR = F_Y - Ye^{\theta(\alpha)} \qquad (23)$$

where F_Y is the forward level of Y (in the risk-neutral world) and $Ye^{\theta(\alpha)}$ is the point at the α percentile of the lower tail distribution of Y_T (in the real world). Using Ito's lemma, we have $F_y = E(Y_T) \equiv Ye^{H_Y}$ and

$$\theta(\alpha) = (\mu_S + \mu_{Q^i} - \eta)(T - t) - (\delta - \eta)\frac{1 - e^{-k(T-t)}}{k} - \frac{1}{2}(\sigma_S^2 + \sigma_{Q^i}^2)(T - t) + z(\alpha)\Sigma_Y \qquad (24)$$

where

$$H_Y = \left(\mu_{Q'} + r + \sigma_S\sigma_{Q'}\rho_{SQ'} - \bar{\eta} - \frac{\sigma_{Q'}\sigma_\delta\rho_{\delta Q'}}{k} + \frac{\sigma_\delta^2}{2k^2} - \frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k}\right)(T-t) + \left(\bar{\eta} - \delta - \frac{\sigma_\delta^2}{k^2} + \frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k} + \frac{\sigma_{Q'}\sigma_\delta\rho_{\delta Q'}}{k}\right)\frac{1-e^{-k(T-t)}}{k} + \frac{\sigma_\delta^2}{4k^3}(1 - e^{-2k(T-t)})$$

and

$$\Sigma_Y^2 = \left(\frac{\sigma_\delta^2}{2k^2} - \frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k} - \frac{\sigma_{Q'}\sigma_\delta\rho_{\delta Q'}}{k} + \frac{1}{2}(\sigma_s^2 + \sigma_{Q'}^2 + 2\sigma_s\sigma_{Q'}\rho_{SQ'})\right)(T-t) + \left(\frac{\sigma_S\sigma_\delta\rho_{S\delta}}{k} + \frac{\sigma_{Q'}\sigma_\delta\rho_{\delta Q'}}{k} - \frac{\sigma_\delta^2}{k^2}\right)\frac{1 - e^{-k(T-t)}}{k} + \frac{\sigma_\delta^2}{4k^3}(1 - e^{-2k(T-t)})$$

Here, we assume that the drifts of the asset price and the firm quantity, μ_S and $\mu_{Q'}$, respectively, are constant, and $z(\alpha)$ is a standard normal variate corresponding to the α percentile of the distribution. For example, $z(0.5) = 0$, $z(1.0) = +\infty$, $z(0.025) = -1.96$, and so on.

For a typical commodity producer, price risk exhibits a stronger impact on revenue than does quantity risk. This is why put options on the asset price, or plain-vanilla puts, are usually used for hedging. Our framework expands the set of hedging tools and allows us to ascertain whether a quantity-triggered put is superior in hedging given the same hedging cost. Before going further, we must decide on the category of quantity-triggered put options to be used for hedging. Since the most devastating scenario for a commodity producer is the combination of a lower price and a lower quantity, a low-quantity triggered put seems to be the right choice for hedging. But this conclusion is erroneous since our objective is to minimize VaR at a particular confidence level, which is different from avoiding the worse case scenario. With a negative correlation between price and quantity, the chance for a lower price and a lower quantity to occur simultaneously is extremely low. Therefore, hedging with a low-quantity triggered put will leave the price risk uncovered most of the time. In contrast, a high-quantity triggered put is the ideal choice given the negative correlation between price and quantity. When the realized price is low, the quantity tends to be high and the option is triggered, which completes the hedge. Of course, a high-quantity triggered put will "miss" when the realized price and quantity are both low. But as mentioned before, this is a low probability event. The key question is will the cost savings from using a high-quantity triggered put more than offset the potential loss due to the uncovered exposure in the overall VaR?

Let B be the total hedging cost in dollars and h $(0 < h < 1)$ be the hedge ratio, or the fraction of a put for one unit of the underlying asset. Then, when using a high-quantity triggered put option to hedge, the total hedging cost can be expressed as $B = Q^i h(QTP_H)$ and the VaR can be expressed as follows:

$$\begin{aligned} VaR &= F_Y - Ye^{\theta(\alpha)} + Q^i h(QTP_H)e^{r(T-t)}, \quad \text{if } Q_T \leq Q^* \\ &= F_Y - Ye^{\theta(\alpha)} + Q^i h(QTP_H)e^{r(T-t)} - Q^i h\left(X - \frac{Ye^{\theta(\alpha)}}{Q_T^i}\right), \quad \text{if } Q_T \geq Q^* \end{aligned}$$

Integrating over the two quantity variables conditional on the α percentile of the distribution for Y_T, and following the spirit of Ahn et al. [1999], our optimization problem becomes

$$\min_{h,X} VaR = F_Y - Ye^{\theta(\alpha)} + Q^i h(QTP_H)e^{r(T-t)} - Q^i h\left[X\frac{\Phi_2}{\Phi_1} - Se^{\theta(\alpha)-(\mu_{Q'}-\sigma_{Q'}^2)(T-t)}\frac{\Phi_3}{\Phi_1}\right] \quad (25)$$

$$\text{s.t. } B = Q^i h(QTP_H)$$

where

$$\Phi_1 = N(d_2' - \Sigma_Y)$$

$$\Phi_2 = N(d_1' - \Sigma_Y\bar{\rho}_{YQ}, d_2' - \Sigma_Y, \bar{\rho}_{YQ}) \quad (26)$$

$$\Phi_3 = N\left(d_1' - \Sigma_Y\bar{\rho}_{YQ} - \frac{\sigma_Q\sigma_{Q'}\rho_{QQ'}(T-t)}{\Sigma_Q}, d_2' - \Sigma_Y - \frac{\sigma_{YQ'}}{\Sigma_Y}, \bar{\rho}_{YQ}\right)$$

$$d_1' = \frac{\ln(Q/Q^*) + \mu_Q(T-t)}{\Sigma_Q} + \Sigma_Y\bar{\rho}_{YQ} - \frac{1}{2}\Sigma_Q$$

$$d_2' = \frac{\ln[F_Y/(Ye^{\theta(\alpha)})] + \frac{1}{2}\Sigma_Y^2}{\Sigma_Y} \quad (27)$$

$$\sigma_{YQ'} = \frac{\sigma_\delta \sigma_{Q'} \rho_{\delta Q'}}{k^2}(1 - e^{-k(T-t)})$$

$$+ \left(\sigma_{Q'}^2 + \sigma_S \sigma_{Q'} \rho_{SQ'} - \frac{\sigma_\delta \sigma_{Q'} \rho_{\delta Q'}}{k} \right)(T-t)$$

$$\bar{\rho}_{YQ}$$

$$= \frac{\sigma_\delta \sigma_Q \rho_{\delta Q}(1 - e^{-k(T-t)})/k_2 + (\sigma_Q \sigma_{Q'} \rho_{QQ'} + \sigma_S \sigma_Q \rho_{SQ} - (\sigma_\delta \sigma_Q \rho_{\delta Q})/k)(T-t)}{\Sigma_Q \Sigma_Y}$$

$$F'_Y = Ye^{H'_Y} \tag{28}$$

$$H'_Y = \Bigg(\mu_{Q'} + \mu_S + \sigma_S \sigma_{Q'} \rho_{SQ'} - \eta$$

$$- \frac{\sigma_{Q'} \sigma_\delta \rho_{\delta Q'}}{k} + \frac{\sigma_\delta^2}{2k^2} - \frac{\sigma_S \sigma_\delta \rho_{S\delta}}{k} \Bigg)(T - t)$$

$$+ \left(\eta - \delta - \frac{\sigma_\delta^2}{k^2} + \frac{\sigma_S \sigma_\delta \rho_{S\delta}}{k} + \frac{\sigma_{Q'} \sigma_\delta \rho_{\delta Q'}}{k} \right)$$

$$\times \frac{1 - e^{-k(T-t)}}{k} + \frac{\sigma_\delta^2}{4k^3}(1 - e^{-2k(T-t)})$$

Notice that hedging with the quantity-triggered put option augments the expression in Equation (23) in two ways. First, VaR increases by the terminal value of the hedging cost (compounded at the riskfree rate). Second, VaR is reduced by the option's payoff (conditional on $Y_T > Ye^{\theta(\alpha)}$ and $Q_T \geqslant Q^\star$) multiplied by the hedge ratio h and the current quantity of the underlying asset Q^i. It can be seen that when $Q^\star = 0$, the VaR expression in Equation (25) reduces to the one corresponding to hedging with a plain-vanilla put option. The answer to the question "which type of option is more effective in reducing VaR" lies in the trade-off apparent in Equation (25). When we replace the plain-vanilla put with the quantity-triggered put (under the same hedging cost), the hedge ratio increases, thanks to the lower cost of the option; this benefit comes with a cost: the lower payoff due to conditioning on the aggregate quantity. The net result of the trade-off will determine the effectiveness when the exercise price is optimized. To this end, the first-order condition leads to

$$X^\star \Phi_2 - Se^{\theta(\alpha) - (\mu_{Q'} - \sigma_{Q'}^2)(T-t)} \Phi_3 = \frac{(QTP_H)\Phi_2}{(\partial(QTP_H)/\partial X)}$$

$$= \left[X^\star - \frac{F \cdot N(d_1, -d_2, -\bar{\rho}_{SQ})}{N(d_1 - \Sigma_S \bar{\rho}_{SQ}, -d_2 + \Sigma_S, -\bar{\rho}_{SQ})} \right] \Phi_2$$

Equivalently,

$$Se^{\theta(\alpha) - (\mu_{Q'} - \sigma_{Q'}^2)(T-t)} \frac{\Phi_3}{\Phi_2}$$

$$= \frac{F \cdot N(d_1, -d_2, -\bar{\rho}_{SQ})}{N(d_1 - \Sigma_S \bar{\rho}_{SQ}, -d_2 + \Sigma_S, -\bar{\rho}_{SQ})} \tag{29}$$

Unfortunately, the optimal exercise price cannot be solved analytically and we must resort to numerical procedures. While one can express all the derivatives of the optimal exercise price with respect to the other parameters, it is hard to sign the said derivatives. Thus, we elect to make plots of the optimal exercise price, $X^\star$, against other parameters.

As pointed out earlier, when we remove the quantity condition, i.e., when $Q^\star = 0$, our case degenerates to minimizing VaR using a plain-vanilla put. In this case, the optimal exercise price is solved from the following equation:

$$Se^{\theta(\alpha) - (\mu_{Q'} - \sigma_{Q'}^2)(T-t)} \frac{N(d'_2 - \Sigma_Y - \sigma_{YQ'}/\Sigma_Y)}{\Phi_1}$$

$$= \frac{F \cdot N(-d_2)}{N(-d_2 + \Sigma_s)} \tag{30}$$

Notice that even with the plain-vanilla options, our result is still not exactly the same as Ahn et al.'s, since we allow a stochastic convenience yield and consider both the price risk and the quantity risk. If we further restrict the convenience yield and the firm-level quantity to be constant, then it is easy to see that our solution reduces to that of Ahn et al. [1999].

Optimal Exercise Price Versus Other Key Parameters

To study how the optimal exercise price is affected by various parameters, we solve for the optimal exercise price, $X^\star$ from Equations (29) and (30) at different levels of parameter inputs. For all parameters, we use the same base values as in the previous section. In addition, the asset drift μ_S is set at 0.2 and the VaR percentile α is set at 0.025. For the firm-quantity variable, we set $\mu_{Q'} = 0.20$, $\sigma_{Q'} = 0.10$, $\rho_{SQ'} = -0.40$, $\rho_{\delta Q'} = -0.50$, and $\rho_{QQ'} = 0.60$. Without loss of generality, we set $Q^i = 1.0$.[8] Exhibit 2 contains the results.

Panel A plots the optimal exercise price against the asset drift, μ_S. It is seen that regardless of which option

Exhibit 2

Optimal Exercise Price Versus Other Parameter Values

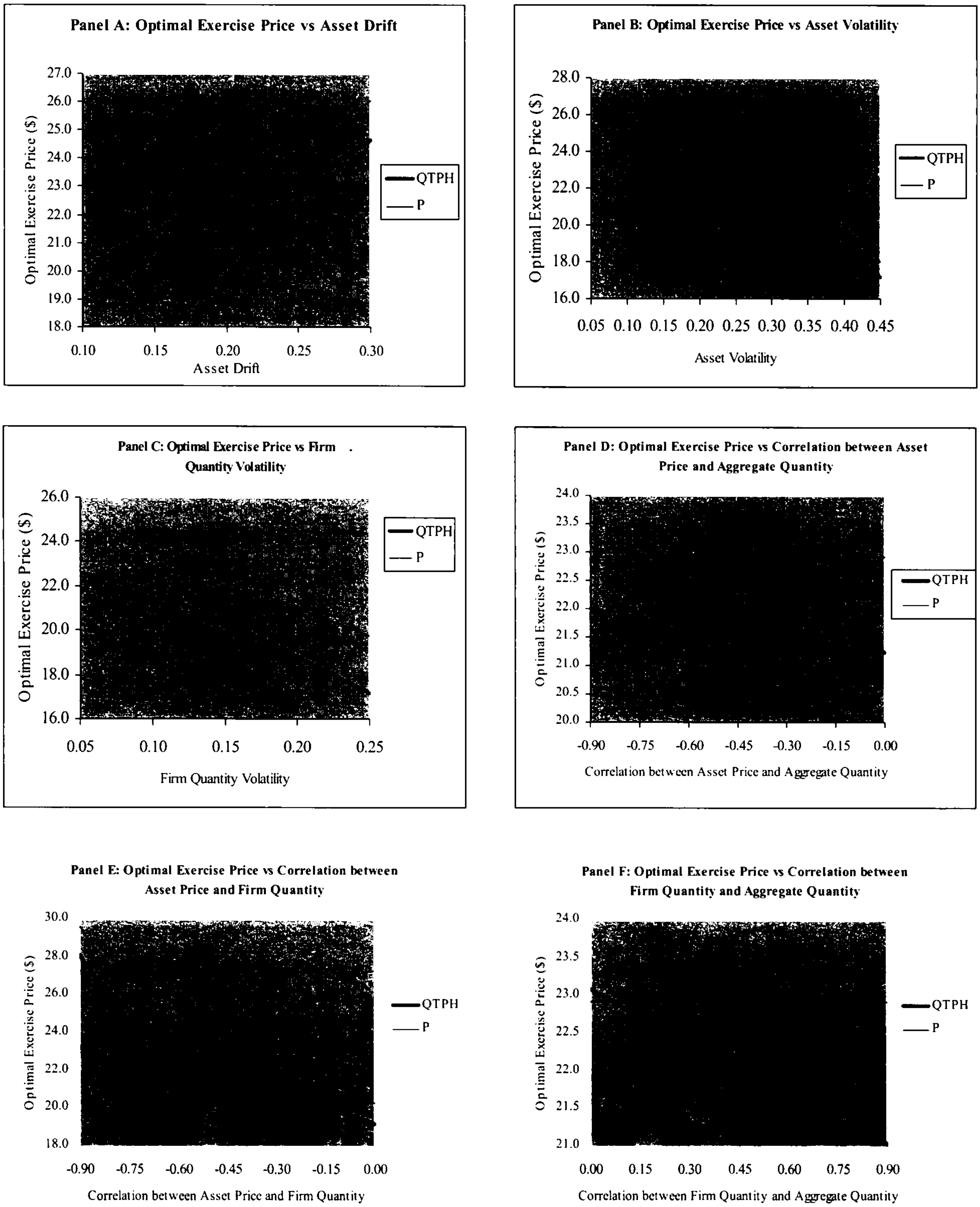
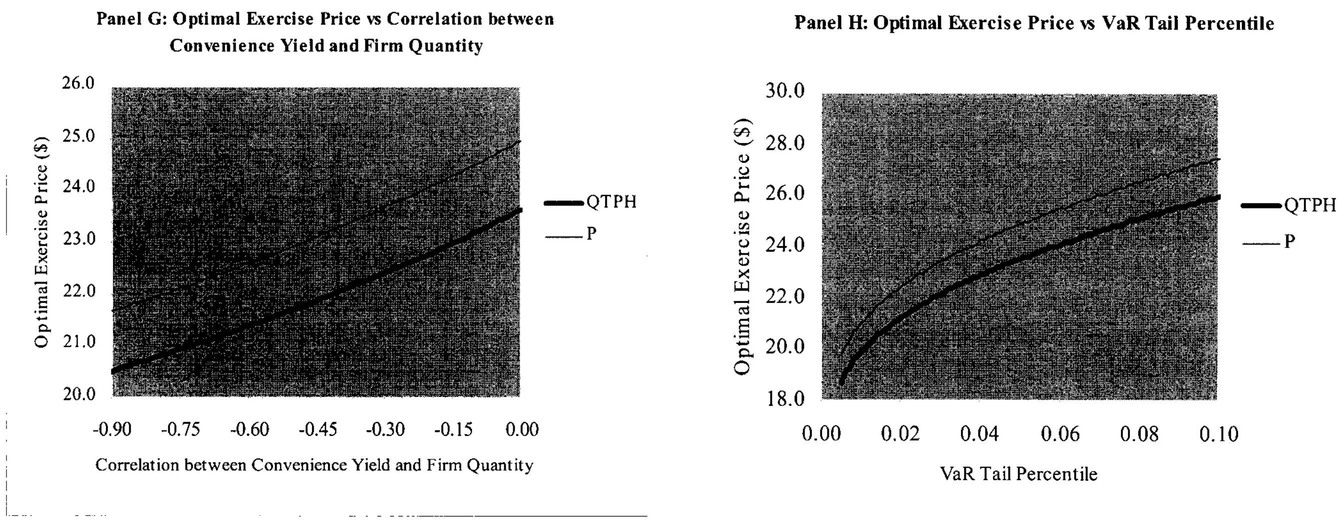


Exhibit 2
Continued



Note:
1. Optimal exercise price is the exercise price that minimizes VaR. The curve with the legend *QTPH* is for the case of using a quantity-triggered put option (when the quantity needs to exceed a critical level), *P* is for the case of using a vanilla put option.
2. Base parameter values: S = \$30, $Q^i = 1.0$, $Q = 100$, $Q^* = 122.14$ (the forward level), $T - t = 1$, $r = 0.06$, $\sigma_S = 0.30$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\sigma_{Q^i} = 0.10$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$, $\rho_{SQ^i} = -0.40$, $\rho_{\delta Q^i} = -0.50$, $\rho_{QQ^i} = 0.60$, $\alpha = 0.025$.

is used to minimize VaR, a higher asset drift requires a higher exercise price, similar to the finding by Ahn et al. [1999]. Intuitively, a higher drift shifts the return distribution to the right, and as a result, to achieve the same level of protection, the exercise price of the put option must rise accordingly. It is also interesting to see that for all levels of the asset drift, the optimal exercise price for the quantity-triggered option is lower than that for the plain-vanilla option.

Panels B and C plot the optimal exercise price against the asset volatility, σ_S, and the firm quantity volatility, σ_Q, respectively. In Panel B, for both types of options, the optimal exercise price goes up first, reaches to a peak, then goes down as the asset volatility increases. Moreover, the optimal exercise price is lower for the quantity-triggered put. The downward sloping part of the curve is similar to the relationship uncovered by Ahn et al. [1999]. As shown in Exhibit 1, when the asset volatility is not very low and increasing, the value of a high-quantity triggered put will increase; meanwhile, a higher volatility also means a bigger dispersion in the price distribution, i.e., the critical asset price corresponding to the same level of α will shift to the left. To keep the same level of statistical confidence, the exercise price must also fall, and hence the observed downward sloping pattern in Panel B. Now, when the asset volatility is in the low range, the dynamic of the convenience yield dominates the standard deviation of the terminal asset price, Σ_S, which results in the observed upward sloping pattern. As for the firm quantity volatility, a higher volatility leads to lower optimal exercise prices for both options. Since firm quantity is the other source of the overall risk, its volatility plays a similar role as the asset price volatility, except that the convenience yield dynamic does not affect firm quantity volatility. As a result, the firm quantity volatility only serves to increase the dispersion of the total revenue's distribution, and hence the downward pattern in the optimal exercise price. The impact of the aggregate quantity's volatility is very minimal on the optimal exercise for the quantity-triggered put option. We therefore omit the plot for brevity.

Panels D through G plot the optimal exercise price against various correlations. To begin with, the optimal exercise price for the quantity-triggered option is lower than the one for the plain-vanilla option in almost all cases. For the correlation between the asset price and the aggregate quantity, ρ_{SQ}, the optimal exercise price for the plain-vanilla option is not affected by this correlation, as we would expect. As for the quantity-triggered option, the optimal exercise price takes a similar pattern as in

Panel B (i.e., a humped pattern). In contrast, a bigger negative correlation between the asset price and the firm quantity will lead to a higher optimal exercise price for both types of options.[9] The correlation between the firm quantity and the aggregate quantity does not affect the optimal exercise price for the plain-vanilla option; however, it negatively affects that for the quantity-triggered option. The result is very intuitive: a lower correlation means a bigger chance that the option may not provide the payoff when needed, and hence a higher exercise price is required for an adequate protection. Finally, a less negative correlation between the convenience yield and the firm quantity leads to a higher optimal exercise price for both options. In addition, although not shown, for the quantity-triggered option, the optimal exercise price is almost invariant to the correlation between the convenience yield and the aggregate quantity.

Panel H of Exhibit 2 plots the optimal exercise price against the VaR percentile, α. The optimal exercise price for the quantity-triggered option is always lower, and for both options, a higher α corresponds to a higher exercise price. This makes perfect intuitive sense, since a higher α means a lower VaR or a higher protection, which requires a higher exercise price of the put option.

VaR Comparative Statics and the Effectiveness of the Quantity-Triggered Put Option in Minimizing VaR

Now we address the central question, "How does the VaR depend on the key parameters, and under what circumstances is the quantity-triggered put option more effective than the plain-vanilla put option in minimizing VaR?" We first study how some key parameters impact the VaR value under three scenarios: no hedge, hedging with a high-quantity triggered put, and hedging with a plain-vanilla put. In so doing, we will also be able to determine which option is more effective in hedging. To this end, we calculate the minimized VaR for each parameter value combination. To ensure comparability, for each parameter combination, we apply the same hedging cost to both hedging scenarios. Also, to avoid too low option prices, we set the critical quantity level at $Q^* = 100$ for all calculations. All other parameters take the same base values as before. Exhibit 3 presents plots that are counterparts of Exhibit 2.[10] Before we make close observations, we should realize that the VaR under the two hedging scenarios should always be lower than that without hedging, thanks to the optimization procedure. This is confirmed in all plots, and we will no longer make specific mentions.

Panel A plots the VaR against the asset drift. It is seen that a higher asset drift leads to a lower VaR, as we would expect. But the performances of the two types of options are almost identical.

Panel B contains the plots for the asset price volatility. The VaR value exhibits a U-shaped pattern at the lower end of the volatility range. More interestingly, when the volatility is low, the quantity-triggered put dominates the plain-vanilla put in reducing VaR; otherwise, the two types of options have similar performances. The U-shaped pattern is largely due to the stochastic nature of the convenience yield. With a constant convenience yield, the curves are upward monotone. We will postpone to Exhibit 4 the detailed investigations regarding the impact of the asset volatility. Panel C plots the VaR against the firm quantity volatility. Again, the performances of the two types of options are indistinguishable. Nonetheless, a higher volatility leads to a higher VaR, as expected.

Panels D through G plots the VaR against various correlations. Panel D is for the correlation between the asset price and the aggregate quantity, ρ_{SQ}. Understandably, when no hedge is in place or when the hedge is conducted with the plain-vanilla put, the VaR value is not affected by this correlation. As for hedging with the quantity-triggered put, although it is not visually clear, the VaR value goes down as the correlation becomes lower.[11] Moreover, the quantity-triggered option becomes more effective when the correlation is close to zero. Intuitively, as the correlation goes down, the put option becomes cheaper, resulting in a higher hedge ratio, which more than offsets the potentially lower payoff of the put. Again, we will make further investigations on this point in Exhibit 4. For the remaining three correlations in Panels E, F and G, only the correlation between the firm quantity and the aggregate quantity leads to mildly discernible differences in performance between the two types of options, which will be investigated further in Exhibits 4 and 5.

Finally, Panel H is for the VaR percentile variable, α. The VaR goes down as α increases, which is purely definitional. Moreover, the plain-vanilla put and the quantity-triggered put have very similar performances in reducing VaR.

So far we have identified three situations where the quantity-triggered put dominates the plain vanilla put: 1) the asset volatility is low, 2) the correlation between the asset price and the aggregate quantity is low, and

EXHIBIT 3
VaR Versus Other Key Parameters

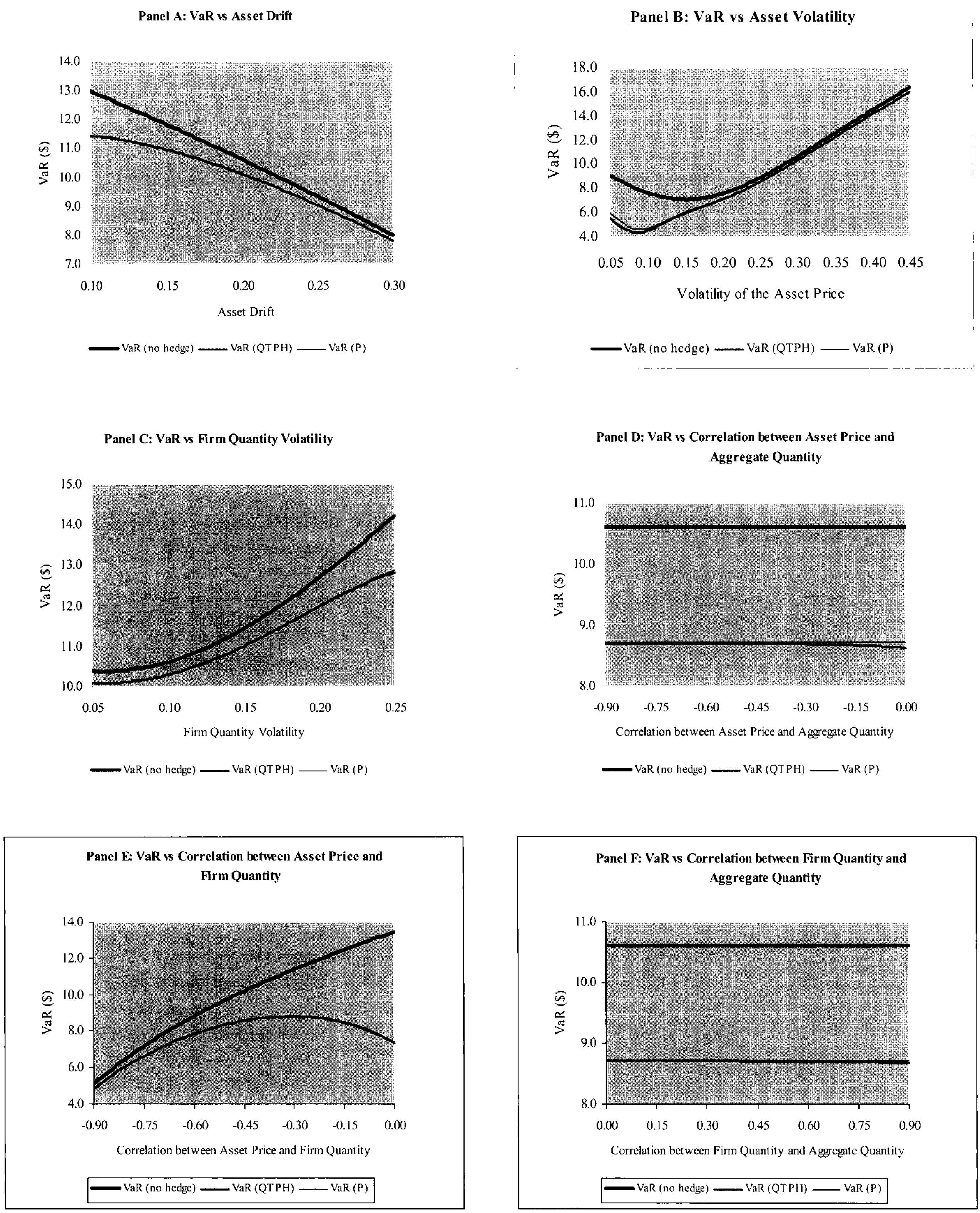
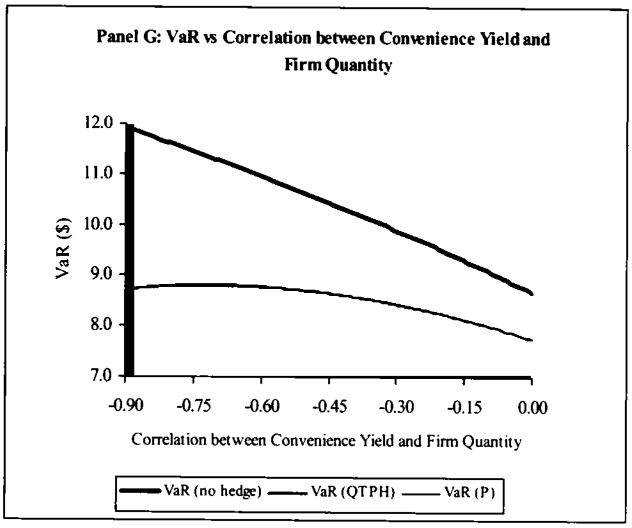
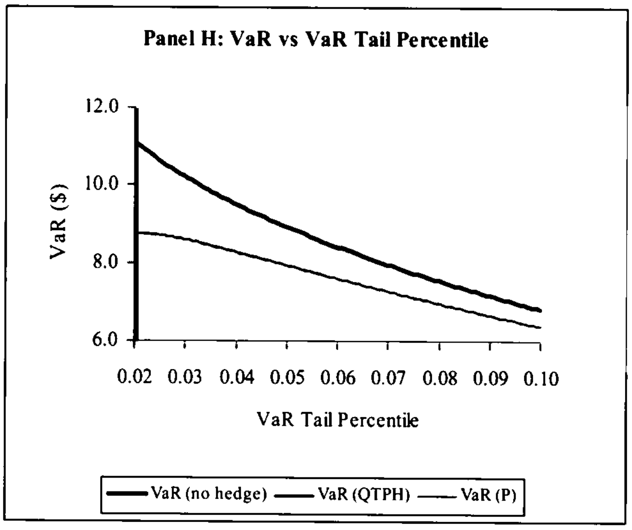


Exhibit 3
Continued





Note:

1. VaR takes the optimized value at each level of the parameter in question. Each graph contains three plots of VaR: no hedge, hedging with a quantity-triggered put option (*QTPH*), and hedging with a plain-vanilla put option (*P*). The hedging cost in each graph is set equal to the lowest value of the quantity-triggered put option within the parameter range considered in order to ensure that the hedge ratio is always less than one. The critical quantity level, Q^*, is set at 100.00 (instead of the forward level, 122.14) to avoid too low option prices in certain cases.
2. Base parameter values: $S = \$30$, $Q^i = 1.0$, $Q = 100$, $T - t = 1$, $r = 0.06$, $\mu_S = 0.20$, $\sigma_S = 0.30$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\mu_{Q^i} = 0.20$, $\sigma_{Q^i} = 0.10$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$, $\rho_{SQ^i} = -0.40$, $\rho_{\delta Q^i} = -0.50$, $\rho_{QQ^i} = 0.60$, $\alpha = 0.025$.

3) the correlation between the firm quantity and the aggregate quantity is high. But this still does not present a complete picture, since we keep all the other parameters at the base level while varying a particular one. In order to fully identify the situations where the quantity-triggered option is superior in reducing VaR, we first examine in Exhibit 4 the relative effectiveness of the two options under the full range of each correlation coefficient at different levels of the asset volatility.

Here, the relative hedging performance of the two types of options is measured by $(VaR_2 - VaR_1)/VaR_0 \times 100\%$, where VaR_0, VaR_1, and VaR_2 are, respectively, the VaR values without hedging, hedging with the quantity-triggered put option, and hedging with the plain-vanilla put option. For comparability, the hedging cost is fixed at $B = \$0.05$ for all cases. It should be noted that the above performance measure is purely for comparison purposes, and it does not necessarily measure the maximum possible improvement. In fact, the higher the hedging cost, the bigger the improvement. The hedging cost of \$0.05 is the lowest denominator to ensure a hedge ratio below one. By the same token, while we examine the range of one correlation, we keep the other at the base level, and as a result, we are not searching the maximum possible improvement in terms of full range parameter combinations. With the above notes and caveats, we are now ready to delineate the results.

Panel A contains the results for the correlation between the asset price and the aggregate quantity, ρ_{SQ}. It confirms the observations in Panels B and D of Exhibit 3, namely, the quantity-triggered put is more effective than the plain-vanilla put when the asset volatility is low and/or when the correlation between the asset price and the aggregate quantity is low. The improvement is quite sizeable judged by the percentage measure we devised. As for the correlation between the firm quantity and the aggregate quantity, Panel B confirms the observations in Panel F of Exhibit 3. Here, a combination of a very low correlation and an asset volatility above 20% will render the quantity-triggered option less effective in reducing VaR. Otherwise, the quantity-triggered option is always more effective.

When we study the impact of the correlation between the firm quantity and the aggregate quantity, we need to realize that this correlation exerts its impact through the covariance. In other words, the volatilities of

EXHIBIT 4

Improvement of QTP_H over Plain-Vanilla Put Option in Minimizing VaR: Asset Volatility and Correlations

	σ_S							
	0.05	0.08	0.11	0.14	0.17	0.20	0.23	0.26
ρ_{SQ}				Panel A				
0.0	14.47	28.03	25.34	8.69	2.92	1.35	0.79	0.53
-0.1	11.61	17.17	12.63	4.83	1.80	0.86	0.51	0.34
-0.2	9.48	11.59	7.57	2.95	1.15	0.55	0.32	0.21
-0.3	7.83	8.22	4.88	1.89	0.74	0.36	0.20	0.13
-0.4	6.53	6.00	3.27	1.25	0.50	0.24	0.13	0.07
-0.5	5.48	4.47	2.27	0.87	0.35	0.17	0.09	0.05
-0.6	4.62	3.38	1.65	0.66	0.28	0.14	0.07	0.03
-0.7	3.92	2.61	1.28	0.57	0.26	0.13	0.06	0.03
-0.8	3.34	2.06	1.07	0.54	0.26	0.13	0.06	0.03
-0.9	2.85	1.68	0.99	0.55	0.26	0.13	0.06	0.03
ρ_{QQ^i}				Panel B				
0.0	4.13	2.91	1.02	0.13	-0.06	-0.09	-0.09	-0.08
0.1	4.37	3.19	1.24	0.26	0.01	-0.05	-0.06	-0.06
0.2	4.60	3.46	1.45	0.38	0.08	0.00	-0.03	-0.04
0.3	4.83	3.72	1.67	0.51	0.15	0.04	0.00	-0.02
0.4	5.06	3.98	1.87	0.63	0.22	0.08	0.03	0.00
0.5	5.27	4.23	2.08	0.75	0.29	0.12	0.06	0.02
0.6	5.48	4.47	2.27	0.87	0.35	0.17	0.09	0.05
0.7	5.67	4.69	2.45	0.99	0.42	0.21	0.12	0.07
0.8	5.85	4.89	2.62	1.09	0.49	0.25	0.15	0.09
0.9	6.00	5.06	2.77	1.20	0.55	0.29	0.18	0.11

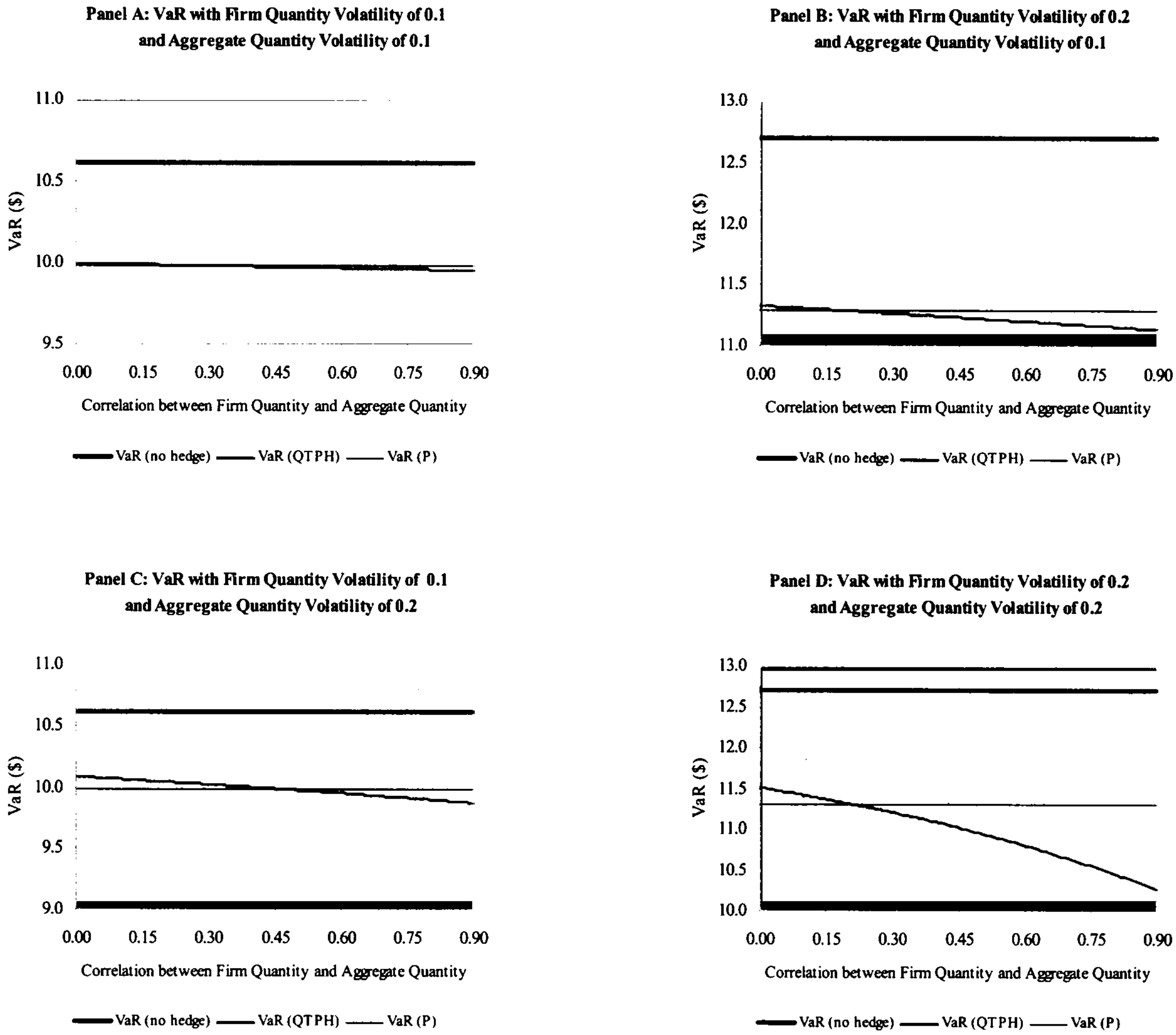
Note:

1. This exhibit reports the improvements of using quantity-triggered put options (QTP_H) over using plain-vanilla put options in minimizing VaR. The entries are calculated as $(VaR_2 - VaR_1)/VaR_0 \times 100\%$, where VaR_0, VaR_1, and VaR_2 are, respectively, the VaR values without hedging, hedging with the quantity-triggered put option, and hedging with the plain-vanilla put option. The hedging cost is fixed at B = \$0.05 for all cases.
2. Base parameter values: $S = \$30$, $Q^i = 1.0$, $Q = 100$, $Q^* = 100$, $T - t = 1$, $r = 0.06$, $\mu_S = 0.20$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\mu_{Q^i} = 0.20$, $\sigma_{Q^i} = 0.10$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$, $\rho_{SQ^i} = -0.40$, $\rho_{\delta Q^i} = -0.50$, $\rho_{QQ^i} = 0.60$, $\alpha = 0.025$.

the two quantity variables will determine the extent of the correlation's impact. To further appreciate this point, we re-generate Panel F of Exhibit 3 for four different combinations of the two volatilities and present the plots in Exhibit 5. Comparing Panel D with Panel A, we see that when the correlation is high, higher volatilities improve the performance of the quantity-triggered option considerably. For instance, calculations show that, at a correlation level of 0.6, the effective measure, $(VaR_2 - VaR_1)/VaR_0 \times 100\%$ is about 4% in Panel D while it is only 0.1% in Panel A. Similarly, comparing Panels B and C with Panel A, it is seen that, when we increase one volatility while keeping the other fixed, the performance of the quantity-triggered option also improves when the correlation is high. More interestingly, compared with the aggregate quantity volatility, the firm quantity volatility has a bigger impact (which is apparent by comparing Panel B with Panel C). This is expected since firm quantity volatility directly represents the quantity risk in our VaR definition, while the aggregate quantity volatility only affects the option value. It is also interesting to see that when the correlation is very low, the plain-vanilla option is more effective than the quantity-triggered option. The lower the firm quantity volatility relative to the aggregate quantity volatility, the wider the correlation range within which the quantity-triggered option is less effective. These results make intuitive sense in that the option is contingent on the aggregate quantity and it will be effective to hedge firm-level risk only when the firm quantity is more volatile and is highly correlated with the aggregate quantity. It should be pointed out that the lower correlation between

Exhibit 5

Improvement of QTP_H over Plain-Vanilla Put Option in Minimizing VaR: Quantity Volatilities and Correlation



Note:

1. VaR takes the optimized value at each level of the parameter in question. Each graph contains three plots of VaR: no hedge, hedging with a quantity-triggered put option (*QTPH*), and hedging with a plain-vanilla put option (*P*). The hedging cost in each graph is set equal to the lowest value of the quantity-triggered put option within the parameter range considered in order to ensure that the hedge ratio is always less than one. The critical quantity level, Q^*, is set at 100.00 (instead of the forward level, 122.14) to avoid too low option prices in certain cases.
2. Base parameter values: $S = \$30$, $Q^i = 1.0$, $Q = 100$, $T - t = 1$, $r = 0.06$, $\mu_S = 0.20$, $\sigma_s = 0.30$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\mu_{Q'} = 0.20$, $\sigma_{Q'} = 0.10$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$, $\rho_{SQ'} = -0.40$, $\rho_{\delta Q'} = -0.50$, $\rho_{QQ'} = 0.60$, $\alpha = 0.025$.

the two quantities does not always negate the effectiveness of the quantity-triggered option. For instance, when the asset price volatility and the correlation between the asset price and the aggregate quantity are both very low, the quantity-triggered option is still more effective even when the quantity correlation is very low.

The above results also bring out another important insight: the quantity-triggered option is very effective in reducing VaR when the price risk is low (indicated by a lower asset price volatility) and the quantity risk is high (indicated by higher quantity volatilities). Most previous studies only focus on price risk that can be easily managed through financial derivatives such as plain-vanilla put options. Our study identifies a vehicle through which both the price risk and the quantity risk can be managed in a combined fashion, and it is especially effective when the quantity risk is relatively high.

By now we can see that, under certain conditions, a quantity-triggered put is superior to a plain-vanilla put in hedging. Although a high-quantity triggered put will fail to hedge the price risk when the realized quantity is not high, we have demonstrated that this loss of hedge could be more than offset by the benefit of the increased hedge ratio. Not surprisingly, a quantity-triggered put is more effective when the quantity risk is significant. The value of a plain-vanilla put is the upper bound for quantity-triggered puts; yet a vanilla put can only hedge one dimension of the overall risk in revenue. The higher cost does not always justify the full coverage of the price risk. When the quantity risk is significant, the benefit of lower hedging costs will more than offset the loss of price hedge. This is the intuitive reason why a quantity-triggered put can be superior to a plain-vanilla put.

To this point, we can draw the following conclusion from the above analyses: *the quantity-triggered put is more effective than the plain-vanilla put in reducing VaR that contains both the price risk and the quantity risk when one of the following conditions is met: 1) the asset price volatility is low, 2) the correlation between the asset price and the aggregate quantity is low, and 3) the correlation between the firm quantity and the aggregate quantity is high, coupled with high volatilities of the two quantity variables; when all the three conditions are met at the same time, the improvement of the quantity-triggered option over its plain-vanilla counterpart in reducing VaR is the most pronounced; in particular, the quantity-triggered option is superior to its plain-vanilla counterpart when the price risk is low while the quantity risk is high.*

An astute reader will realize that the above conclusion is based on $Q^* = 100$. What about at other threshold levels of the quantity variable? Furthermore, why do we not jointly search for the optimal levels of the exercise price X and the quantity threshold level Q^* in minimizing VaR? Indeed, the joint optimization is the most logical route to take and it does not present any technical challenge at all—all we need is to solve a two-variable, non-linear simultaneous equation system resulting from the first-order conditions. It turns out that there is no interior solution for the quantity threshold level Q^*. When the above three conditions are met individually or in combination, the corner solution is the highest threshold imposed to the system (e.g., the optimal level of Q^* will be 150 if that is the highest level at which a sensible contract can be struck); when the above three conditions are not met, the corner solution is $Q^* = 0$, in which case the entire solution degenerates to the plain-vanilla option scenario. To illustrate this point, in Exhibit 6, we present plots of VaR against Q^* for difference levels of the asset price volatility and the hedging cost. It is apparent that the VaR value is monotone (increasing or decreasing, depending on whether the volatility is high or low) in the threshold level of the quantity variable, Q^*.

We stress again that the corner solution for the optimal threshold level of quantity should be understood in the practical context. Theoretically, when the quantity-triggered option is superior, the optimal threshold is infinity, which implies an infinite hedge ratio. In reality though, the quantity threshold for a sensible contract is not too far away from the forward level, and as a result, the hedge ratio will be a finite number.

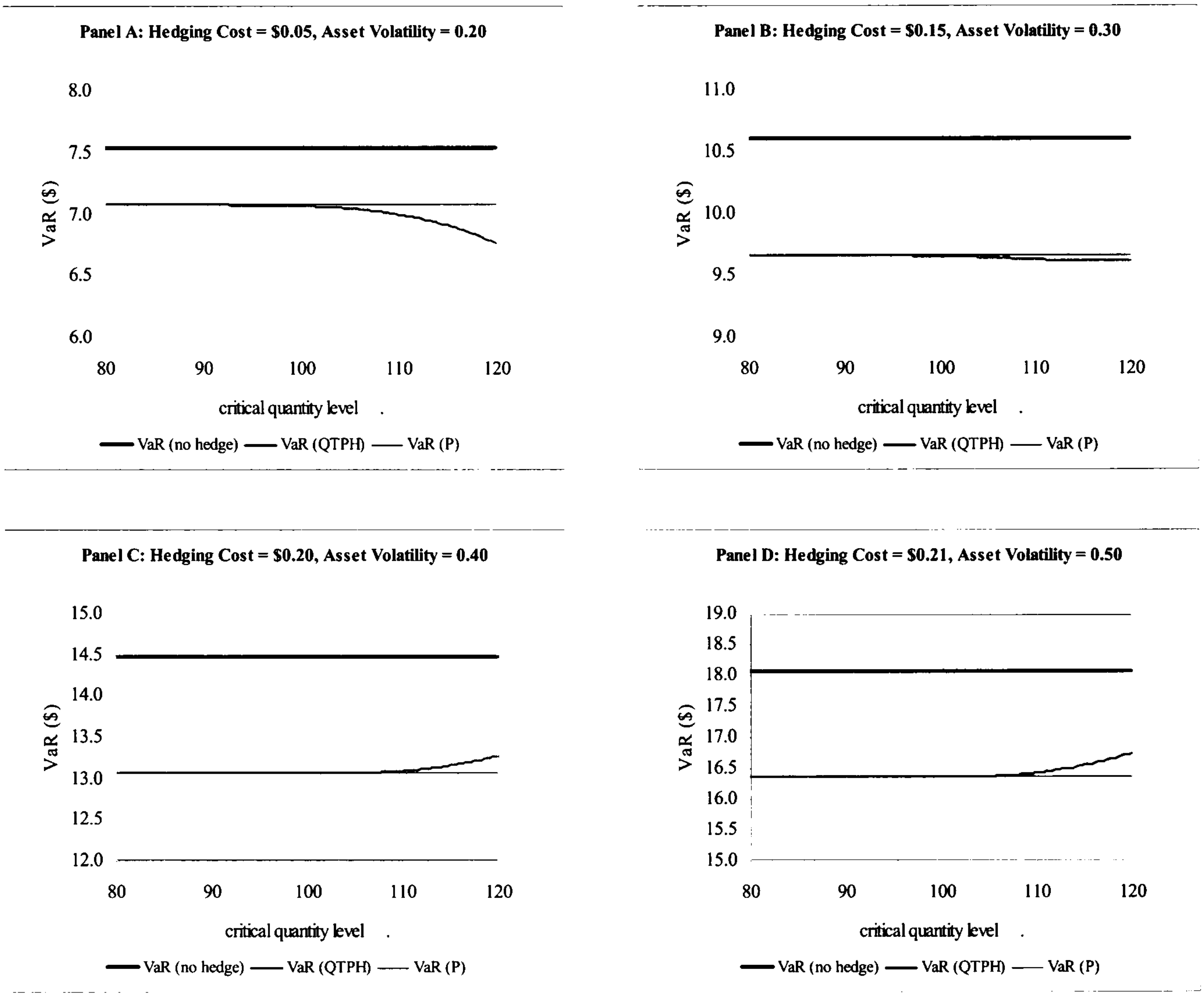
VI. CONCLUSION

Ahn [1999] provide an optimal hedging strategy under price risk only. Brown and Toft [2002] accommodate price risk and local quantity risk, but not the important interaction between local and aggregate quantity risk and its effects on optimal hedging. We provide an optimal hedging technique that accommodates price risk and local and aggregate quantity risk. Specifically, we provide a method to minimize VaR where the hedger faces stochastic and correlated price risk, local quantity risk, and aggregate quantity risk. The commodity at hand can be an asset exhibiting a stochastic convenience yield.

Based on closed-form valuation formulas provided herein for Quantity-Triggered Options, we provide a numerical solution to the problem of optimally managing the combined risk from price and quantity uncertainties by minimizing Value-at-Risk (VaR) using quantity-triggered put options. The quantity in the VaR definition is the firm-level quantity, and it is assumed to be positively correlated with the aggregate quantity. The VaR minimization is achieved by choosing the optimal exercise price of the put option. We then compare the effectiveness of the quantity-triggered put with its plain-vanilla counterpart in reducing VaR.

We find that the quantity-triggered put option is more effective than the plain-vanilla put in reducing the comprehensive VaR when one of the following conditions

Exhibit 6
VaR Versus Critical Quantity Level under Different Asset Volatilities



Note:
1. VaR takes the optimized value at each critical level of the quantity variable. Each graph contains three plots of VaR: no hedge, hedging with a quantity-triggered put option (*QTPH*), and hedging with a plain-vanilla put option (*P*). In each graph, the hedging cost, *B*, is set close to the lowest option value so that the hedge ratio is less than one.
2. Base parameter values: $S = \$30$, $Q^i = 1.0$, $Q = 100$, $T - t = 1$, $r = 0.06$, $\mu_S = 0.20$, $\sigma_s = 0.30$, $\delta = 0.10$, $k = 1.5$, $\eta = 0.15$, $\sigma_\delta = 0.40$, $\mu_Q = 0.20$, $\sigma_Q = 0.08$, $\mu_{Q'} = 0.20$, $\sigma_{Q'} = 0.10$, $\lambda = 0.2$, $\rho_{SQ} = -0.50$, $\rho_{S\delta} = 0.85$, $\rho_{\delta Q} = -0.60$, $\rho_{SQ'} = -0.40$, $\rho_{\delta Q'} = -0.50$, $\rho_{QQ'} = 0.60$, $\alpha = 0.025$.

is met: 1) the asset price volatility is low, 2) the correlation between the asset price and the aggregate quantity is low, and 3) the correlation between the firm quantity and the aggregate quantity is high, coupled with high volatilities of the two quantity variables. When all the three conditions are met at the same time, the quantity-triggered option exhibits its biggest improvement over its plain-vanilla counterpart in reducing VaR. The above findings imply that the quantity-triggered option is superior in risk management when the price risk is low while the quantity risk is high.

Intuitively, for a given level of desired protection and a given hedging cost, the cheaper quantity-trigger put will allow the hedger to acquire more of it; the associated cost is the payoff risk due to its contingency on the aggregate quantity and the imperfect correlation between the firm-level quantity and the aggregate quantity. The net result of the trade-off is not always in favor of the quantity-triggered option as far as reducing VaR is concerned. Nonetheless, in the situations identified earlier, quantity-triggered options are more effective than their plain-vanilla counterparts in reducing VaR.

ENDNOTES

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[1]Chowdhry [1995] studies price hedge with quantity uncertainty in a different context. By examining the joint effect of quantity risk (uncertainty in the amount of foreign currency cash flow) and the price risk (uncertainty in the exchange rate) on the firm's profit, he finds that the probability of bankruptcy is lower when there is some price risk than when there is not.

[2]A quantity-triggered call can also be used for corporate hedging. For instance, firms that consume soybeans need to protect costs against a negative quantity shock. These firms should purchase a call option on the price of soybeans whose payoff is triggered by the aggregate quantity below some threshold level.

[3]These correlation effects on the firm's hedging strategy are also accommodated in Brown and Toft [2002]. These authors mainly address different levels of correlation between price and firm-level quantity.

[4]While the correlation between the asset price and the convenience yield, $\rho_{S\delta}$ is set according to Schwartz [1997], the other two correlations are set based on common sense. For instance, price and quantity ought to be negatively correlated, and as a result, the correlation between the convenience yield and the quantity variable should be negative.

[5]Note that, in a Black-Scholes world, it is also true that a higher volatility will lead to a lower risk-neutral probability for the stock price to be above the exercise price when the exercise price is set equal to the stock's forward price. But the overall call value would increase thanks to the higher expected stock price conditional on the price being above the exercise price. Our quantity variable does not enjoy this part of the benefit.

[6]As willl be seen, the choice of benchmark does not affect the solution as far as choosing the optimal exercise price for the put option is concerned. It will only affect the absolute size of VaR. But for comparison purposes, the absolute size does not even matter. Therefore, the choice of benchmark is not very consequential for our analysis.

[7]The correlations are the same in the real world and the risk-neutral world. In addition, technically, the firm-level quantity and the aggregate quantity cannot both follow a lognormal process, since the latter is the sum of all firm-level quantities. However, when the firm-level quantity is small, this becomes a convenient and well-justified approximation.

[8]Note that the optimal exercise price is independent of the hedging budget B and the firm quantity drift μ_Q. However, they both affect VaR as shown in the next section.

[9]Note that a negative correlation will lead to both a lower drift and a lower overall volatility for the variable Y. As discussed before, a lower drift means a lower optimal exercise price, while a lower volatility means a higher optimal exercise price. In this case, the volatility impact dominates the drift's, and hence the pattern.

[10]We are again omitting the plots for the aggregate quantity volatility and the correlation between the convenience yield and the quantity variable. The VaR value is almost invariant to these two parameters.

[11]Although the trend continues when the correlation becomes positive, it is not meaningful to show it for two reasons: first, it is very rare for commodity prices to be positively correlated with the quantity; second, it makes no sense to use high-quantity triggered put to hedge if the correlation is positive.

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