# Conditions for No Triangular Arbitrage With Transaction Costs: A Pedagogical Note

**JASON Z. WEI**
University of Saskatchewan
Saskatoon, Canada

**ABSTRACT.** In this article, I identify the conditions under which a foreign exchange market is in equilibrium in the presence of transaction costs. Because the topic of triangular arbitrage with transaction costs is an important one and is not adequately dealt with in any international finance textbooks, this article will be useful for finance instructors. Having identified easy-to-check conditions, I also provide pedagogical suggestions on how to teach this topic effectively.

My main objective in this article is to identify and discuss the conditions for the absence of triangular arbitrage among any three currencies' quotes with a bid-ask spread. After defining the term "transaction costs," I briefly survey the textbook treatment (or lack of it) of the topic. Then, as a starting point, I describe the conditions for no triangular arbitrage without transaction costs. Finally, I lay out and discuss the main contents of the article: the conditions for no triangular arbitrage with transaction costs.

It is useful to first define the term "transaction costs." Transaction costs typically include commissions and bid-ask spread. For example, let us suppose that Mary Jones of the United States is visiting Paris for one day (perhaps taking advantage of a flight layover). In the morning, she exchanges dollars for French francs (FF), and the exchange outlet may charge a flat fee of five francs and make the exchange at the posted rate of, say, \$0.2/FF. After a full day's tour, Mary must catch the flight to New York, so she has to exchange the leftover francs back to dollars. She goes to the same exchange outlet, and pays another five francs as a flat fee, and is able to exchange the francs to dollars at \$0.195/FF. In this case, the five francs for each transaction is the commission fee. But Mary has also incurred transaction costs of a different form—the bid-ask spread. Even though the exchange rate between the dollar and the franc remains the same during the day, the exchange outlet uses two different rates for buying and selling francs for a very simple reason: to make a profit. For every 10 francs to be exchanged back and forth, Mary loses 5 cents, which is the outlet's profit. This is the bid-ask spread. In foreign exchange markets (i.e., interbank markets), the first form of transaction costs—commission fees—typically does not exist. As a result, in this article, I only focus on the second form. So "transaction cost" and "bid-ask spread" will be used as synonyms.

## Nature of the Problem and Textbook Treatments

Most professors who teach international finance would cover the topic of triangular arbitrage when introducing the foreign exchange markets to students. Existing textbooks discuss cross-rates and triangular arbitrage only for the case of no transaction costs or zero bid-ask spread. (Examples include Eiteman, Stonehill, & Moffet 1995; Giddy, 1994; Madura, 1995; and Shapiro, 1996.) Those that do discuss triangular arbitrage with transaction costs stop short of giving a full set of conditions (e.g., Levi, 1996; Sercu & Uppal, 1995). To the best of my knowledge, there has been no published account for the full conditions of no triangular arbitrage when transaction costs are present.

Without transaction costs or bid-ask spread, we can, by way of eliminating potential arbitrage, infer a cross-rate between currencies A and B if we know the two currencies' quotes vis-à-vis the third currency, C. For example, if we observe the quotes for the French franc and the British pound vis-à-vis the dollar as $S_{\$/FF}$ and $S_{\$/£}$, respectively, then we know that, in the absence of transaction costs and arbitrage, the cross-rate between the franc and the pound must be $S_{FF/£} = S_{\$/£}/S_{\$/FF}$. What can we conclude about the no-arbitrage conditions with transaction costs? In this article, I attempt to address that question. Specifically, I identify and discuss the conditions for the absence of triangular arbitrage among any three currencies' quotes with bid-ask spreads.

## Triangular Arbitrage Without Transaction Costs

To begin with, we assume for now that there is no transaction cost or bid-ask spread. Suppose the three currencies of concern are the dollar ($), the French franc (FF), and the British pound (£). If we observe $S_{\$/FF} = 0.2046$ and $S_{\$/£} = 1.5876$, then the cross-rate, $S_{FF/£}$, is 1.5876/0.2046 = 7.7595. This is the rate a bank will quote, if it offers the service of exchanges between the franc and the pound. Any other quote would imply an arbitrage opportunity that can be realized by a triangular operation. To illustrate, suppose a bank quotes $S_{FF/£} = 7.800$, higher than the implied cross-rate. Then we would (a) sell £, buy FF, (b) sell FF, buy $, and (c) sell $, buy back £. For each pound, we will make an arbitrage profit of £0.005215, or FF0.0405. The amount of profit available is the size of the deviation: 7.800 –7.7595 = 0.0405 FF/£. In what follows, I will describe what happens if all rates are quoted with a bid-ask spread.

## Triangular Arbitrage With Transaction Costs

As shown before, in the case of no transaction costs, given the quotes of two currencies vis-à-vis the same third currency, the cross-rate is unique and can be inferred. But when bid-ask spreads are present, the cross-rate is no longer unique. As a matter of fact, even the bid and ask rates are not unique. All we can infer in this case is an allowable range within which the cross-rate's bid and ask can be quoted. Given three currencies, there are three possible cross-rates and, as a result, we can identify three allowable ranges for bid and ask rates. To this end, let $S_{A/ask\ B}$ and $S_{A/bid\ B}$ denote the rates at which the bank sells and buys currency B vis-à-vis currency A, respectively. We then have $S_{B/ask\ A} = 1/S_{A/bid\ B}$, and $S_{B/bid\ A} = 1/S_{A/ask\ B}$. Furthermore, we assume that all cross-rates as well as regular rates are quoted by banks. Thus, using the three currencies in the previous illustration, the allowable quotes for the three cross-rates, FF/£, $/£ and $/FF, are summarized in the following three inequalities:

(1) $S_{FF/bid\$} * S_{\$/bid£} \leq S_{FF/bid£} \leq S_{FF/ask£} \leq S_{FF/ask\$} * S_{\$/ask£}$,

**TABLE 1.—Examples 1, 2, and 3**

*Example 1.* Suppose we observe the following rates:

$S_{\$/bid\ FF} = 0.2046$, $S_{\$/bid\ £} = 1.5876$, $S_{FF/bid\ £} = 7.7595$,
$S_{\$/ask\ FF} = 0.2055$, $S_{\$/ask\ £} = 1.5961$, $S_{FF/ask\ £} = 7.7912$.

Substituting the rates into (1), (2), and (3), we have

(1–1) 7.7255 < 7.7595 < 7.7912 < 7.8011,
(2–1) 1.5976 = 1.5876 = 1.5961 < 1.6011,
(3–1) 0.2038 < 0.2046 < 0.2055 < 0.2057.

Clearly, all three conditions are satisfied.

*Conclusion:* The observed rates are in equilibrium. No triangular arbitrage.

*Example 2.* Suppose we observe the following rates:

$S_{\$/bid\ FF} = 0.2011$, $S_{\$/bid\ £} = 1.5876$, $S_{FF/bid\ £} = 7.7595$,
$S_{\$/ask\ FF} = 0.2041$, $S_{\$/ask\ £} = 1.5961$, $S_{FF/ask\ £} = 7.7912$.

Substituting the rates into (1), (2), and (3), we have

(1–2) *7.7785 > 7.7595* < 7.7912 < 7.9368,
(2–2) 1.5604 < 1.5876 < *1.5961 > 1.5902*,
(3–2) *0.2038 > 0.2011* < 0.2041 < 0.2057

All three conditions are violated, and the parts in italic type indicate where the violations occurred. It is not hard to see that all three violations indicate the same triangular loop in order to take advantage of potential arbitrage: (a) sell FF, buy £, (b) sell £, buy $, and (c) sell $, buy back FF. Suppose we carry out the above triangular operation by starting with 7.7912 francs (so that we get exactly one pound after completing the first transaction). Then we will end up with 7.7785 francs after we complete the triangular loop of transactions.[a] We lose 7.7912 – 7.7785 = 0.0127 francs. The reason for the loss is that the size of violation, or potential profit, is not as big as the bid-ask spread. Specifically, (1–2) shows that the size of the violation is 7.7785 – 7.7595 = 0.0190, but the bid-ask spread on the pound is 7.7912 – 7.7595 = 0.0317. The difference is exactly the amount of the loss. Similar observations can be made when examining (2–2) and (3–2).

*Conclusion:* The observed rates are still in equilibrium. Triangular arbitrage does not exist, because the size of the violation is smaller than the bid-ask for all three violations. But some of the rates are irrational in that customers will never transact them. For example, if a customer wishes to sell £ for FF, then the customer is better off by first selling £ for $ and then selling $ for FF.

*Example 3.* Suppose we observe the following rates:

$S_{\$/bid\ FF} = 0.2001$, $S_{\$/bid\ £} = 1.5876$, $S_{FF/bid\ £} = 7.7595$,
$S_{\$/ask\ FF} = 0.2012$, $S_{\$/ask\ £} = 1.5961$, $S_{FF/ask\ £} = 7.7912$.

Substituting the rates into (1), (2), and (3), we have

(1–3) *7.8907 > 7.7595* < 7.7912 < 7.9765,
(2–3) 1.5527 < 1.5876 < *1.5961 > 1.5676*,
(3–3) *0.2038 > 0.2001* < 0.2012 < 0.2057.

We have exactly the same violation as in Example 2, except for the size. By starting with 7.7912 francs and carrying out the same triangular operations as in Example 2, we will end up with 7.8907 francs. Here in (1–3), the size of the violation is 7.8907 – 7.7595 = 0.1312, yet the bid-ask spread is only 7.7912 – 7.7595 = 0.0317. The difference is the arbitrage profit, 0.0995 francs per pound. Again, similar observations can be made in (2–3) and (3–3).

*Conclusion:* The observed rates are not in equilibrium. Triangular arbitrage exists, because the size of violation is bigger than the bid-spread for all three violations.

[a]The detailed transactions are as follows: First, sell 7.7912 francs at FF 7.7912/ £ to get one pound; then sell one pound at $1.5876/ £ to get 1.5876 dollars; and finally sell 1.5876 dollars ar $0.2041/ FF to get 7.7785 francs.

(2) S$/bidFF * SFF/bid£ ≤ S$/bid£ ≤ S$/ask£ ≤ S$/askFF * SFF/ask£, and
(3) S$/bid£ * S£/bidFF ≤ S$/bidFF ≤ S$/askFF ≤ S$/ask£ * S£/askFF.

To understand the meaning of the inequalities, let us look at (1) for the cross-rate FF/£. The middle part of the inequalities simply says that the ask rate is higher than the bid rate. The first part of the inequality identifies the lowest bid on the pound, which essentially says, if an investor wants to sell £ for FF, he or she will not get a better deal by first selling £ for $ at $S_{\$/bid £}$ and then selling $ for FF at $S_{FF/bid \$}$. He or she will end up with fewer francs by going the roundabout way. By the same token, the last part of the inequality identifies the highest ask rate on the pound. It means that, if another investor wants to buy £ with FF, she or he will not get a better deal by first buying $ with FF at $S_{FF/ask \$}$ and then buying £ with $ at $S_{\$/ask £}$. He or she will end up spending more francs for the same amount of pounds. Therefore, in a well-functioning market, the exchange rate between the franc and the pound should obey the relationships in (1). The inequalities in (2) and (3) can be understood in a similar fashion. (Levi, 1996, pp. 52–54, has identified similar conditions in his textbook.)

Because there are two ways to quote the same exchange rate (e.g., FF/£ and £/FF), we can write out another three inequality relationships simply by taking reciprocals of all terms in (1), (2), and (3) and reversing the direction of the inequality signs. The three extra inequalities are

(4) S$/askFF * S£/ask$ ≥ S£/askFF ≥ S£/bidFF ≥ S$/bidFF * S£/bid$,
(5) SFF/ask$ * S£/askFF ≥ S£/ask$ ≥ S£/bid$ ≥ SFF/bid$ * S£/bidFF, and
(6) S£/ask$ * SFF/ask£ ≥ SFF/ask$ ≥ SFF/bid$ ≥ S£/bid$ * SFF/bid£.

Because (4), (5), and (6) are derived from (1), (2), and (3), they do not represent independent conditions. For any three currencies, we need only three mutually independent relationships to describe no-arbitrage conditions. In the remainder of the article, I will make use of (1), (2), and (3).

I will show that the above inequalities are only sufficient conditions for the absence of arbitrage. The key insight is that the inequalities only outline the relationships among the rates a rational bank will quote, and irrational rates do not necessarily lead to arbitrage. Only when the "size of irrationality" is bigger than the relevant bid-ask spread can an investor take advantage of the irrationality, (i.e., arbitrage opportunities). We will look at a series of examples to understand the point (see Table 1 for examples 1, 2, and 3).

The examples illustrate a very important point: The three conditions in (1), (2), and (3) are only sufficient conditions for the absence of triangular arbitrage. Violations of the conditions do not always lead to arbitrage opportunities. *In order for a triangular operation to be profitable, the size of violation must be bigger than the bid-ask spread.*

When an instructor lectures on this topic in class, this is the point at which to pause. By now, some students may have gotten lost in the inequality conditions or in the examples. It is advisable to tell a joke at this point and regain the students' attention. After that, the instructor may reiterate the conditions and try several more examples while stressing the conclusions of each example. Then the instructor may pose the following question to the class, "Is it possible that only one or two (of the three) conditions are violated, which still leads to a triangular arbitrage?" The question can be answered by introducing two more examples (see Table 2, examples 4 and 5).

Examples 4 and 5 (Table 2) indicate that it is possible to have just one or two conditions violated, and that arbitrage opportunities do not exist in those cases. Can we generalize this observation to state that all three conditions in (1), (2), and (3) must be violated before triangular arbitrage is possible? It turns out that this statement is indeed valid. A formal proof is given in the Appendix. I will only offer some intuitions here. Recall that the three conditions together in (1), (2), and (3) are sufficient to eliminate any potential triangular arbitrage, with each condition describing the rational bounds for the quotes of a cross-rate, given the individual quotes of the two currencies vis-à-vis the third one. By virtue of the fact that a triangular arbitrage operation involves all three currencies, it must have the "blessing" of all three conditions (violated) in order to be profitable. If only one or two conditions are satisfied, you will never come out with a positive bottom line after you have completed the triangular loop of transactions.

## Summary

What can we conclude on the central issue: conditions for no triangular arbitrage with transaction costs? Based on the above discussions, either of the following two conditions will ensure the absence of arbitrage:

1. All of conditions (1), (2), and (3) are satisfied simultaneously; or

**TABLE 2.—Examples 4 and 5**

*Example 4.* Suppose we observe the following rates:

$S_{\$/bid\ FF} = 0.2001$, $S_{\$/bid\ £} = 1.5905$, $S_{FF/bid\ £} = 7.8010$,
$S_{\$/ask\ FF} = 0.2041$, $S_{\$/ask\ £} = 1.5961$, $S_{FF/ask\ £} = 7.8214$.

Substituting the rates into (1), (2), and (3), we have

(1–4) 7.7928 < 7.8010 < 7.8214 < 7.9765,
(2–4) 1.5610 < 1.5905 < 1.5961 < 1.5964,
(3–4) *0.2034 > 0.2001* < 0.2041 < 0.2046.

Here, only one condition is violated in (3–4). But there is no triangular arbitrage, because the size of the violation is smaller than the bid-ask spread.

*Example 5.* Suppose we observe the following rates:

$S_{\$/bid\ FF} = 0.2001$, $S_{\$/bid\ £} = 1.5905$, $S_{FF/bid\ £} = 7.8010$,
$S_{\$/ask\ FF} = 0.2041$, $S_{\$/ask\ £} = 1.5961$, $S_{FF/ask\ £} = 7.8030$.

Substituting the rates into (1), (2), and (3), we have

(1–5) 7.7927 < 7.8010 < 7.8030 < 7.9765,
(2–5) 1.5610 < 1.5905 < *1.5961 > 1.5926*,
(3–5) *0.2038 > 0.2001* < 0.2041 < 0.2046

Here, two conditions are violated in (2–5) and (3–5). But there is no triangular arbitrage, because the size of the violation is smaller than the bid-ask spread for both violations.

2. Some, or all three conditions in (1), (2), and (3) are violated, but none of the violations is bigger than the corresponding bid-ask spread.

When violations are detected, the process of determining if arbitrage actually exists can be simplified by keeping in mind the following:

- As long as one of the three inequality conditions is satisfied, then there is no arbitrage.
- If all three inequalities are violated, then it is only necessary to examine the size of one violation. If the size is smaller than the corresponding bid-ask spread, then the same must be true for the other two violations. The reverse is also true if the size is bigger than the corresponding bid-ask spread.

Finally, it should be mentioned that an instructor can generate many exercise questions to help students digest the above materials. For instance, one may give five bid-and-ask quotes for three currencies and ask students to identify the range for the sixth quote for all the rates to be rational ones. One can also ask students to identify the range for the sixth quote that is consistent with absence of arbitrage.

## REFERENCES


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## APPENDIX

***Proposition:*** *For triangular arbitrage profits to exist among three currencies, all three inequality relationships that describe the equilibrium conditions must be violated, and the size of the violation must be bigger than the corresponding bid-ask spread.*

***Corollary:*** *Triangular arbitrage profits do not exist if only one or two inequality relationships are violated.*

***Proof:*** Without loss of generality, we use the three currencies in the text and rewrite the three relationships in (1), (2), and (3) as follows:

(A1) $S_{FF/bid\ \$} * S_{\$/bid\ £} \le S_{FF/bid\ £} \le S_{FF/ask\ £} \le S_{FF/ask\ \$} * S_{\$/ask\ £}$,

(A2) $S_{\$/bid\ FF} * S_{FF/bid\ £} \le S_{\$/bid\ £} \le S_{\$/ask\ £} \le S_{\$/ask\ FF} * S_{FF/ask\ £}$, and

(A3) $S_{\$/bid\ £} * S_{£/bid\ FF} \le S_{\$/bid\ FF} \le S_{\$/ask\ FF} \le S_{\$/ask\ £} * S_{£/ask\ FF}$.

Furthermore, for ease of exposition, the three above sets of inequalities are represented by block letters as follows:

(AA1) $A_1 \le A_2 \le A_3 \le A_4$,

(AA2) $B_1 \le B_2 \le B_3 \le B_4$, and

(AA3) $C_1 \le C_2 \le C_3 \le C_4$.

Because an ask rate is always bigger than a bid rate, the middle parts of the inequalities are always satisfied. Thus, there are altogether six potential violations:

$$A_1 > A_2,\ A_3 > A_4,\ B_1 > B_2,\ B_3 > B_4,\ C_1 > C_2, \text{ and } C_3 > C_4.$$

In the remainder of the proof, I will show that in order for arbitrage profits to exist, there must be three violations, one in each relationship, and the three violations are related in a triangular fashion. For example, if we observe $A_1 > A_2$ and arbitrage profits exist, then the other two violations must be $B_3 > B_4$ and $C_1 > C_2$ (this is the case in Example 3). By the same token, $A_3 > A_4$, $B_1 > B_2$, and $C_3 > C_4$ will have to occur simultaneously in order for arbitrage profits to exist. In other words, the three violations are equivalent.

Suppose we observe A > A, or $S_{FF/bid\ \$} * S_{\$/bid\ £} > S_{FF/bid\ £}$. Because the quoted rate $S_{FF/bid\ £}$ is lower than the calculated rate, the pound is undervalued relative to the franc. Therefore we should sell the franc and buy the pounds. To complete the triangular operation, we then need to sell the pound and buy the dollar, and sell the dollar and buy the franc back. If we start out with one franc, then the resulting amount of francs is

$$(1/S_{FF/ask\ £}) * S_{\$/bid\ £} * S_{FF/bid\ \$}.$$

To earn an arbitrage profit, the above amount must be greater than one, that is,

$$(1/S_{FF/ask\ £}) * S_{\$/bid\ £} * S_{FF/bid\ \$} > 1.$$

Rearranging the above we obtain,

(A4) $S_{\$/bid\ £} * S_{FF/bid\ \$} > S_{FF/ask\ £}$,

which is equivalent to saying that the size of violation is bigger than the bid-ask spread. (This becomes apparent when subtracting $S_{FF/bid\ £}$ from both sides of the last inequality.) Now, if we divide both sides of (A4) by $S_{FF/bid\ \$}$ and realize that $S_{\$/askFF} = 1/S_{FF/bid\ \$}$, we have

(A5) $S_{\$/bid\ £} > S_{\$/ask\ FF} * S_{FF/ask\ £}$.

Comparing (A5) with (A2) and (AA2) reveals that (A5) implies $B_3 > B_4$. In other words, the second patrt of (A2) is also violated. Similarly, if we divide both sides of (A4) first by $S_{FF/bid\ \$}$ and then by $S_{FF/ask\ £}$ and make use of the reciprocal relationships, we obtain

(A6) $S_{\$/bid\ £} * S_{£/bid\ FF} > S_{\$/ask\ FF}$.

which implies that thefirst part of (A3) is violated. We have therefore shown that if the first part of (A1) is violated and arbitrage profits exist, then the second part of (A2) and the first part of (A3) must also be violated. Because they are all equivalent, we would have reached the same conclusion had we started with the second part of (A2) or the first part of (A3).

To this point, we have dealt with three of the six potential violations and shown that if one occurs the other two must occur for arbitrage profits to exist. Using exactly the same logic and procedures, similar conclusions can be drawn for the other three violations: $A_3 > A_4$, $B_3 > B_4$, and $C_3 > C_4$. For brevity, this part of the proof is omitted. We have therefore covered all six potential violations.

Q.E.D.

***Discussions:*** Because one violation in a particular relationship implies a violation in each of the other two relationships (when arbitrage exists), it can be inferred that a violation in only one or two relationships will not lead to any arbitrage profits. It is also important to remember that the proposition is true only when arbitrage profits exist. Another way of saying the same thing: if (A1), (A2), and (A3) are not violated simultaneously, then the size of the violations is necessarily smaller than the corresponding bid-ask spread and arbitrage profits do not exist.