Competing against a socially concerned firm when capacities are limited

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Abstract

The government may regulate a market by obtaining partial ownership in a firm, which induces the firm to maximize a weighted sum of its profits and the social surplus. We study price competition of a private firm against such a socially concerned firm with the novel and realistic assumption of capacity constraints. We highlight how the aggressive pricing of a publicly owned firm induces the private firm to *in*crease its price. Therefore, in contrast to other results in the literature that abstract from capacity constraints, we find that full privatization is the socially best outcome, that is the optimal level of public ownership is equal to zero.

Keywords:

pricing, capacity constraints, public ownership, mixed duopoly, price competition.

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1 Introduction

In many markets, purely private firms compete with firms that are partially or wholly owned by the government. Such publicly owned companies often take consumer surplus into account, while privately owned firms set prices to maximize profits. A recent literature analyzed the question of optimal government ownership but under the assumption that firms can produce without capacity constraints. However, in many industries, including the natural gas and electricity industries, air and rail transportation, and pharmaceuticals the public-private competition takes place under capacity constraints. For example, in the case of LNG (Liquefied Natural Gas) markets capacity bottlenecks are present at least in the short to medium run due to the limited capacity of LNG terminals that receive the liquefied gas supplies. Many of the current competitors and planned new terminals in the European LNG sector are (or will be) partially owned by governments including the German Federal Government, France, Finland and the German State of Baden-Württemberg.[1] In the capacity constrained air travel industry, several leading airlines including Aeroflot, Air China, Air New Zealand, Finnair, ITA Airways, LOT Polish Airlines are wholly or partially owned by governments. In airplane manufacturing, Airbus whose 26% is owned by European governments, is competing with privately owned (American) Boeing. Lei (2016) indicates that this industry is capacity constrained and the two firms vigorously compete in prices with Airbus being somewhat more aggressive in pricing.[2] In the pharmaceutical industry, capacity constraints in medicine production give rise to incentives to establish nationally owned production units to speed up production in crisis situations but also to reduce prices of medicines. In the wake of the COVID-19 crisis, in 2021 Canada has discussed plans to establish a government owned vaccine company.[3]

[1]Leading LNG firm GRTGaz is partially controlled by the French government through company Engie. New German LNG projects are run by partly publicly owned giants Uniper and RWE (Finnish and German shares), and utility company EnBW -owned mostly by the German public sector. On the other hand, other European players like Fluxys and Snam are fully private.

[2]The fierce price competition through discounts is discussed in the Wall Street Journal article *The Secret Price of a Jet Airliner*. In line with our findings, that in case of capacity constraints the semi-public firm sets lower prices, when it comes to the lower range the Airbus 320 was slightly cheaper than the comparable Boeing 737-800, while for the longer range the Airbus 330 was significantly cheaper than its direct competitor Boeing 777.

[3]Several emerging economies have also set up public production units to decrease medicine prices, including India and especially China. A WHO report states that "Many

Governments all over the world consider increasing the role of the public sector in these industries to improve supply security, and reaction time or simply to reduce consumer prices. However, such policies are costly, and their effects must be carefully analyzed in each situation. To provide a framework for such analysis, we provide a simple duopoly model and focus on the important impact of public ownership share on market performance as measured by social surplus. It is intuitively plausible that if the government acquires shares in one of the firms this makes that firm more socially concerned, which leads to a lower price charged by that firm. Our contribution shows that there is a strategic effect that can more than counterbalance this direct effect, and as a result prices may in fact increase once one of the firms is (wholly or partially) owned by the government. In particular, we show that the intensity of competition is adversely affected by the combination of public ownership and capacity constraints.

To capture the nature of competition in the studied industries, we set up a model where two firms compete in prices in a Bertrand-Edgeworth game, and the firm with the higher price obtains a leftover demand according to the efficient rationing rule. We determine the pure-strategy equilibrium whenever it exists, derive certain properties of the mixed-strategy equilibrium and determine the mixed-strategy equilibrium in explicit form for the case of symmetric capacity constraints and linear demand. The main difficulty of the analysis involves deriving a novel form of the mixed strategy equilibrium that takes into account the incentives of the semi-public firm under capacity constraints, and the optimal reaction of the private firm.

In contrast to Matsumura (1998) and Barcéna-Ruiz and Sedano (2011) who study the same question under no capacity constraints, we find that the purely private price-setting duopoly maximizes social surplus. Intuitively, an increase in the government's share in the semi-public firm induces the semi-public firm to lower its price to reflect the higher share of consumer surplus in its objectives. However, in response to such a price decrease, the private firm gives up on competing in price and sets a price closer to the monopoly price on the residual demand curve. As a result, the private firm produces an equilibrium price-distribution that is stochastically increasing in the governmental share. This strategic effect is responsible for our main

of the largest domestic pharmaceutical companies in China are majority state-owned, but with shares listed on public securities exchanges, i.e. mixed public and private ownership."

result that social surplus is maximized when both firms are private. The reason the strategic effect outweighs the direct effect is that social surplus depends more on the higher of the two prices than it depends on the lower of the prices because extra purchases are hard to make at the more congested cheaper firm.[4]

2 Literature review

First, we review the literature of mixed duopolies in the absence of capacity constraints. In their seminal paper, Merrill and Schneider (1966) investigated the welfare effect of a public firm in a quantity-setting oligopoly. The case of a semi-public firm with an objective function obtained as a weighted sum of firm's profit and social surplus was analyzed by Matsumura (1998) in a homogeneous good quantity-setting duopoly. He determined the optimal governmental share and found an interior solution, that is the pure public firm case and the standard profit-maximizing case do not emerge as an optimal solution and a governmental share in a firm is beneficial. Similar investigations have been carried out for the heterogeneous goods price-setting duopoly game by Barcéna-Ruiz and Sedano (2011) in which again the optimal governmental share was positive.[5] Our result is different from the case where firms are competing in quantities (as in Matsumura (1998)) or if products are differentiated (as in Barcéna-Ruiz and Sedano (2011)); in both those models competition is imperfect and public ownership helps in reducing prices. In our model, lower prices are enforced in the purely private duopoly game through the incentive of undercutting the opponent's price, an incentive that is weakened by semi-public ownership. Such an incentive for privately owned firms to compete by undercutting is absent with quantity-competition, and is reduced with differentiated products.

A vast literature studies the role of public ownership and the question of privatization for other policy objectives than price/quantity effects. For instance, Kalashnikov et al. (2009) investigated the so-called conjectural variation equilibrium in which each firm selects its optimal action in a way that

[4]However, the strategic effect itself may be smaller when the rationing rule is changed, an idea we further study in our Discussion Section.

[5]Fujiwara (2008) and Ishiwara et al. (2007) also study the effects of (vertical or horizontal) product differentiation on the optimal public share in the case of no capacity constraints, while Casadesus-Masanell and Ghemawat (2006) study a model of open source software in a mixed duopoly.

its competitors' strategies are a conjectured function of its own strategy in a mixed duopoly. Lee and Tomaru (2017) find that through output subsidy and R&D tax the first-best allocation can be achieved. Gil-Moltó et al. (2020) show among other results that the optimal R&D subsidy is reduced by privatization in a mixed oligopoly.

Closest to our paper in assuming capacity constraints on technology, Zhou et al. (2022) determine the optimal governmental share in a semi-mixed duopoly in which there is quality-differentiation, queuing of costumers and congestion in the consumption of goods. They propose their main applications in service industries like health care while our main area is the manufacturing and energy where rationing is more prevalent than queueing. Just like our work, Zhou et al. (2022) also takes capacities as given but, unlike our paper, they focus on the more tractable case of quantity competition. In our different framework, we are able to solve the more commonly observed case where firms set prices and not quantities. Their main finding is similar to ours in that they find that full privatization is the optimal solution if the customers are delay sensitive; however they can show this statement only numerically. Their result that a fully public firm is not welfare enhancing also involves a strategic effect where the private firm reacts in an adverse way to the more aggressive strategy (in terms of quantities) of the public or semi-public firm, although the exact mechanism of this strategic effect is quite different from ours.[6] The simpler mixed duopoly game with a purely public firm was investigated by Balogh and Tasnádi (2012) for which they found that an equilibrium in pure strategies always exists in contrast to the duopoly with a purely private firm, henceforth referred to as the standard case. However, since in the semi-public setting both firms' objective functions have a profit component, there is a capacity region for which a pure-strategy equilibrium does not exist. Hence, the analysis of the semi-public case becomes much more difficult.

Energy markets fit our stylized model due to the role of public ownership and capacity constraints as it is also highlighted by the literature. For example, the capacity-constrained Bertrand-Edgeworth model is used in the modeling of energy markets (e.g. Vickers and Yarrow, 1990) and mixed duopoly models are also employed in energy markets (e.g. Escrihuela-Villar

[6]In a related paper, Hua et al. (2016) study the role of production subsidies to the public firm instead of the effects of (the level of) privatization.

et al., 2020). The more recent work of Fabra and Llobet (2021) emphasize the role of capacity constraints under uncertain capacity levels. Due to recent instabilities caused by deteriorating supply conditions, the role of governments in energy markets has become an important question in European countries. Our paper informs this discussion by emphasizing the role of capacity constraints in determining the welfare effects of public-private competition.

3 The framework

There are two firms with capacities k_1, $k_2 > 0$ who produce homogenous products and compete in prices. We denote the set of firms by $\{1, 2\}$, where 1 is the semi-public firm and 2 is the private firm. The industry faces a downward sloping demand function D that satisfies the following standard assumption:

Assumption 1. (i) D intersects the horizontal axis at quantity a and the vertical axis at price b; (ii) D is strictly decreasing, concave and twice-continuously differentiable on $(0, b)$; (iii) D is right-continuous at 0 and left-continuous at b; and (iv) $D(p) = 0$ for all $p \geq b$.

Production costs are neglected except that firms cannot produce more than their capacities:

Assumption 2. The firms face zero unit cost up to their capacity constraints k_1 and k_2.[7] We assume that the semi-public firm is not capable of serving the entire demand, i.e. $k_1 < a$.[8]

Since for the interesting price region the low-price firm cannot satisfy the whole demand, its consumers have to be rationed so that the residual demand of the high-price firm is a function of the consumers served by the low-price firm. The most frequently employed rationing rule is the so-called efficient rationing rule, which is reasonable if there is a secondary market for the duopolists' products. In what follows $p_1, p_2 \in [0, b]$ stand for the prices set by the firms.

[7]The main assumption here is that firms have identical unit costs, assuming zero unit costs is just a matter of normalization since firms will produce to order.

[8]In case of $k_1 \geq a$ a pure-strategy equilibrium exists, which is not necessarily unique; however, sales happen only at price zero.

Assumption 3. We assume efficient rationing on the market; that is, the demand faced by the firms $i \in \{1, 2\}$ equals

$$\Delta_i (D, p_1, k_1, p_2, k_2) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j, \\ \frac{k_i}{k_1+k_2} D(p_i) & \text{if} \quad p = p_i = p_j, \\ (D(p_i) - k_j)^+ & \text{if} \quad p_i > p_j. \end{cases}$$

The assumption of efficient rationing is widely used for its tractability in the literature, and it means that the consumers with the highest valuations buy at the cheaper store first.[9]

We turn to specifying the firms' payoff functions. Recall that social surplus is equal to the sum of consumer surplus and profits in a single market, which is also the area under the demand curve in our case with zero costs. Let $\alpha \in (0, 1)$ be the weight of the social surplus-maximizing component in the payoff function of the semi-public firm, which might be a function of the governmental share in the equity of firm 1. The extreme cases of $\alpha = 0$ and $\alpha = 1$ correspond to the already analyzed cases of the standard Bertrand-Edgeworth game and to the mixed version of the Bertrand-Edgeworth game investigated by Balogh and Tasnádi (2012). Let P denote the inverse demand function, that is $P(q) = D^{-1}(q)$ for $0 < q \leq a$, $P(0) = b$, and $P(q) = 0$ for all $q > a$. Denote the residual demand curves of firm i by $D_i^r(p) = (D(p) - k_j)^+$, and denote the inverse of these residual demand curves by R_1 and R_2. The payoff function of the semi-public firm is given by

$$\begin{aligned} \pi_1(p_1, p_2) \quad = \quad & (1-\alpha)p_1 \min\{k_1, \Delta_1(D, p_1, k_1, p_2, k_2)\} + \\ & \alpha \int_0^{\min\left\{(D(p_j)-k_i)^+, k_j\right\}} R_j(q)dq + \alpha \int_0^{\min\{a, k_i\}} P(q)dq, \quad (1) \end{aligned}$$

where $0 \leq p_i \leq p_j \leq b$. Observe that because of efficient rationing, social surplus is only a function of the largest price at which sales are realized. The private firm's payoff is equal to its profits:

$$\pi_2(p_1, p_2) = p_2 \min\{k_2, \Delta_2(D, p_1, k_1, p_2, k_2)\}. \quad (2)$$

[9]In case of equal prices we assume for simplicity that firms split demand in proportion to their capacities. However, we could have admitted a large class of tie-breaking rules, the only tie-breaking rules that have to be avoided are the ones that give full priority to one of the two firms.

Upon describing the setup, we define some useful price levels that characterize the incentives of the two firms. Let $p^c = P(k_1 + k_2)$ the market clearing price, and by p^M the price set by a monopolist without capacity constraints, and by p_i^M the price set by a monopolist with capacity constraint k_i, where $i \in \{1, 2\}$, i.e. $p^M = \arg\max_{p\in[0,b]} pD(p)$, and $p_i^M = \arg\max_{p\in[0,b]} p \min\{D(p), k_i\}$.

For $i \in \{1, 2\}$ let

$$p_i^m = \arg\max_{p\in[0,b]} pD_i^r(p)$$

be the unique revenue maximizing price on the firms' residual demand curves $D_i^r(p) = (D(p) - k_j)^+$, where $j \in \{1, 2\}$ and $j \neq i$, if $D_i^r(0) > 0$. Let $p_i^m = 0$ if $D_i^r(0) = 0$. Clearly, p^c and p_i^m are well defined whenever Assumptions 1 and 2 are satisfied. We have $p_i^M \geq p^M > p_i^m$. Furthermore, $k_1 < a$ implies $p_2^m > 0$. It can be easily verified that from $k_i > k_j$ it follows that $p_i^m > p_j^m$.

Let us denote by p_i^d the smallest price p_i for which $p_i \min\{k_i, D_i(p_i)\} = p_i^m D_i^r(p_i^m)$, whenever this equation has a solution.[10] Provided that the private firm has 'sufficient' capacity (i.e. $p^c < p_2^m$), then the private firm is indifferent between serving residual demand at price level p_i^m or selling $\min\{k_i, D_i(p_i^d)\}$ at the lower price level p_i^d.[11] By Deneckere and Kovenock (1992, Lemma 1) we know that $p_i^d > p_j^d$ if $k_i > k_j$. We define the payoff maximizing price p_1^s for the semi-public firm when it faces residual demand:

$$p_1^s = \arg\max_{p_1\in[0,b]} \left\{ (1-\alpha)p_1 D_1^r(p_1) + \alpha \int_0^{D(p_1)} P(q)dq \right\}.$$

It can be checked that p_1^s is determined uniquely and that $p_1^s < p_1^m$ under Assumptions 1-3.

4 Equilibrium existence and characterization

Concerning the pure-strategy equilibrium of the capacity constrained Bertrand-Edgeworth game with a socially concerned firm, henceforth called the semi-public Bertrand-Edgeworth game, the following holds:

Proposition 1. *Under Assumption 1-3, the semi-public Bertrand-Edgeworth*

[10]The equation defining p_i^d has a solution if, for instance, $p_i^m \geq p^c$, which will be the case in our analysis when we will refer to p_i^d.

[11]The superscript d comes from the third letter of the word indifference.

game has a pure-strategy equilibrium if and only if $\max\{p_1^s, p_2^m\} \leq p^c$. *If a pure-strategy equilibrium exists, then it is given by*

$$p_1^* = p_2^* = p^c = P(k_1 + k_2). \tag{3}$$

Proof. First, we show that whenever a pure-strategy equilibrium exists it can only be given by (3). Suppose that $p_1^* < p_2^*$. We start with the case of $D(p_1^*) > k_1$. If $D(p_2^*) > k_1$, then the semi-public firm can increase its profit by increasing its price such that social surplus will not change. If $D(p_2^*) \leq k_1$, then the private firm can gain profits by decreasing its price sufficiently. Turning to the case of $D(p_1^*) \leq k_1$, the private firm can make again profits by decreasing it price because of $D(0) > k_1$. Hence, an equilibrium in which $p_1^* < p_2^*$ does not exist.

Showing that in a pure-strategy-equilibrium, we cannot have $p_1^* > p_2^*$ is a bit simpler. If $p_1^* > p_2^*$, then in case of $D(p_2^*) > k_2$ the private firm can sell its entire capacity at prices above p_2^*, while in case of $D(p_2^*) \leq k_2$ the semi-public firm can increase its payoff by setting a price below p_2^* since this will not change social surplus, while it can earn profits.[12]

Thus, in a pure-strategy equilibrium both firms must set the same price $p_1^* = p_2^*$. However, there cannot be an equilibrium with $p_1^* = p_2^* > p^c$ because in this case at least one firm can benefit from unilaterally undercutting its opponent price. Clearly, $p_1^* = p_2^* < p^c$ cannot be an equilibrium neither.

Finally, by the concavity of the residual payoff functions and the definitions of p_1^s and p_2^m it follows that $p_1^* = p_2^* = p^c$ is a pure-strategy equilibrium if and only if $\max\{p_1^s, p_2^m\} \leq p^c$. $\square$

The existence of a mixed-strategy equilibrium can be established by employing a recent existence theorem demonstrated by Prokopovych and Yannelis (2014, Theorem 3).

If a pure-strategy equilibrium exists, the standard, the mixed and the semi-public Bertrand-Edgeworth games all result in the same outcome in which the firms produce at their capacity constraints and the equilibrium price is the market clearing price. Therefore, in what follows we focus on the case in which a pure-strategy equilibrium does not exist.

Now, we turn to characterizing the mixed strategy equilibrium. The goal

[12] Note that $p_2^* = 0$ cannot be in an equilibrium by $k_1 < D(0)$.

is twofold: besides getting a better understanding of the properties of the mixed-strategy equilibrium for the case of general demand functions, the results below also help characterizing the mixed-strategy equilibrium derived in the next section under linear demand.

To make matters interesting, we assume that a pure-strategy equilibrium does not exist, i.e. $\max\{p_1^s, p_2^m\} > p^c$. We shall denote by (φ_1, φ_2) an arbitrary mixed-strategy equilibrium. Let $\overline{p}_i = \max \mathrm{supp}(\varphi_i)$ and $\underline{p}_i = \min \mathrm{supp}(\varphi_i)$, where $i \in \{1, 2\}$. Observe that $p_2^m > p^c$ implies $\underline{p}_2 \geq p_2^d > p^c$ because the private firms profits at price p_2^m are at least as large as at price p_2^d. Hence, $\underline{p}_1 \geq p_2^d$. Furthermore, if $p_1^s > p^c \geq p_2^m$, then $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$.

We present several results concerning the mixed-strategy equilibrium.[13] Lemma 1 shows that ties cannot occur with a positive probability, which derives from the fact that each firm would like to undercut the price of the other, a result that shows that competition is intense in prices.

Lemma 1. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$, *we obtain that* φ_1 *and* φ_2 *cannot both have an atom at the same price.*

The next two results characterize the upper bounds of the equilibrium price distributions. It shows that the firm with higher upper bound chooses a price that maximizes its payoffs conditional on losing the price war for sure. For firm two, this implies maximizing profits on the residual demand curve ($D_2^r = D(p_2) - k_1$), and for firm one it means maximizing its weighted objective of profits on the residual demand and consumer surplus.

Lemma 2. *Under Assumptions 1, 2, 3, and* $\max\{p_1^s, p_2^m\} > p^c$, *for any mixed-strategy equilibrium* (φ_1, φ_2) *we have* $\overline{p}_1 = p_1^s > \overline{p}_2$, $\overline{p}_1 < \overline{p}_2 = p_2^m$ *or* $\min\{p_1^s, p_2^m\} \leq \overline{p}_1 = \overline{p}_2 \leq \max\{p_1^s, p_2^m\}$.

We have the following result that proves useful in Section 4 where we provide our main comparative statics results on social surplus:

Lemma 3. *Under Assumptions 1, 2, 3,* $p_1^s \leq p_2^m$, *and* $\max\{p_1^s, p_2^m\} > p^c$, *for any mixed-strategy equilibrium* (φ_1, φ_2) *it holds that* $\overline{p}_2 = p_2^m$.

It is useful to note that $p_1^s \leq p_2^m$ holds under mild conditions. First, $\lim_{\alpha \to 1} p_1^s = 0$ and thus a firm with a high level of public share satisfies

[13] The proofs are in the Appendix.

$p_1^s \leq p_2^m$. Second, when $k_1 \leq k_2$ it holds by construction that $p_1^m \leq p_2^m$, and under Assumptions 1-3 it holds that $p_1^s \leq p_1^m$. In Section 4, we provide a comparative statics result with linear demand where $k_1 = k_2$.

Finally, we show that the lower bounds of the equilibrium price distributions are identical, and feature no atoms.

Lemma 4. *Let Assumptions 1, 2, and 3 be satisfied and let* (φ_1, φ_2) *be a mixed-strategy equilibrium. If* $\max\{p_1^s, p_2^m\} > p^c$*, then* $\underline{p}_1 = \underline{p}_2$ *and* $\varphi_1(\underline{p}_1) = \varphi_2(\underline{p}_2) = 0$.

5 Welfare analysis

5.1 An (almost) purely public firm

Our first welfare result states that social surplus is minimized as $\alpha \to 1$ if a pure strategy equilibrium fails to exist, so having a purely public firm minimizes welfare:

Proposition 2. *Let* $SS(x)$ *denote the equilibrium level of social surplus when* $\alpha = x$*. Then under Assumptions 1, 2, 3 and* $k_1 \leq k_2$*, for any* $x < 1$ *it holds that* $SS(x) > \lim_{\alpha \to 1} SS(\alpha)$ *if* $p_2^m > p^c$.

Proof. Given that the social surplus depends only on the higher of the two prices, it is sufficient to prove that the price of the private firm, p_2 converges to p_2^m in distribution as $\alpha \to 1$. This follows because under $k_1 \leq k_2$ it holds that $p_2^m \geq p_1^m$, and firm two never prices above p_2^m regardless of firm one's strategy.

In the rest of the proof, we show that p_2 converges to p_2^m in distribution as $\alpha \to 1$. Take a $p^* < p_2^m$ and suppose that $\lim_{\alpha \to 1} \varphi_2(p^*) > 0$. Given that $\lim_{\alpha \to 1} p_1^s = 0$, we obtain through Lemma 3 that $\overline{p}_2 = p_2^m$. By the best reply of firm one, $\lim_{\alpha \to 1} \overline{p}_1 \leq p^*$ otherwise firm one would have a profitable deviation for a high enough α. Taking p^* close to the lower bound of the price distribution of φ_2 for any $\alpha < 1$, we obtain that $\lim_{\alpha \to 1} \overline{p}_1 \leq \lim_{\alpha \to 1} \underline{p}_2$. By Lemma 4, $\lim_{\alpha \to 1} \underline{p}_2 = \lim_{\alpha \to 1} \underline{p}_1$, which then implies $\lim_{\alpha \to 1} \overline{p}_1 = \lim_{\alpha \to 1} \underline{p}_1 = \lim_{\alpha \to 1} \underline{p}_2 = \widetilde{p} = p_2^d < p_2^m$. Moreover, given profit maximization of firm two, the limiting distribution of φ_2 shrinks to a support that only contains $\widetilde{p}$ and p_2^m.

Our argument leads to atoms in both of the limiting distributions of φ_1 and φ_2 at $\widetilde{p}$. By a similar argument as in the proof of Lemma 4, if there are atoms in either φ_1 or φ_2 at $\widetilde{p}$ in the limiting distributions, then both firms would like to undercut the other firm, and charge the lower price with probability one, which cannot occur. In particular, it is clear that the private firm would either charge p_2^m or would win almost surely as $\alpha \to 1$. To see this, take a price path $p_2(\alpha)$ such that $\lim_{\alpha\to 1} \Pr(\varphi_1(p_2(\alpha))) > 0$ and $\lim_{\alpha\to 1} p_2(\alpha) = \widetilde{p}$. Then switching to the strategy that chooses $p_2(\alpha) = \underline{p}_1 - \varepsilon(\alpha)$ with $\lim_{\alpha\to 1} \varepsilon(\alpha) = 0$ is profitable if α is high enough because it charges the same price but sells strictly more in the limit. For the semi-public firm, a similar argument implies that undercutting firm two is optimal because both the profit component and the welfare component are increased by undercutting firm 2's price for a high enough α. $\square$

Balogh and Tasnádi (2012) has analyzed this game under the assumption that $\alpha = 1$, and (depending on parameter values) obtained two or three pure-strategy equilibria. Our result in effect selects one of those equilibria as $\alpha \to 1$.[14] More importantly for our welfare analysis, we can obtain the following intuition for this result. When firm one becomes fully public, firm two loses its chance to become the firm with the lower price as $\alpha \to 1$ because firm one has a strong incentive to undercut firm 2's price no matter what that price is. Therefore, since it is losing the price war anyway, firm two sets the monopoly price on the residual demand curve p_2^m in the limit. Furthermore, given that under the efficient rationing rule only the higher of the two prices matter for the social surplus, the fact that $p_2 \to p_2^m$ implies that social surplus is minimized when α approaches one.

Proposition 2 has already important implications for the welfare analysis but in the next Section we strengthen this result for the case of linear demand by showing that the private firm increases its equilibrium price (in the sense of first-order stochastic dominance) when the public share in the semi-public firm increases for any value of α.

5.2 Comparative statics under linear demand

Determining the mixed-strategy equilibrium of the standard Bertrand-Edgeworth duopoly under general demand conditions is a difficult task. The

[14] In particular, their NE_2-type equilibrium is approached.

semi-public version of this game appears to be even more difficult. Therefore, in this section we focus on the case of linear demand and symmetric capacities to provide a comparative statics result for how social surplus changes with the key parameter α. After the analysis, we provide discussion of the robustness of the main comparative statics results for different rationing rules and other considerations.

Let $D(p) = (1-p)^+$, $P(q) = (1-q)^+$ and $k = k_1 = k_2$. Then by Proposition 1, the equilibrium is in mixed strategies if and only if $k \in (1/3, 1)$. For the latter capacity region we get:

$$p_1^m = p_2^m = \frac{1-k}{2}, \quad p_1^d = p_2^d = \frac{(1-k)^2}{4k}, \quad \text{and} \quad p_1^s = \frac{1-\alpha}{2-\alpha}(1-k).$$

It can be verified that for $\alpha > 0$ we have $p_i^s < p_i^m$. Social surplus equals:

$$SW(p_1, p_2) = \begin{cases} \frac{1}{2}(1+p_1)(1-p_1) = \frac{1}{2}(1-p_1^2) & \text{if} \quad p_1 \geq p_2; \\ \frac{1}{2}(1+p_2)(1-p_2) = \frac{1}{2}(1-p_2^2) & \text{if} \quad p_1 < p_2. \end{cases}$$

To simplify notation, we shall denote the cumulative distribution functions of the semi-public and the private firms by F and G, respectively. From Lemma 3 it follows that $\overline{p}_2 = p_2^m$. The private firm's objective function, supposed that the semi-public firm plays its mixed strategy F, is given by

$$\pi_2(F, p_2) = p_2 k(1 - F(p_2)) + p_2(1 - p_2 - k)F(p_2) = \overline{\pi}_2 \tag{4}$$

if F does not have an atom at price p_2. Rearranging (4), we get

$$F(p_2) = \frac{p_2 k - \overline{\pi}_2}{p_2(2k - 1 + p_2)}. \tag{5}$$

It can be verified that $F(p_2^d) = 0$, $F(p_2^m) = 1$ and F is strictly increasing on $[p_2^d, p_2^m]$ if $\overline{\pi}_2 = p_2^d k = p_2^m(1 - p_2^m - k)$. Observe that the private firm can guarantee itself $p_2^m D_2^r(p_2^m) = p_2^m(1 - p_2^m - k)$ profits, and therefore it could achieve only higher profits in a mixed-strategy equilibrium if $\underline{p}_2 > p_2^d$, which can only be the case if the semi-public firm has an atom at price p_2^m since $\overline{p}_2 = p_2^m$. However, then the private firm cannot have an atom at price p_2^m, and thus, the semi-public firm would be better off by setting lower prices than p_2^m since $p_1^s < p_2^m$. To summarize our findings, we have shown that p_2^d and p_2^m are the lowest and highest prices the private firms sets, respectively, furthermore, the private firm serves residual demand at price p_2^m.

The objective function of the semi-public firm, supposing that the private firm plays its mixed strategy G, is given by

$$\begin{aligned}\pi_1(p_1, G) &= (1-\alpha)p_1 k(1-G(p_1)) + (1-\alpha)p_1(1-p_1-k)G(p_1) + \\ &\quad \alpha\frac{1}{2}(1-p_1^2)G(p_1) + \alpha\frac{1}{2}\int_{p_1}^{p_2^m}(1-p_2^2)dG(p_2) = \overline{\pi}_1, \end{aligned} \tag{6}$$

where the first line of (6) contains the profit component and the second line of (6) the social surplus component of the semi-public firm's payoff function.

The private firm's equilibrium strategy can be obtained by solving $\frac{\partial \pi_1}{\partial p_1}(p_1, G) = 0$ and $G(p_2^d) = 0$. In the Appendix, we show that -upon solving a simple differential equation- that $G(p)$ can be written as follows:

$$G(p) = \frac{k}{p}(1-\alpha)\left(1 - \left(\frac{\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^2}{2k+p-1}\right)^{\frac{1}{1-\alpha}}\right). \tag{7}$$

By construction, $G(p_2^d) = 0$ and $G(p) > 0$ for all $p > p_2^d$ when $k > 1/3$, as it can be easily verified.

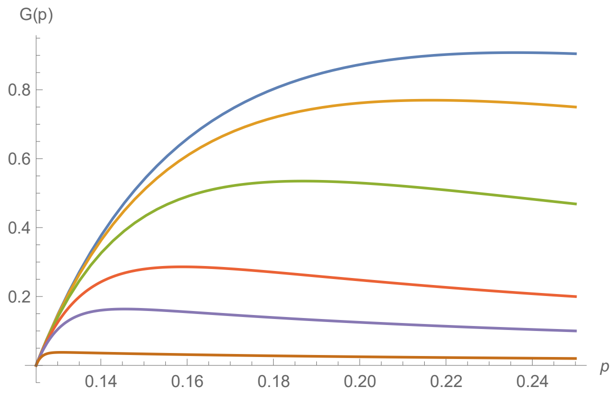

Figure 1: G in case of $\alpha \in \{0.25, 0.5, 0.75, 0.9, 0.95\}$ and $k = 0.5$.

Unfortunately, G does not specify a cumulative probability distribution function because it fails to be increasing on $[p_2^d, p_2^m]$, which is illustrated in Figure 1. Therefore, we need to derive the equilibrium distributions in such a way that their monotonicity are respected. It can be shown that G

is increasing at p_2^d. We shall denote by $p_0 \in (p_2^d, p_2^m)$ the price for which $g(p_0) = G'(p_0) = 0$ and p_0 is a local maximum of G, which is shown to be uniquely defined in the proof.[15] The equilibrium takes the following form:

Proposition 3. *$\overline{F}(p)$ and $\overline{G}(p)$ given by*

$$\overline{F}(p) = \begin{cases} 0 & \text{if } p \in [0, p_2^d], \\ F(p) & \text{if } p \in (p_2^d, p_0), \\ 1 & \text{if } p \in [p_0, b] \end{cases} \quad \text{and} \quad \overline{G}(p) = \begin{cases} 0 & \text{if } p \in [0, p_2^d], \\ G(p) & \text{if } p \in (p_2^d, p_0], \\ G(p_0) & \text{if } p \in (p_0, p_2^m), \\ 1 & \text{if } p \in [p_2^m, b], \end{cases}$$

is the unique mixed-strategy equilibrium of the semi-public Bertrand-Edgeworth game, where F and G stand for the functions determined by (5) and (8), respectively. In particular, $\overline{F}$ has an atom at p_0, while $\overline{G}$ has an atom at p_2^m.

The following key result (Lemma 5) shows that the equilibrium price distribution produced by the private firm is increasing in α in the first-order stochastic dominance sense. The intuition for this result is the generalization of the strategic effect mentioned after Proposition 2. In particular, when firm one has a higher public share, its price decreases, which means that firm two has a lower chance to become the lower priced firm as α increases. Therefore, firm two sets a price that aims more and more to maximize profits on the residual demand curve rather than to undercut firm one's price to increase its sales as α increases.

Lemma 5. *$G(p, \alpha)$ is strictly decreasing in $\alpha \in [0, 1]$ for any $p \in (p_2^d, p_2^m]$.*

Based on Lemma 5, whose proof is in the Appendix, our main result states that social surplus is decreasing in the governmental influence in the semi-public firm, and therefore full privatization is the socially optimal solution. As discussed in the next Section, since social surplus depends entirely on the higher price, Lemma 5 implies that a higher governmental share decreases this surplus.

Proposition 4. *The standard Bertrand-Edgeworth game yields the highest social surplus, which means that even partial privatization would be harmful in the semi-public framework. Furthermore, social surplus is decreasing in α.*

[15] The proof of this result appears in the Appendix.

The proof of the Proposition, which appears in the Appendix, is a straightforward application of Lemma 5, and the formulas derived for functions F and G above. This result is starkly different from what was found in the literature in that increasing public shares in the semi-public firm reduces social surplus for any level of α.

5.3 Discussion

The role of capacities and the demand function. We discuss robustness of the results on welfare with respect to general demand conditions, and with respect to capacities first. First, note that purely public ownership minimizes social surplus regardless of the demand function as long as $p_1^m \leq p_2^m$, which is equivalent to $k_1 \leq k_2$ under our assumptions on the demand function. However, this is only a strong sufficient condition for our result, and the result still holds if k_1 is not much higher than k_2.

Second, note that the proof of Lemma 5 requires explicit solutions, which were derived under linear demand only. Therefore, while Lemma 5 and Proposition 4 hold under linear demand and other nearby demand functions, it is unclear how general concave demand affects their validity. Future results should be able to identify sufficient conditions for Proposition 4 as it does not seem to be a knife-edge result.

The role of the rationing rule. We have identified two effects: the direct effect implies that the price of the semi-public firm decreases in α, while the strategic effect means that the price of the private firm increases in α due to its lower incentive to compete with the semi-public firm. These two effects generalize to other rationing rules, including the proportional rationing rule, the other major rationing rule used in the literature. The unique feature of the efficient rationing rule is that only the higher of the two price matters, which means that the social surplus is completely governed by the strategic effect. For a rationing rule that is similar to the efficient rule mathematically, we can still expect that the social surplus is decreasing in α. However, when the rationing rule is changed to a rule where social surplus depends on the lower price to a larger extent, the result may change.

To illustrate this possibility, take the case of linear demand and assume that rationing is proportional. Assuming $p_2 > p_1$, proportional rationing means that $D_2^r = (1 - \frac{k}{1-p_1})(1 - p_2)$. In this case, the equilibrium is in pure strategies when α is close to one for a certain region of capacities. We identify such a region, and show that social surplus is increasing in α in that region because $p_2 = p^M = 1/2$ and p_1 is decreasing in α.

In this case, the private firm maximizes $pD(p)(1 - \frac{k}{1-p_1})$, which is maximized at $p^M = 1/2$. The rest of the analysis first derives the best response of the semi-public firm (firm 1), and then we check back whether the private firm (firm 2) has an incentive to deviate.

When $\alpha = 1$ firm 1 maximizes social surplus. If firm 1 chooses $p_1 < p_2 = p^M = 1/2$, then all consumers with valuations higher than 1/2 trade with probability one, and all consumers with valuations between p_1 and 1/2 buy with probability $k/(1-p_1)$. Therefore, social surplus, the sum of consumer and producer surplus is

$$SS = \frac{1}{2}\frac{3}{4} + \frac{k}{1-p_1}(\frac{1}{2} - p_1)\frac{1/2 + p_1}{2}$$

by using the gross consumer surplus formulation. The problem of firm 1 becomes

$$\max_{p_1} \frac{1/4 - (p_1)^2}{1 - p_1}$$

with solution $p_1 = 1 - \sqrt{3}/2 = 0.13$.

Now, let us check if firm 2 has an incentive to undercut firm 1's price. The equilibrium profit is $\frac{1}{4}(1 - \frac{k}{\sqrt{3}/2})$. The profit from undercutting and charging $p_1 - \varepsilon$ is equal to $kp_1 = k(1 - \sqrt{3}/2)$, and thus the incentive condition becomes $\frac{1}{4}(1 - \frac{k}{\sqrt{3}/2}) \geq k(1 - \sqrt{3}/2)$ or $k \leq \frac{1}{4 + \frac{2\sqrt{3}}{3} - 2\sqrt{3}} = 0.59151$. Focusing on the range where $k \leq 0.59151$ the equilibrium is in pure strategies when $\alpha = 1$. Suppose that k is (much) lower than this cutoff and thus there is a pure strategy equilibrium for a range where $\alpha < 1$. Analyzing the equilibrium in this case, it still holds that $p_2 = 1/2$. By construction, firm 1 maximizes social surplus when $\alpha = 1$. Given that firm 2 sets $p_2 = 1/2$ for any α it must be that in this range TW is maximized when $\alpha = 1$ given that there is no strategic effect that affects firm 2's choice.

In summary, in the range where α is high social surplus is maximized when $\alpha = 1$. The reason is the lack of strategic effect in this case: firm two sets the monopoly price in equilibrium regardless of α, which then implies that the direct effect through p_1 makes social surplus increase in α. When α

is close to zero, the equilibrium is in mixed strategies, see Beckman (1965). In this case, a change in α changes the equilibrium price distribution of firm two, and further calculations are necessary to settle the question even for the case of linear demand.[16]

6 Conclusions

We have analyzed the difficult question of a semi-public firm competing with a private firm under capacity constraints. We have shown that under the commonly used efficient rationing rule, social surplus is decreasing in the public concern of the semi-public firm, and discussed robustness of this finding. We also highlighted that for general rationing rules there are two opposing effects. First, a higher level of α directly increases social surplus by the action of the semi-public firm. On the other hand, α also indirectly affects the price set by firm two as it was highlighted under efficient rationing. This second effect is ambiguous, and can be surplus reducing especially if the rationing rule is close to the efficient rationing rule. It remains for future research to study general conditions for comparative statics in terms of demand conditions and rationing rules.

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7 Appendix

7.1 Derivation of function G from the indifference condition

By differentiation we get

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1}(p_1, G) &= (1-\alpha)k(1-G(p_1)) - (1-\alpha)p_1 k g(p_1) \\
&\quad +(1-\alpha)\left[(1-p_1-k)G(p_1) - p_1 G(p_1) + p_1(1-p_1-k)g(p_1)\right] \\
&\quad -\alpha p_1 G(p_1) + \frac{1}{2}\alpha(1-p_1^2)g(p_1) \\
&\quad -\frac{1}{2}\alpha(1-p_1^2)g(p_1) \\
&= \left[(1-\alpha)(1-2p_1-2k) - \alpha p_1\right] G(p_1) \\
&\quad +(1-\alpha)p_1(1-p_1-2k)g(p_1) + (1-\alpha)k = 0,
\end{aligned}$$

where g is the derivative of G, and the expression is just defined where G is differentiable. Solving the first-order linear differential equation, we get[17]

$$G(p_1) = C\frac{1}{p_1}\left(\frac{1}{2k+p_1-1}\right)^{\frac{1}{1-\alpha}} + \frac{k(1-\alpha)}{p_1} \tag{8}$$

and employing $G(p_2^d) = 0$ we arrive at

$$C = -k(1-\alpha)\left(\frac{3}{2}\sqrt{k} - \frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}.$$

Upon substituting C in (8), we obtain (7).

7.2 Proof of Lemmas and Propositions

Proof of Lemma 1:

Proof. Suppose that there exists a price $p \in [0, b]$ for which $\varphi_1(p) > 0$ and $\varphi_2(p) > 0$. However, this would imply because of $\underline{p}_1 > p^c$ and $\underline{p}_2 > p^c$ that both firms $i \in \{1, 2\}$ would be better off by unilaterally shifting probability mass from price p to $p - \varepsilon$; a contradiction. $\square$

Proof of Lemma 2:

[17]Under our assumptions we have that $2k + p - 1 = p - p^c > 0$ since $p_1^d = p_2^d > p^c$.

Proof. Let $\overline{p}_1 > \overline{p}_2$. If $\overline{p}_1 > p_1^s$, then the semi-public firm could benefit from setting a price below $\overline{p}_1$ because of the strict concavity of its residual payoff function. If $\overline{p}_1 < p_1^s$, then the semi-public firm would make more profits by setting price p_1^s than setting any other price in $(\overline{p}_2, p_1^s)$; a contradiction. Hence, in case of $\overline{p}_1 > \overline{p}_2$ we must have $\overline{p}_1 = p_1^s$.

In an analogous way it can be shown that if $\overline{p}_1 < \overline{p}_2$, we must have $\overline{p}_2 = p_2^m$.

Suppose that $\min\{p_1^s, p_2^m\} > \overline{p}_1 = \overline{p}_2$. Then since in equilibrium at least one of the mixed strategies cannot have an atom at $\overline{p}_1 = \overline{p}_2$, say $\varphi_i(\{\overline{p}_i\}) = 0$, firm $j \neq i$ could increase its payoff by setting either price p_1^s or p_2^m; a contradiction.

Suppose that $\overline{p}_1 = \overline{p}_2 > \max\{p_1^s, p_2^m\}$. Then since in equilibrium at least one of the mixed strategies cannot have an atom at $\overline{p}_1 = \overline{p}_2$, say $\varphi_i(\{\overline{p}_i\}) = 0$, and thus firm $j \neq i$ would serve residual demand with probability one at price $\overline{p}_j$ from which it follows that its payoff would be higher at price p_1^s or p_2^m (in the former case this would result for the semi-public firm both an increase in profit and social surplus); a contradiction. □

Proof of Lemma 3:

Proof. From Lemma 2 and $p_1^s < p_2^m$ it follows that there can be three cases: (i) $\overline{p}_1 = p_1^s > \overline{p}_2$, (ii) $\overline{p}_1 < \overline{p}_2 = p_2^m$ or (iii) $p_1^s \leq \overline{p}_1 = \overline{p}_2 \leq p_2^m$.

In Case (i) firm 1 faces residual demand on $(\overline{p}_2, \overline{p}_1)$, and therefore it will not set prices within that interval and must have an atom at price $\overline{p}_1 = p_1^s$. Then for prices in $(\overline{p}_2, \overline{p}_1)$ the profit function of firm 2 is given by

$$R(p) = \varphi_1(p_1^s)pk + (1 - \varphi_1(p_1^s))p\,(D(p) - k)$$

and its derivative

$$R'(p) = \varphi_1(p_1^s)k + (1 - \varphi_1(p_1^s))\left(D(p) - k + pD'(p)\right)$$

is positive on $(\overline{p}_2, \overline{p}_1)$ since $p < \overline{p}_1 < p_2^m$. Therefore, if firm 1 does not have an atom at $\overline{p}_2$, then firm 2 benefits from setting prices above $\overline{p}_2$; a contradiction. If firm 1 does have an atom at price $\overline{p}_2$, then firm 2 does not have one at $\overline{p}_2$. However, then firm 1 would be better of by setting price p_1^s instead of price

$\overline{p}_2$, where it is supposed to have an atom, yielding another contradiction. We conclude that Case (i) cannot occur.

Taking Cases (i) and (ii) into consideration, Case (iii) simplifies to $p_1^s < \overline{p}_1 = \overline{p}_2 < p_2^m$. If $\varphi_1(\overline{p}_2) = 0$, then firm 2 would be better off by setting price p_2^m; a contradiction. If $\varphi_1(\overline{p}_2) > 0$, then $\varphi_2(\overline{p}_2) = 0$, and therefore firm 1 would be better off by setting price p_1^s since both its profit and social surplus component would be higher. □

Proof of Lemma 4:

Proof. First, we establish that $\overline{p}_i \leq p_i^M$. Clearly, the semi-public firm's prices above p_1^M would be strictly dominated by price p_1^M (i.e. $\pi_1(p_1^M, \varphi_2) > \pi_1(p_1, \varphi_2)$ for any $p_1 > p_1^M$ and any mixed strategy φ_2 played by the private firm). The case of $\overline{p}_2 \leq p_2^M$ is even more obvious. Hence, the firms' do not set 'extremely' high prices.

Second, we demonstrate that $\underline{p}_1 \leq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 > \underline{p}_2$. Then by $\underline{p}_2 < \overline{p}_2 \leq p_2^M$ the private firm would benefit from switching from φ_2 to any price $p_2 \in (\underline{p}_2, \underline{p}_1)$; a contradiction.

Third, we demonstrate that $\underline{p}_1 \geq \underline{p}_2$. Suppose to the contrary that $\underline{p}_1 < \underline{p}_2$. Then by $\underline{p}_1 < \overline{p}_1 \leq p_1^M$ the public firm would benefit from switching from φ_1 to any price $p_1 \in (\underline{p}_1, \underline{p}_2)$ since the profit component of its payoff function would increase and the social surplus component of its payoff function would not change; a contradiction.

Forth, suppose that $\varphi_1(\underline{p}_1) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_1 - \varepsilon$ would strictly dominate price $\underline{p}_1 + \varepsilon$ for the private firm; a contradiction.

Finally, suppose that $\varphi_2(\underline{p}_2) > 0$. Then for a sufficiently small $\varepsilon > 0$ price $\underline{p}_2 - \varepsilon$ would strictly dominate price $\underline{p}_2 + \varepsilon$ for the semi-public firm since its profit component would be radically larger at the former price than at the latter one by its discontinuity at $\underline{p}_2$, while the social surplus component would be just slightly lower by the continuity of the social surplus component; a contradiction. □

Proof of Proposition 3:

Proof. First, we establish that $G(p) \leq 1$ for any $p \in [p_2^d, p_2^m]$ by showing that

$$G(p) \leq F(p) \quad \text{for any } p \in [p_2^d, p_2^m]. \tag{9}$$

Note that (9) holds with equality in case of $p = p_2^d$. (9) is equivalent with

$$C\left(\frac{1}{2k+p-1}\right)^{\frac{1}{1-\alpha}} + k(1-\alpha) \leq \frac{pk - \overline{\pi}_2}{(2k+p-1)}, \tag{10}$$

which we show by verifying that the derivative of the LHS is smaller than that of the RHS for any $p \in [p_2^d, p_2^m]$:

$$\begin{aligned}
C\frac{1}{1-\alpha}\left(\frac{1}{2k+p-1}\right)^{\frac{1}{1-\alpha}-1}\frac{-1}{(2k+p-1)^2} &\leq \frac{k(2k+p-1)-\left(pk-\frac{(1-k)^2}{4}\right)}{(2k+p-1)^2} \\
k\left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{2k+p-1}\right)^{\frac{\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}k-\frac{1}{2}\right)^2,
\end{aligned}$$

where the LHS of the last inequality achieves its maximum value on $[p_2^d, p_2^m]$ for a given $k \in [1/3, 1]$ at $p = p_2^d = (1-k)^2/(4k)$, and therefore

$$\begin{aligned}
\left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{2k+\frac{(1-k)^2}{4k}-1}\right)^{\frac{\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^2 \\
\left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^{\frac{2}{1-\alpha}}\left(\frac{1}{\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}}\right)^{\frac{2\alpha}{1-\alpha}} &\leq \left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^2,
\end{aligned}$$

and we can see that the last inequality holds with equality, and (9) follows. Verifying that $0 \leq G(p)$ for any $p \in [p_2^d, p_2^m]$, can be obtained through simple rearrangements, which leads to

$$\left(\frac{3}{2}\sqrt{k}-\frac{1}{2\sqrt{k}}\right)^2 \leq 2k+p-1,$$

and by employing again $p \geq p_2^d = (1-k)^2/(4k)$.

By (6) and (4) firms 1 and 2 achieve $\overline{\pi}_1$ and $\overline{\pi}_2$ payoffs, respectively, when playing any of their pure strategies $p \in [p_2^d, p_0]$ against their opponents' above specified strategies ($\overline{G}$ and $\overline{F}$). Clearly, playing a price below p_2^d results in less payoff than $\overline{\pi}_1$ and $\overline{\pi}_2$, respectively. It is straightforward that the private firm makes less profit by setting a price $p_2 \in (p_0, p_2^m) \cup (p_2^m, b]$ than by setting price p_2^m, when playing against mixed strategy $\overline{F}$, since for prices in $(p_0, b]$ it serves residual demand with probability one.

We check that the socially concerned firm achieves less than $\overline{\pi}_1$ payoff when playing any pure strategy $p_1 \in (p_0, p_2^m]$ against mixed strategy $\overline{G}$.

Starting with (6) and as a slight modification of its solution G, we define $\widetilde{G}$ for any pure strategy $p_1 \in [0, b]$ by $\widetilde{G}(p) = 0$ on $[0, p_2^d]$, by $G(p)$ on $(p_2^d, p_1]$, by $\widetilde{G}(p) = G(p_1)$ on $(p_1, p_2^m]$, and by $\widetilde{G}(p) = 1$ on $(p_2^m, b]$ (which is not necessarily a mixed strategy). Assuming that the mixed strategy of the private firm would be mainly determined by G, but remains constant on $[p_1, p_2^m)$ and jumps up to 1 at p_2^m, which gives us function $\widetilde{G}$, we determine the price level p^* for the socially concerned firm at which its payoff is maximized. However, since G is not necessarily a cumulative distribution function because it might have decreasing parts before p_1, we just call the payoff function, which we are maximizing based on $\widetilde{G}$, as a 'virtual payoff' function and we will find out that G is indeed increasing on $[p_2^d, p^*]$, and therefore $\widetilde{G}$ specifies a cumulative distribution function $p_1 = p^*$. The virtual payoff function of the socially concerned firm is given by

$$\begin{aligned} z(p_1) = \pi_1^g(p_1, G(p_1)) &= (1-\alpha)p_1 k(1 - G(p_1)) \\ &\quad +(1-\alpha)p_1(1 - p_1 - k)G(p_1) \\ &\quad +\alpha\frac{1}{2}(1 - p_1^2)G(p_1) \\ &\quad +\alpha\frac{1}{2}(1 - (p_2^m)^2)(1 - G(p_1)), \end{aligned} \tag{11}$$

where $\pi_1^g(p_1, G(p_1))$ equals its real payoff if G is increasing until p_1.

First, we will show that the sign of g equals the sign of $z'(p_1) = d\pi_1^g(p_1, G(p_1))/dp_1$, which then would imply that $p^* = p_0$ and a function of type $\widetilde{G}$ cannot be a cumulative distribution function in case of $p_1 > p_0$. To establish that $p^* = p_0$ we consider the derivative of the difference of (11) and (6)

$$z(p_1) - \pi_1(p_1, G) = \frac{\alpha}{2}(1 - G(p_1))\left(1 - (p_2^m)^2\right) - \frac{\alpha}{2}\int_{p_1}^{p_2^m} \left(1 - p_2^2\right) dG(p_2), \tag{12}$$

which equals

$$\begin{aligned} \frac{d}{dp_1}\left(z(p_1) - \pi_1(p_1, G)\right) &= -\frac{\alpha}{2}g(p_1)\left(1 - (p_2^m)^2\right) + \frac{\alpha}{2}\left(1 - p_1^2\right)g(p_1) \\ &= -\frac{\alpha}{2}g(p_1)\left[\left(1 - (p_2^m)^2\right) - \left(1 - p_1^2\right)\right], \end{aligned}$$

and therefore, it follows that the signs of g and z are identical since $\pi_1(p_1, G) = \overline{\pi}_1$ and $\left[\left(1 - (p_2^m)^2\right) - \left(1 - p_1^2\right)\right] < 0$ in case of $p_1 < p_2^m$.

We now turn to proving that setting prices above p^* results in less payoff than $\overline{\pi}_1$ for the socially concerned firm. For any $p \in [p^*, p_2^m]$ we have

$$\begin{aligned}\pi_1(p_1, \overline{G}) &= (1-\alpha)p_1\left[k(1-G(p^*)) + (1-p_1-k)G(p^*)\right] \\ &\quad + \frac{\alpha}{2}\left[(1-p_1^2)G(p^*) + (1-(p_2^m)^2)(1-G(p^*))\right] \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{d}{dp_1}\pi_1(p_1, \overline{G}) &= (1-\alpha)\left[k(1-G(p^*)) + (1-2p_1-k)G(p^*)\right] \\ &\quad - \alpha p_1 G(p^*). \end{aligned} \tag{14}$$

In order to employ the results obtained so far we need the derivative of z:

$$\begin{aligned} z'(p_1) &= (1-\alpha)\left[k(1-G(p_1)) + (1-2p_1-k)G(p_1)\right] - \alpha p_1 G(p_1) \\ &\quad + (1-\alpha)\left[-p_1 k g(p_1) + p_1(1-p_1-k)g(p_1)\right] \\ &\quad + \frac{\alpha}{2}\left[(1-p_1^2)g(p_1) - (1-(p_2^m)^2)g(p_1)\right]. \end{aligned} \tag{15}$$

By employing that p^* maximizes z, and therefore $z'(p^*) = 0$, and $g(p^*) = 0$ we obtain from (15) that

$$z'(p^*) = (1-\alpha)\left[k(1-G(p^*)) + (1-2p^*-k)G(p^*)\right] - \alpha p^* G(p^*) = 0. \tag{16}$$

Since the function appearing in (13) is strictly concave in p_1 it has a unique maximum point, which then equals $p^* = p_0$ by taking (14) and (16) into consideration.

Finally, we demonstrate that the mixed-strategy equilibrium given in the statement of Proposition 3 is unique. Suppose that there is a gap $(c, d) \subset [p_2^d, p_2^m]$ in the support of the semi-public firm's equilibrium strategy. Then the private firm would not set a price in that interval neither since its profits would be increasing on (c, d).[18] If the semi-public firm's equilibrium strategy does not have an atom at c, then the private firm could achieve higher profits than $\overline{\pi}_2$ by setting prices in (c, d); a contradiction. If the semi-public firm's equilibrium strategy does have an atom at c, then the private firm's equilibrium strategy does not have one by Lemma 1. The profit component of the semi-public firm could increase radically by setting appropriately

[18]The formal proof of this and the next observation is analogous to the proof of Case (i) in the part on the more favorable residual demand for firm 2 in Lemma 3.

selected prices slightly below c, while the social surplus component is continuous at c, and therefore the semi-public firm would deviate and we cannot have a mixed-strategy equilibrium in which the semi-public firm does have a gap $(c,d) \subset [p_2^d, p_2^m]$ in the support of its equilibrium mixed strategy.

Suppose that the private firm has a gap $(c,d) \subset [p_2^d, \overline{p}_1]$ in its equilibrium strategy. Then the semi-public firm would not set some prices in a subinterval of (c,d), because its profit and social surplus components have different functional forms and an increase in one of them cannot be exactly compensated by a decrease in the other one. Hence, the semi-public firm must have a gap in $[p_2^d, \overline{p}_1]$, which cannot be the case as shown in the previous paragraph; a contradiction. Therefore, the private firm does not have a gap in $[p_2^d, \overline{p}_1]$.

From this and what we have already shown it follows that the private firm's equilibrium profit equals $\overline{\pi}_2$, the support of the private firm equals $[p_2^d, \overline{p}_1] \cup \{p_2^m\}$ and $\overline{p}_1 = p_0$. □

Proof of Lemma 5:

Proof. For any $p \in [p_2^d, p_2^m]$ by differentiation of (7) with respect to α we get

$$\begin{aligned} \frac{\partial}{\partial \alpha} G(p,\alpha) &= -\frac{k}{p}\left[1 - \left(\frac{\frac{9}{4}k - \frac{3}{2} + \frac{1}{4k}}{2k+p-1}\right)^{\frac{1}{1-\alpha}}\right] \\ &\quad + \frac{k(1-\alpha)}{p}\left[\left(\frac{\frac{9}{4}k - \frac{3}{2} + \frac{1}{4k}}{2k+p-1}\right)^{\frac{1}{1-\alpha}} \frac{1}{(1-\alpha)^2} \ln \frac{\frac{9}{4}k - \frac{3}{2} + \frac{1}{4k}}{2k+p-1}\right], \end{aligned}$$

which is negative if and only if

$$1 > \left(\frac{\frac{9}{4}k - \frac{3}{2} + \frac{1}{4k}}{2k+p-1}\right)^{\frac{1}{1-\alpha}} \left(1 + \frac{1}{1-\alpha} \ln \frac{\frac{9}{4}k - \frac{3}{2} + \frac{1}{4k}}{2k+p-1}\right),$$

where the latter inequality holds true for any $p > p_2^d$ by G(p) > 0. □

Proof of Proposition 4:

Proof. We shall denote by H the cumulative distribution function of the higher price. Hence, $H(p,\alpha) = \overline{F}(p,\alpha)\overline{G}(p,\alpha)$ for all $p \in [p_2^d, p_2^m]$ and for any given $\alpha \in [0,1]$, where we emphasize that each $\alpha \in [0,1]$ specifies a different game with the respective equilibrium strategies $\overline{F}(p,\alpha)$ and $\overline{G}(p,\alpha)$ of the two firms.

Then for any $p \in \left[p_2^d, p_0(\alpha)\right)$, where $g(p_0(\alpha),\alpha) = 0$, we have

$$\begin{aligned}
\frac{\partial H}{\partial p}(p,\alpha) &= h(p,\alpha) = f(p)G(p,\alpha) + F(p)g(p,\alpha), && (17)\\
\frac{\partial H}{\partial \alpha}(p,\alpha) &= F(p)\frac{\partial G}{\partial \alpha}(p,\alpha), && (18)\\
\frac{\partial^2 H}{\partial p \partial \alpha}(p,\alpha) &= \frac{\partial h}{\partial \alpha}(p,\alpha) = f(p)\frac{\partial G}{\partial \alpha}(p,\alpha) + F(p)\frac{\partial g}{\partial \alpha}(p,\alpha), && (19)
\end{aligned}$$

where $g(p,\alpha) = (\partial G/\partial p)(p,\alpha)$.

Since for the relevant price region $[p_2^d, p_2^m]$ social surplus is determined by the higher price set by the two firms to prove our theorem it is sufficient to show that $H(\cdot,\alpha)$ first order stochastically dominates $H(\cdot,\alpha')$ for any $1 \geq \alpha > \alpha' \geq 0$. We establish this dominance relationship by showing that $(\partial H/\partial p)(p,\alpha) \leq 0$ for any $p \in [p_2^d, p_2^m]$. Clearly, this inequality holds for any $p \in [p_2^d, p_0(\alpha))$ by Lemma 5 and Equation (18). Therefore, we still have to consider the case of $p \in [p_0(\alpha), p_2^m)$ for which $H(p,\alpha)$ is given by

$$H(p,\alpha) = \int_{p_2^d}^{p_0(\alpha)} f(r)G(r,\alpha) + F(r)g(r,\alpha)dr + \left(1 - F\left(p_0(\alpha)\right)\right) G\left(p_0(\alpha),\alpha\right).$$

By differentiating with respect to α,[19] thereafter by rearrangements and taking $g(p_0(\alpha),\alpha) = (\partial G/\partial p)(p_0(\alpha),\alpha) = 0$ into account, and finally employing

[19] Note that $F(p_2^d) = 0$ and $G(p_2^d,\alpha) = 0$.

Young‘s theorem, we get

$$
\begin{aligned}
\frac{\partial H}{\partial \alpha}(p,\alpha) &= \left[f(p_0(\alpha))G(p_0(\alpha),\alpha)+F(p_0(\alpha))g(p_0(\alpha),\alpha)\right]p_0'(\alpha) \\
&\quad +\int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial g}{\partial \alpha}(r,\alpha)dr \\
&\quad -f(p_0(\alpha))p_0'(\alpha)G(p_0(\alpha),\alpha) \\
&\quad +\left(1-F\left(p_0(\alpha)\right)\right)\left[\frac{\partial G}{\partial p}\left(p_0(\alpha),\alpha\right)p_0'(\alpha)+\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)\right] \\
&= \int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial g}{\partial \alpha}(r,\alpha)dr \\
&\quad +\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right) \\
&= \int_{p_2^d}^{p_0(\alpha)} f(r)\frac{\partial G}{\partial \alpha}(r,\alpha)+F(r)\frac{\partial^2 G}{\partial \alpha \partial p}(r,\alpha)dr \\
&\quad +\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right) \\
&= \left[F(r)\frac{\partial G}{\partial \alpha}(r,\alpha)\right]_{p_2^d}^{p_0(\alpha)}+\left(1-F\left(p_0(\alpha)\right)\right)\frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right) \\
&= \frac{\partial G}{\partial \alpha}\left(p_0(\alpha),\alpha\right)<0.
\end{aligned}
$$

□