Dynamic Attention Behavior under Return Predictability *

Daniel Andrei[†]   Michael Hasler[‡]

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Abstract

We investigate the dynamic problem of how much attention an investor should pay to news in order to learn about stock-return predictability and maximize expected lifetime utility. We show that the optimal amount of attention is U-shaped in the return predictor, increasing with both uncertainty and the magnitude of the predictive coefficient, and decreasing with stock-return volatility. The optimal risky asset position exhibits a negative hedging demand that is hump-shaped in the return predictor. Its magnitude is larger when uncertainty increases, but smaller when stock-return volatility increases. We test and find empirical support for these theoretical predictions.

[†]McGill University, Desautels Faculty of Management, 1001 Sherbrooke Street West, Montréal, QC, H3A 1G5, Canada, daniel.andrei@mcgill.ca, www.danielandrei.info.

[‡]University of Toronto, 1095 Military Trail, Toronto, ON, M1C 1A4, Canada, michael.hasler@utoronto.ca; www.MichaelHasler

1 Introduction

Growing empirical evidence suggests that investors' attention is time-varying (Gargano and Rossi, 2017).[1] This affects trading and asset prices, and therefore significantly impacts financial markets (Fisher et al., 2016).[2] The aim of this paper is to show that these fluctuations result from a rational information gathering behavior.

We build a dynamic portfolio choice model to investigate the problem of how much attention an investor should pay to news. Our setup is based on the incomplete-information literature,[3] which implicitly assumes that the set of available information is exogenously given. The objective here is to relax this assumption and assume that information is optimally acquired at a cost. We consider an investor who faces uncertainty about stock-return predictability (Xia, 2001), and solves her portfolio choice and information acquisition problem accordingly. While information acquisition is a one time choice in, for instance, Huang and Liu (2007), it is dynamic in our framework. This allows us to provide empirically testable predictions on the dynamic relation between attention, risky investments, and the relevant state variables.

In our theoretical model, an agent can invest in one risk-free asset and one risky stock with unobservable expected returns. At each point in time, the investor optimally chooses her consumption, portfolio, and quantity of information needed to estimate expected returns and maximize expected lifetime utility of consumption. Information acquisition regulates both the learning and the investment decisions of the investor. By acquiring more accurate information, i.e. by paying more *attention to news*,[4] the investor is able to better estimate expected returns and, therefore, to increase her expected utility, but at the expense of decreasing her current wealth. In other words, the investor faces a dynamic trade-off problem of asset and attention allocation.

The investor assumes that expected returns are a linear function of an observable predictive variable (Xia, 2001).[5] The focus of the paper is on the predictive coefficient, which is stochastic

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[1]See also Barber and Odean (2008); Da, Engelberg, and Gao (2011); Sicherman, Loewenstein, Seppi, and Utkus (2015); Fisher, Martineau, and Sheng (2016).

[2]See also Chien, Cole, and Lustig (2012); Andrei and Hasler (2014); Gargano and Rossi (2017); Hasler and Ornthanalai (2017).

[3]Refer to the seminal papers by Detemple (1986); Gennotte (1986); Dothan and Feldman (1986).

[4]See Sims (2003), Peng and Xiong (2006), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), Mondria (2010), Mondria and Quintana-Domeque (2013) for a similar interpretation of attention.

[5]See also Kandel and Stambaugh (1996), Barberis (2000), and Brandt, Goyal, Santa-Clara, and Stroud (2005) for equivalent assumptions.

and *unobservable*. The investor uses all of the available historical data to estimate the predictive coefficient and then construct a forecast of returns. In additon to historical data, the investor can also choose to improve her estimation by collecting news, but at a cost. Because perfect learning—observing expected returns—has an infinite cost, the investor must choose an optimal finite amount of attention. We characterize this optimal amount of attention and its responsiveness to changes in the state variables of the model.

The predictive variable determines, to a large extent, the optimal amount of attention to news. We show that the investor pays more attention to news the further away the predictive variable is from its long-term mean. In these states, the investor attempts to profit from the reversion to the mean of expected returns and thus acquires additional information to bet on the upcoming trend. Because this arises whenever the predictive variable is far away from its long-term mean, attention exhibits a U-shaped pattern.

Furthermore, we show that greater uncertainty unambiguously increases attention to news. The reason is that greater uncertainty increases the volatility of *expected returns* and therefore increases the likelihood of large future trends. Since the investor can efficiently exploit these trends only if her estimate of the predictive coefficient is accurate, the optimal decision is to pay more attention to news.

We also find that attention to news decreases with the volatility of *realized returns*, which we assume to be stochastic (Chacko and Viceira, 2005; Liu, 2007). When the volatility of realized returns increases, the quality of information provided by those returns deteriorates, which decreases the volatility of expected returns. This, in turn, prompts the investor to decrease her attention to news.[6]

Finally, attention to news increases with the magnitude of the estimated predictive coefficient. A predictive coefficient of large magnitude implies highly volatile expected returns, which prompts the investor to be particularly attentive to news.

Investor's optimal risky investment share increases with the Sharpe ratio of the stock, and features a negative hedging demand because expected returns are positively correlated to returns.

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[6]Note that in our model there is a distinction between "expected return volatility" and "realized return volatility." The former depends on all state variables and can be thought of as a forward looking measure (which can be proxied by the VIX index), whereas the latter is a contemporaneous measure. As such, our model predicts that attention to news increases when the expected return volatility is high (see also Fisher et al., 2016) but decreases when the realized return volatility is high.

The hedging demand is hump-shaped in the predictive variable, it is more negative when uncertainty increases, and it is less negative when stock-return volatility increases. This is because expected returns are particularly sensitive to return shocks when the predictive variable is far from its long-term mean, uncertainty is high, and stock-return volatility is low.

In the empirical section of the paper, we test the dependence of attention to news and the risky investment share on the state variables of our model. We first calibrate the model to S&P 500 returns and define the predictive variable as the S&P 500 earnings-to-price ratio. Using this dataset, we then build three *model-implied* time series: uncertainty, attention, and risky investment share. We compare these time series to their empirical counterparts and find a positive correlation in each case. We also provide evidence that our model-implied measure of attention is counter-cyclical and positively predicts the VIX index. Consistent with the predictions of the model, we find that the empirical proxy for attention to news is indeed U-shaped in the predictive variable, it increases with both uncertainty and the magnitude of the predictive coefficient, and it decreases with stock-return volatility. In addition, we show that the empirical proxy for the risky investment share increases with the Sharpe ratio of the stock and features a negative hedging demand whose shape mirrors the model's prediction. We therefore conclude that our rational setup accurately describes investors' dynamic asset and attention allocation behavior.

This paper complements a large body of literature that considers portfolio selection problems with stochastic expected returns, stochastic volatility, incomplete information, and uncertainty about predictability.[7] In the costly information acquisition literature, Detemple and Kihlstrom (1987) analyze the demand for information and the equilibrium price of information in the context of a production economy. Veldkamp (2006a,b) shows that costly information acquisition helps explain excess co-movement and the simultaneous increases in emerging markets' media coverage and equity prices. The most closely related paper is Huang and Liu (2007), who consider a dynamic portfolio choice problem with static costly information choice. That is, the investor dynamically chooses her portfolio, but the accuracy and frequency of information is chosen at time zero only. The

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[7]For portfolio selection problems with stochastic expected returns see Kim and Omberg (1996), Brandt (1999), Campbell and Viceira (1999), Ait-Sahalia and Brandt (2001), Wachter (2002), with stochastic volatility see Chacko and Viceira (2005), Liu (2007); with incomplete information see Detemple (1986), Gennotte (1986), Dothan and Feldman (1986), Feldman (1989), David (1997), Brennan (1998), Veronesi (1999, 2000), Brennan and Xia (2001); and with uncertainty about predictability see Kandel and Stambaugh (1996), Barberis (2000), Xia (2001), and Brandt et al. (2005).

investor optimally acquires a flow of information that has limited accuracy and limited frequency, which potentially makes her under- or over-invest. The frequency and accuracy of information acquisition is shown to be decreasing and increasing with risk aversion, respectively.[8]

Our paper contributes to this literature by considering a dynamic portfolio choice and information acquisition problem in the presence of uncertainty about stock-return predictability. Our study clarifies the dynamic relation between attention, risky investments, and the relevant state variables. More precisely, we show that attention is U-shaped in the predictive variable, increasing with uncertainty about the predictive coefficient, decreasing with stock-return volatility, and increasing with the magnitude of the predictive coefficient. As a result, the relation between attention and the risky investment share is positive when expected returns are high and negative when expected returns are low. These predictions are first described in our theoretical framework, and are then shown to be supported by the data.

The remainder of the paper is organized as follows. Section 2 describes the economy and examines the optimal attention, consumption, and portfolio choice problem of the investor. Section 3 calibrates the parameters of the model and describes the results. Section 4 performs the empirical analysis. Section 5 concludes. The Appendix contains all proofs and computational details.

2 The model

Consider an economy populated by an investor with utility function defined by

$$U(c) = \mathbb{E}\left(\int_0^\infty e^{-\delta s} u(c_s) ds\right),\tag{1}$$

where c_t is the consumption at time t, δ is the subjective discount rate, and $u(c)$ is an increasing and concave function of c differentiable on $(0, \infty)$.

The investor continuously trades one risk-free asset paying a constant rate of return r^f, and

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[8]One related segment of the literature postulates that information has a hedonic impact on utility (Loewenstein, 1987; Caplin and Leahy, 2001, 2004; Brunnermeier and Parker, 2005; Pagel, 2013; Andries and Haddad, 2014). This can generate different levels of attention depending on the state of the world and can thus explain the fluctuations in attention we observe. Another part of the literature postulates that investors have limited ability to process information (Sims, 2003; Van Nieuwerburgh and Veldkamp, 2006, 2010; Kacperczyk et al., 2016). Therefore, information does not directly enter the utility function, but indirectly helps investors make better decisions under uncertainty.

one risky asset (the *stock*) whose return dynamics satisfy

$$\frac{dP_t}{P_t} = \mu_t dt + \sqrt{V_t} dB_{P,t}, \tag{2}$$

where μ_t is the instantaneous expected return on the stock and V_t is the instantaneous variance of stock returns.

The investor operates under partial knowledge of the economy. Specifically, the expected return μ_t is unobservable (Brennan, 1998; Xia, 2001; Ziegler, 2003), but the investor knows that it is an affine function of an observable state variable y_t. That is, the expected return satisfies

$$\mu_t = \bar{\mu} + \beta_t(y_t - \bar{y}), \tag{3}$$

where β_t is an *unobservable* predictive coefficient (Xia, 2001). The observable predictive variable y_t and the unobservable predictive coefficient β_t evolve according to the following diffusion processes:

$$dy_t = \lambda_y(\bar{y} - y_t)dt + \sigma_y dB_{y,t} \tag{4}$$

$$d\beta_t = \lambda_\beta(\bar{\beta} - \beta_t)dt + \sigma_\beta dB_{\beta,t}, \tag{5}$$

where we assume that $\bar{y}$, λ_y, σ_y, $\bar{\beta}$, λ_β, and σ_β are known constants. The variance of stock returns is observable and follows a square-root process (Heston, 1993; Liu, 2007):

$$dV_t = \lambda_V(\bar{V} - V_t)dt + \sigma_V \sqrt{V_t} dB_{V,t}, \tag{6}$$

where $\bar{V}$, λ_V and σ_V are known constants. The four Brownian motions $B_{P,t}$, $B_{y,t}$, $B_{\beta,t}$, and $B_{V,t}$ are mutually independent.

2.1 The inference process: active learning

Given the dynamics of the state variables described above, the investor's problem consists of inferring the predictive coefficient β_t before choosing an optimal portfolio and consumption rule that maximizes the expected lifetime utility of consumption.

The investor has the opportunity to actively learn about return predictability, i.e. to collect

arbitrarily accurate information about the predictive coefficient β_t. This is achieved by acquiring a news signal with the following dynamics:

$$ds_t = \beta_t dt + \frac{1}{\sqrt{a_t}} dB_{s,t}, \tag{7}$$

where $B_{s,t}$ is an Brownian motion independent of $B_{P,t}$, $B_{y,t}$, $B_{\beta,t}$, and $B_{V,t}$.

The dynamics of the news signal (7) are interpreted as follows. Assume the investor acquires n_t signals of equal precision s_t^j, $j = 1, \ldots, n_t$ at time t:

$$ds_t^j = \beta_t dt + \sigma_s dB_t^j, \quad j = 1, \ldots, n_t \tag{8}$$

where B_t^j is independent of $B_{P,t}$, $B_{y,t}$, $B_{\beta,t}$, and $B_{V,t}$ for all j and $B_t^j \perp B_t^i$, $\forall j \neq i$. Aggregating yields the following dynamics of the aggregate signal s_t acquired by the investor

$$ds_t \equiv \frac{1}{n_t} \sum_{j=1}^{n_t} ds_t^j = \beta_t dt + \frac{\sigma_s}{\sqrt{n_t}} dB_{s,t}, \tag{9}$$

where $B_{s,t}$ is independent from $B_{P,t}$, $B_{y,t}$, $B_{\beta,t}$, $B_{V,t}$. Setting $\frac{\sigma_s}{\sqrt{n_t}} \equiv \frac{1}{\sqrt{a_t}}$ in Equation (9) leads to Equation (7). That is, the investor controls the accuracy a_t of the aggregate signal by choosing the number of individual signals n_t she acquires. When the investor is attentive to news, the number of individual signals she acquires is large and the aggregate signal is accurate. When the investor is inattentive to news, the number of individual signals she acquires is small and the aggregate signal is inaccurate. Given this, we call a_t the investor's *attention to news*.

Denote by $\mathcal{F}_t$ the information set of the investor at time t. This information set includes: realized returns defined in (2), changes in the predictive variable defined in (4), changes in the instantaneous variance of stock returns defined in (6), and changes in the signal defined in (7). This last source of information is the focus of our paper. The key feature is that the investor is able to change her information acquisition policy by controlling the magnitude of the noise in the signal (7) *at any point in time*. This results in a control problem with an endogenous information structure.

Let us denote by $\widehat{\beta}_t \equiv \mathbb{E}[\beta_t | \mathcal{F}_t]$ the estimated predictive coefficient and its posterior variance

by $\nu_t \equiv \mathbb{E}[(\beta_t - \widehat{\beta}_t)^2 | \mathcal{F}_t]$. Thus,

$$\beta_t \sim N(\widehat{\beta}_t, \nu_t), \tag{10}$$

where $N(m, v)$ denotes the Normal distribution with mean m and variance v. Henceforth, we refer to the estimated predictive coefficient $\widehat{\beta}_t$ as the *filter* and to the posterior variance ν_t as the *uncertainty about beta*.

The dynamics of the state variables observed by the investor are obtained from standard filtering results (Liptser and Shiryaev, 2001) and are provided in Proposition 1 below.

Proposition 1. *The dynamics of the observed state variables satisfy*

$$\frac{dP_t}{P_t} = \left(\bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}) \right) dt + \begin{bmatrix} \sqrt{V_t} & 0 & 0 & 0 \end{bmatrix} d\widehat{B}_t^{\perp} \tag{11}$$

$$ds_t = \widehat{\beta}_t dt + \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{a_t}} \end{bmatrix} d\widehat{B}_t^{\perp} \tag{12}$$

$$dy_t = \lambda_y(\bar{y} - y_t)dt + \begin{bmatrix} 0 & \sigma_y & 0 & 0 \end{bmatrix} d\widehat{B}_t^{\perp} \tag{13}$$

$$dV_t = \lambda_V(\bar{V} - V_t)dt + \begin{bmatrix} 0 & 0 & \sigma_V\sqrt{V_t} & 0 \end{bmatrix} d\widehat{B}_t^{\perp}. \tag{14}$$

The dynamics of the filter and uncertainty about beta are

$$d\widehat{\beta}_t = \lambda_\beta(\bar{\beta} - \widehat{\beta}_t)dt + \begin{bmatrix} \frac{\nu_t(y_t - \bar{y})}{\sqrt{V_t}} & 0 & 0 & \nu_t\sqrt{a_t} \end{bmatrix} d\widehat{B}_t^{\perp} \tag{15}$$

$$\frac{d\nu_t}{dt} = -\left(\frac{(y_t - \bar{y})^2}{V_t} + a_t \right)\nu_t^2 - 2\lambda_\beta\nu_t + \sigma_\beta^2, \tag{16}$$

where $\widehat{B}_t^{\perp} \equiv \begin{bmatrix} \widehat{B}_{1,t}^{\perp} & \widehat{B}_{2,t}^{\perp} & \widehat{B}_{3,t}^{\perp} & \widehat{B}_{4,t}^{\perp} \end{bmatrix}^{\top}$ is a 4-dimensional vector of independent Brownian motions under the investor's observation filtration.

Proof. See Appendix A.

Equations (15) and (16) describe the investor's updating rule regarding the expectation and variance of the predictive coefficient. The instantaneous change in the filter is driven by four sources of information: realized returns, changes in the predictive variable, changes in volatility, and changes in the news signal. As Equation (15) shows, the investor assigns stochastic weights to

these four sources of information.[9] As we will describe below, the size of these weights depend on the relative informativeness of each source of information.

Equation (16) describes the change in uncertainty about beta when the investor controls her attention to news. Uncertainty about beta is locally deterministic and decreases faster when the investor's attention is high. The decline in uncertainty is weaker when the predictive coefficient is more persistent (i.e. low λ_β) or when the volatility of realized returns V_t is high. Finally, the last term in Equation (16) shows that the larger the volatility of the predictive coefficient, the stronger the increase in uncertainty about beta over time.

The informativeness of the signal depends on investor's attention, which impacts learning in two ways. First, it has a direct impact on the instantaneous volatility of the filter in Equation (15). Second, it drives the drift of uncertainty about beta in Equation (16). We analyze each of these two effects separately. To facilitate our discussion, we refer to $d\widehat{B}_{1,t}^{\perp}$ as *return shocks* and to $d\widehat{B}_{4,t}^{\perp}$ as *news shocks*.

2.1.1 The impact of attention on the filter

The magnitude of the impact of return shocks and news shocks on the filter depend on the uncertainty ν_t, on the difference between the predictive variable and its long-term mean $y_t - \bar{y}$, and on investor's attention a_t. The following example provides insight on how the investor updates her beliefs using each piece of information.

To fix ideas, let y_t be the earnings-to-price ratio, and suppose that at time t the investor's estimate of β_t is positive. That is, $\widehat{\beta}_t > 0$ and therefore expected returns increase with the earnings-to-price ratio. When the earnings-to-price ratio is above its long-term mean ($y_t > \bar{y}$), an unexpectedly high return ($d\widehat{B}_{1,t}^{\perp} > 0$) means that the current estimate of β_t is too low. Therefore, the investor adjusts $\widehat{\beta}_t$ upwards, thereby making the relationship between the earnings-to-price ratio and expected returns stronger than previously thought. The opposite arises when the earnings-to-price ratio is below its long-term mean ($y_t < \bar{y}$); An unexpectedly high return means that the current estimate of β_t is too high, which implies that the investor adjusts $\widehat{\beta}_t$ downwards. Hence, the first coefficient in the diffusion of the filter has the same sign as $y_t - \bar{y}$ (see also Xia (2001) for a similar

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[9]Because the Brownian motions $B_{P,t}$, $B_{y,t}$, $B_{\beta,t}$, $B_{V,t}$, and $B_{s,t}$ are uncorrelated, shocks to the predictive variable y_t and to return variance V_t do not impact the investor's estimate of β_t. Hence, the second and third components of the diffusion of $\widehat{\beta}_t$ are both equal to zero.

interpretation).

An additional component drives the filter through active learning from news shocks. When attention is high, the signal becomes more informative and therefore the investor increases the weight assigned to news shocks, as can be seen from the last coefficient in the diffusion of the filter.

Overall, the instantaneous variance of the filter is an increasing function of attention:

$$\text{Var}[d\widehat{\beta}_t] = \nu_t^2 \left(\frac{(y_t - \bar{y})^2}{V_t} + a_t \right). \tag{17}$$

As attention converges to infinity, the instantaneous variance of the filter converges to its upper bound σ_β^2. This upper bound represents the variance of the filter when the predictive coefficient β_t is perfectly observable.

2.1.2 The impact of attention on uncertainty about beta

The predictive variable y_t is a key driver of the dynamics of uncertainty about beta. Intuitively, if y_t is close to its long-term mean, learning from realized returns becomes ineffective in estimating β_t because the signal-to-noise ratio is very low. Therefore, the reduction in uncertainty about beta is weak when $y_t - \bar{y} \approx 0$. In contrast, when y_t is far away from its long-term mean, realized returns offer valuable information on the predictive coefficient β_t and uncertainty about beta decreases faster.

Furthermore, Equation (16) shows that uncertainty about beta decreases faster when attention is high. It is worth noting that uncertainty about beta does not converge to a "steady state" in this model because three stochastic variables, namely the predictive variable y_t, the volatility of asset returns V_t, and investor's attention a_t, drive its dynamics.

2.2 Properties of the estimated expected returns

The following lemma describes the properties of the *estimated expected return* of the stock, defined as:

$$\widehat{\mu}_t \equiv \bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}). \tag{18}$$

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Lemma 1. *The dynamics of the estimated expected return follow*

$$d\widehat{\mu}_t = (\lambda_y + \lambda_\beta)\left(\bar{\mu} + \frac{\bar{\beta}\lambda_\beta(y_t - \bar{y})}{\lambda_y + \lambda_\beta} - \widehat{\mu}_t\right)dt + \left[\frac{(y_t - \bar{y})^2\nu_t}{\sqrt{V_t}} \quad \sigma_y\widehat{\beta}_t \quad 0 \quad \nu_t\sqrt{a_t}(y_t - \bar{y})\right]d\widehat{B}_t^\perp. \quad (19)$$

The mean square error of this estimate (i.e. the uncertainty about expected returns, which we denote hereafter by η_t) satisfies

$$\eta_t \equiv \mathbb{E}\left[(\mu_t - \widehat{\mu}_t)^2\big|\mathcal{F}_t\right] = (y_t - \bar{y})^2\nu_t. \quad (20)$$

The instantaneous variance of the estimated expected return is

$$\text{Var}[d\widehat{\mu}_t] = \nu_t^2(y_t - \bar{y})^2\left(a_t + \frac{(y_t - \bar{y})^2}{V_t}\right) + \sigma_y^2\widehat{\beta}_t^2. \quad (21)$$

$\text{Var}[d\widehat{\mu}_t]$ *is a monotone increasing function of attention. Its maximum depends on both $\widehat{\beta}_t$ and y_t and is given by*

$$\lim_{a_t \to \infty} \text{Var}[d\widehat{\mu}_t] = \sigma_\beta^2(y_t - \bar{y})^2 + \sigma_y^2\widehat{\beta}_t^2. \quad (22)$$

Proof. See Appendix B.

Equations (19)-(22) result from an application of Itô's lemma to the expected return defined in (18). Together they characterize the dynamics of expected returns under the information set of the agent. Equation (19) shows that expected returns mean-revert at speed $\lambda_y + \lambda_\beta$ to a stochastic level that depends on the predictive variable. If the long-term mean $\bar{\beta}$ is assumed to be zero (which means that there is no predictability on average) then the stochastic level simplifies to the constant $\bar{\mu}$.

When the filter $\widehat{\beta}_t$ is large, expected returns react to changes in the predictive variable y_t (the second component of the diffusion). Furthermore, if investor's attention is high, expected returns react to news shocks, but *only* when $y_t \neq \bar{y}$ (the fourth component of the diffusion). This concurs with the relation between returns and the predictive variable described in Equation (3), whereby more news on the predictive coefficient β_t—no matter how accurate—is not going to change investor's view about expected returns if $y_t - \bar{y} = 0$.

High uncertainty about beta ν_t magnifies the sensitivity of expected returns to both return shocks and news shocks. As shown in Equation (21), when $y_t - \bar{y} \neq 0$, an increase in uncertainty about beta increases the instantaneous variance of expected returns. Equation (21) further shows that higher attention (or more accurate news) increases the variance of expected returns. When attention is high, expected returns are more sensitive to news shocks and therefore become more volatile. In the limit, when attention tends to infinity, the variance of expected returns reaches its maximum, as stated in Equation (22).[10]

To summarize, attention to news drives two important factors, the variance of expected returns and the drift of uncertainty about beta. As elaborated above, more attention increases the sensitivity of expected returns to news shocks and therefore augments their variance. At the same time, Proposition 1, Equation (16), shows that more attention yields lower future uncertainty about beta.

2.3 Optimal attention, portfolio choice, and consumption

Turning now to the investor's optimization problem, we consider $\widehat{\mu}_t$ as a state variable instead of $\widehat{\beta}_t$, with the aim to better interpret and characterize our results. The investor's problem is to choose consumption c_t, attention to news a_t, and the risky investment share w_t so as to maximize her expected lifetime utility of consumption conditional on her information set at time t, $\mathcal{F}_t$. That is, the investor's maximization problem is written

$$J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t) \equiv \max_{c,a,w} \mathbb{E}\left[\int_t^\infty e^{-\delta(s-t)} u(c_s) ds \middle| \mathcal{F}_t\right], \tag{23}$$

subject to the budget constraint

$$dW_t = \left[r^f W_t + w_t W_t \left(\widehat{\mu}_t - r^f\right) - c_t - \overline{K}_t\right] dt + w_t W_t \left[\sqrt{V_t} \quad 0 \quad 0 \quad 0\right] d\widehat{B}_t^\perp. \tag{24}$$

We assume that the total information cost, $\overline{K}_t$, is linear in wealth:

$$\overline{K}_t = K(a_t) W_t. \tag{25}$$

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[10]We derive the latter equation by applying Itô's lemma to the expected return in Equation (3) and by assuming that the predictive coefficient β_t is perfectly observable. See Appendix B.

This assumption reflects the fact that, similar to the price of financial securities, the price of information tends to increase as time passes (i.e., the price of information features an exponential time trend). Furthermore, the above cost function preserves the homogeneity of the value function in wealth and therefore implies that attention, the consumption-to-wealth ratio, and the information cost-to-wealth ratio are independent of wealth at the optimum. Finally, the per-unit of wealth cost function, $K(a_t)$, is assumed to be increasing and convex in attention, which implies that perfect information ($a_t \to \infty$) cannot be attained, and thus the investor is never able to observe the true level of expected returns.

Since attention does not depend on wealth at the optimum, it is a function of the expected return $\widehat{\mu}_t$, the predictive variable y_t, stock-return variance V_t, and uncertainty about beta ν_t. These state variables do not feature any trend and thus neither does attention. The stationary dynamics of optimal attention implied by the cost function in (25) are therefore consistent with Sims (2003), who argues that investors have limited information-processing capacity, i.e. investors' attention is bounded. If instead attention were increasing with wealth, then it would tend to increase as time passes and would therefore not be bounded.

Two additional considerations about the information cost function (25) are worth mentioning. First, this specification implies that it is more costly to acquire information when one gets wealthier. In reality, it could be the case that the cost is actually decreasing in wealth, because a wealthier investor may acquire more information and thus may get a volume discount on the cost. In Appendix F, we discuss the implications of a more general information cost specification, which allows $\overline{K}_t$ to be *ex ante* either increasing with wealth, independent of wealth, or decreasing with wealth. We show that such a specification does not change the qualitative results of our paper.

Second, the specification (25) implies that if two investors (A and B) start with different levels of wealth W_0^A and W_0^B, then they will incur different costs of information at any given time t (because the wealth at time t will be different across investors). To avoid such situations in which there is a discount or an overcharge across investors, the cost function can be normalized by dividing by the initial level of wealth. This modification clearly does not alter our results, but ensures that poor and rich investors pay the same price for the same piece of information at any given time t.

To summarize, given the total information cost (25), if the investor chooses to be inattentive to news—and therefore to learn using only the information provided by the price P_t, the predictive

variable y_t, and the variance of stock returns V_t—then her entire wealth is invested in the financial market. In contrast, if the investor decides to pay attention to news ($a_t > 0$), then a positive fraction of her wealth flows to the information market in order to pay for research expenditures. Attention, therefore, can be perceived as a non-financial security in the investor's portfolio.

Proposition 2. *The optimal consumption c_t^*, risky investment share w_t^*, and attention to news a_t^* are given by*

$$c_t^* = u_c^{-1}(J_W) \tag{26}$$

$$w_t^* = \frac{\widehat{\mu}_t - r^f}{V_t} \frac{-J_W}{J_{WW}W_t} + \frac{\nu_t(y_t - \bar{y})^2}{V_t} \frac{-J_{W\mu}}{J_{WW}W_t} \tag{27}$$

$$a_t^* = \Phi\left(\frac{1}{2J_W W_t}\left(\nu_t^2(y_t - \bar{y})^2 J_{\mu\mu} - 2\nu_t^2 J_\nu\right)\right), \tag{28}$$

where $\Phi(\cdot)$ is a positive and increasing function defined as the inverse of the derivative of the cost function, $\Phi(\cdot) \equiv K'(\cdot)^{-1}$.

Proof. See Appendix C.

Equation (26) is the standard optimal consumption derived by Merton (1971). The optimal risky investment share, expressed in Equation (27), comprises a myopic and a hedging portfolio (Merton, 1971). The hedging term represents the effect of parameter learning and significantly impacts the asset allocation decision (Brennan, 1998; Xia, 2001). It is positive if $J_{W\mu} > 0$ and negative if $J_{W\mu} < 0$.[11] It vanishes when all the state variables are observable (i.e. when $\nu_t = 0$) because, by assumption, none of these variables are correlated to returns.

Our object of focus is the optimal attention a_t^*, expressed in Equation (28). Since the function $\Phi(\cdot)$ is positive and increasing, we can directly analyze the term in brackets. Two factors drive the optimal level of attention: the *state risk aversion factor* $J_{\mu\mu}$, which measures the extent to which the investor (dis)likes variations in expected returns and the *uncertainty factor* J_ν, which measures the extent to which the investor (dis)likes uncertainty. Recall from Section 2.2 that attention drives both the variance of expected returns and the drift of uncertainty about beta. The state risk aversion factor $J_{\mu\mu}$ is multiplied by $\nu_t^2(y_t - \bar{y})^2$, which is the marginal effect of attention on the

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[11]The value function J will be concave in wealth whenever u is concave in current consumption (Benveniste and Scheinkman, 1979), and thus $-J_{W\mu}/(J_{WW}W_t)$ has the same sign as $J_{W\mu}$.

variance of expected returns (see Equation (21)). The uncertainty factor J_ν is multiplied by $-\nu_t^2$, which is the marginal effect of attention on the drift of ν_t (see Equation (16)).

Because our setup features mean-reverting expected returns, the value function is convex in $\widehat{\mu}_t$, which yields $J_{\mu\mu} > 0$ (Kim and Omberg, 1996). That is, the investor prefers expected return volatility because it creates the possibility of trends that she can exploit. The higher the expected return volatility, the higher the convexity of the value function. This in turn implies that the investor pays higher attention to accurately estimate the predictive coefficient and efficiently exploit future trends.

Two opposing forces determine the sign of the uncertainty factor J_ν. First, the investor dislikes uncertainty; second, the investor likes uncertainty because it leads to higher expected return volatility and a higher likelihood of trends. Depending on which of these two forces dominates, the uncertainty factor J_ν is positive or negative. If it is negative, the investor acquires more information to reduce uncertainty. If it is positive, the investor acquires less information in order to keep expected return volatility high and take advantage of trends.

Since $J_{\mu\mu} > 0$, the first term in Equation (28) implies that the investor optimally chooses to be more attentive to news when (i) uncertainty about beta is high and (ii) the predictive variable moves away from its long-term mean. These results are independent on the investor's utility function. The effects of uncertainty about beta and the predictive variable on attention reinforce each other, yielding high attention in environments characterized by high uncertainty about beta and by a large difference between the predictive variable and its long-term mean. However, the sign of the second term in Equation (28) is ambiguous because J_ν can either be positive or negative.

2.4 CRRA utility and quadratic attention cost

To illustrate the effects of optimal learning about predictability, we assume that the investor has a CRRA utility function with risk aversion parameter γ. In addition, the per-unit of wealth information cost function is specified in quadratic form:

$$K(a_t) = ka_t^2, \tag{29}$$

where $k > 0$ is the information cost parameter. In this case, the inverse of the derivative of the function $K(.)$ satisfies

$$\Phi(x) = \frac{x}{2k}.\tag{30}$$

Under these assumptions, the value function J can be written as

$$J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \phi(\widehat{\mu}_t, y_t, V_t, \nu_t).\tag{31}$$

Computing the partial derivatives of J as a function of the partial derivatives of ϕ and substituting them into the first-order conditions yields the optimal consumption, risky investment share, and attention to news provided in Proposition 3 below.

Proposition 3. *With CRRA utility and quadratic attention cost, the optimal consumption c_t^*, risky investment share w_t^*, and attention to news a_t^* are given by*

$$c_t^* = \phi^{-1/\gamma} W_t \tag{32}$$

$$w_t^* = \frac{\widehat{\mu}_t - r^f}{\gamma V_t} + \frac{\nu_t (y_t - \bar{y})^2}{\gamma V_t} \frac{\phi_\mu}{\phi} \tag{33}$$

$$a_t^* = \nu_t^2 \left(\frac{\phi_\nu}{\phi} \frac{1}{2k(\gamma-1)} + \frac{-\phi_{\mu\mu}}{\phi} \frac{(y_t - \bar{y})^2}{4k(\gamma-1)} \right).\tag{34}$$

The optimal consumption defined in Equation (32) is well-known (Merton, 1971) and does not require any further analysis. The optimal risky investment share defined in Equation (33) is analyzed by Xia (2001) in a setup with constant stock-return volatility.[12] We discuss the dependence of the risky investment share and its hedging components on the state variables in Section 3.3.

The optimal attention defined in Equation (34) becomes a strictly increasing quadratic function of uncertainty about beta. We discuss the dependence of the investor's attention on the state variables in Section 3.2.

Although an indirect dependence of attention on the variance of stock returns, V_t, arises through

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[12]In Xia (2001), the risky investment share features additional terms resulting from the correlations between realized returns and the predictive variable y_t, and between returns and the predictive coefficient β_t. These correlations are set to zero in the present case. Note also that, although a similar decomposition appears in Xia (2001), the attention level affects the shape of the value function, causing differences between the portfolio holdings obtained in the two models.

the function ϕ, the variance of stock return has no direct impact on the investor's attention. In Section 3.2 we show that the indirect impact of the return variance on attention is quantitatively weak, as opposed to the impact of uncertainty about beta and of the predictive variable.

3   Numerical results

In this section, we investigate the determinants of optimal attention and risky investment share. We first calibrate the parameters of the model. Then, we show that attention is a U-shaped function of the predictive variable, a decreasing function of the stock-return variance, and an increasing function of both the squared predictive coefficient and the uncertainty about beta. The risky investment share increases with the Sharpe ratio of the stock and features a negative hedging demand that is hump-shaped in the predictive variable. Furthermore, the size of the hedging demand increases with uncertainty about beta and decreases with stock-return volatility.

3.1   Calibration

We calibrate the parameters of the model as follows. First, we consider the S&P 500 earnings-to-price ratio to be the predictive variable y_t of S&P 500 returns r_t.[13] This dataset is available at monthly frequency from 02/1871 to 03/2018, and is obtained from Robert Shiller's website. The dynamics of the predictive variable provided in Proposition 1 imply that the long-term mean of y_t is $\bar{y}$, the long-term variance of y_t is $\sigma_y^2/(2\lambda_y)$, and $\text{cov}(y_{t+\Delta}, y_t)/\text{var}(y_t) = e^{-\lambda_y \Delta}$, where $\Delta = 1/12 = 1$ month. Solving these three moment conditions yields the parameters $\bar{y}$, σ_y, and λ_y. Second, the monthly variance of S&P 500 returns, $V_t \Delta$, is estimated with a GARCH(1,1) model (Bollerslev, 1986). The dynamics of the stock return variance provided in Proposition 1 imply that the long-term mean of V_t is $\bar{V}$, the long-term variance of V_t is $\sigma_V^2 \bar{V}/2\lambda_V$, and $\text{cov}(V_{t+\Delta}, V_t)/\text{var}(V_t) = e^{-\lambda_V \Delta}$. Solving these three moment conditions yields the parameters $\bar{V}$, σ_V, and λ_V. Third, we jointly estimate the parameters $\bar{\mu}$, $\bar{\beta}$, σ_β, and λ_β and the time-varying predictive coefficients $\widehat{\beta}_t$ by Kalman filter maximum likelihood (Hamilton, 1994). Additional details about the estimation are provided in Appendix D.

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[13]Several other predictive variables have been identified. They include past market returns, the dividend yield, nominal interest rates, and expected inflation among others. See Goyal and Welch (2008) for a comprehensive survey.

Parameter	Symbol	Value
Mean-reversion speed of stock-return variance	λ_V	0.9215*
		(1.858)
Long-term mean of stock-return variance	$\bar{V}$	0.0181***
		(10.617)
Volatility of stock-return variance	σ_V	0.2262***
		(2.578)
Mean-reversion speed of earnings-to-price ratio	λ_y	0.0958
		(1.368)
Long-term mean of earnings-to-price ratio	$\bar{y}$	0.0738***
		(32.964)
Volatility of earnings-to-price ratio	σ_y	0.0118***
		(2.686)
Long-term expected return	$\bar{\mu}$	0.0975***
		(14.181)
Mean-reversion speed of β	λ_β	0.0503***
		(3.230)
Long-term mean of β	$\bar{\beta}$	0.5890**
		(1.865)
Volatility of β	σ_β	0.1682***
		(4.567)

Table 1: Parameter Values.
We report t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***.

The estimated parameter values and t-statistics are provided in Table 1. In addition, we set the risk-free rate to its historical mean $r^f = 4.57\%$. The coefficient of relative risk aversion is $\gamma = 6$, which implies a mean risky investment share of about 0.5 (see Section 3.3). The cost parameter is $k = 10^{-3}$ and the subjective discount rate is $\delta = 0.02$.

Using the parameter values reported in Table 1, we numerically solve the partial differential equation resulting from specification (31) by applying the Chebyshev collocation method (Judd, 1998). Further details are provided in Appendix E, where we specify boundary conditions, we discuss the transversality condition (Merton, 1998), and we measure the accuracy of the solution algorithm.

3.2 Optimal attention

Four state variables impact the optimal attention: the predictive variable y_t, the uncertainty about beta ν_t, the stock-return variance V_t, and the predictive coefficient $\widehat{\beta}_t$. Since the predictive variable is the main determinant of expected returns, we choose to plot the optimal attention against the
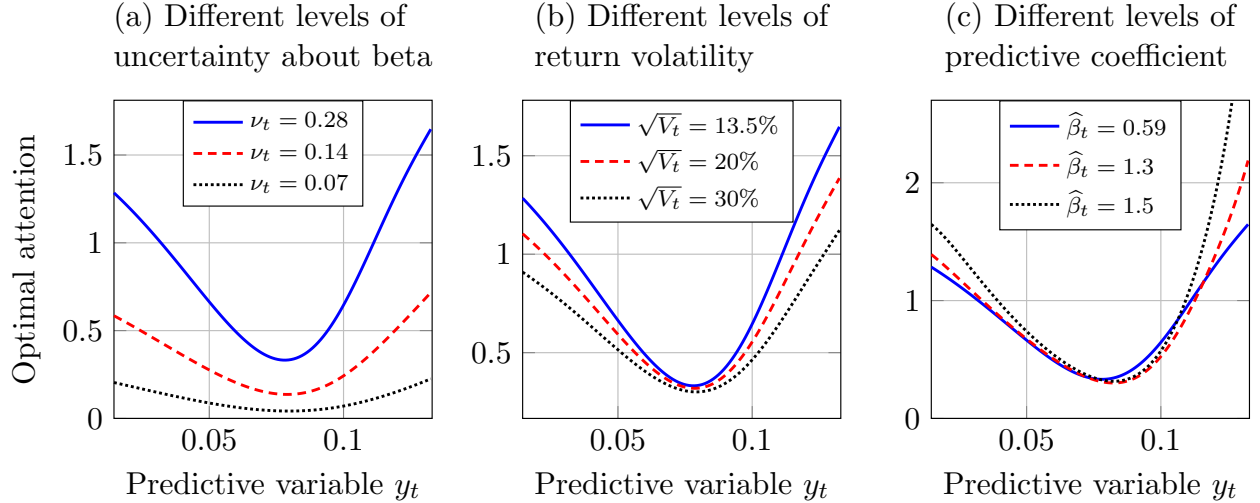
Figure 1: Impact of the predictive variable on attention.
The three panels depict the relation between attention and the predictive variable. We plot three curves corresponding to three different levels of uncertainty about beta in panel (a), return volatility in panel (b), and the predictive coefficient in panel (c). The range of values on the horizontal axis corresponds to the 95% confidence interval for the process y_t. If not stated otherwise, the state variables are $\widehat{\beta}_t = \bar{\beta} = 0.59$, $\sqrt{V_t} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_t = \bar{\nu} = 0.28$. Parameter values are provided in Table 1.

predictive variable for different values of the other state variables.

Figure 1 reports plots for three different levels of uncertainty about beta in panel (a), return volatility in panel (b), and the estimated predictive coefficient in panel (c). The benchmark solid blue line is obtained by setting ν_t, V_t, and $\widehat{\beta}_t$ to their long-term levels $\bar{\nu}$, $\sqrt{\bar{V}}$, and $\bar{\beta}$.[14]

All three panels of Figure 1 confirm the intuition provided in Equation (34) that attention is a U-shaped function of the demeaned predictive variable $y_t - \bar{y}$. This is because when the predictive variable is close to its long-term mean, the investor knows that the expected return is equal to $\bar{\mu}$ and therefore has weak incentives to pay for information. In contrast, when the predictive variable is far from its long-term mean, there is a trend in expected returns that the investor can efficiently exploit, but only if the predictive coefficient is accurately estimated. The investor's optimal reaction to this situation is to pay attention, efficiently exploit the trend, and profit from it.

Panel (a) of Figure 1 shows that uncertainty about beta drives investor attention in two ways. First, it increases the curvature of the U-shaped relation between attention and the predictive

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[14]The long-term uncertainty about beta $\bar{\nu}$ is determined by solving $d\bar{\nu}/\bar{\nu} = 0$ conditional on setting $y_t = \bar{y}$, $V_t = \bar{V}$, and $a_t = 0$ in the dynamics of ν_t, which yields $\bar{\nu} = \sigma_\beta^2/2\lambda_\beta$. This is an upper bound of uncertainty about beta because all the sources of information (i.e. the stock return, the predictive variable, the return variance, and the signal) are uninformative when $y_t = \bar{y}$ and $a_t = 0$.

variable through the presence of ν_t^2 in Equation (34). Second, it increases the level of the U-shaped relation between attention and the predictive variable. That is, higher uncertainty about beta leads to higher attention. This is because when uncertainty about beta is close to zero, the investor observes the predictive coefficient and feels no incentive to pay attention and learn about it. The greater the uncertainty about beta, the less accurate the investor's estimate of the expected return, and therefore the larger her incentive to pay attention to news.

Panel (b) of Figure 1 shows that an increase in stock-return volatility decreases the curvature of the U-shaped relation between attention and the predictive variable. This is because an increase in the return volatility generates less informative returns, which decreases the volatility of expected returns (see Equation (21)). Because trends in expected returns are more difficult to detect in this case, the convexity of the value function in expected returns is less pronounced. Since the convexity of the value function determines the curvature of the U-shaped relation between attention and the predictive variable (see Equation (34)), more return volatility leads to a weaker curvature and therefore to lower attention.[15]

Panel (c) of Figure 1 shows that an increase in the (squared) predictive coefficient increases the curvature of the U-shaped relation between attention and the predictive variable. This is because large positive or negative values of the predictive coefficient imply a high expected return volatility (see Equation (21)). Higher expected return volatility implies more opportunities to exploit trends in expected returns, and thus yields a highly convex value function. This translates into a more pronounced U-shaped relation between attention and the predictive variable (see Equation (34)).

We now turn to the relation between attention and risk aversion, which is depicted in Figure 2. Consistent with Equation (34), an increase in risk aversion scales down the level of attention. This decreasing relation between attention and risk aversion comes from the fact that an increase in risk aversion decreases the investor's risky investment share (see the myopic component in Equation (33)). The smaller the risky investment share, the lower the investor's incentive to pay attention to news about expected stock returns.

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[15]This prediction is similar to the "ostrich effect," documented by Galai and Sade (2006), Karlsson, Loewenstein, and Seppi (2009) and Sicherman et al. (2015). The "ostrich effect" states that investors prefer to pay attention to their portfolios following positive news and "put their heads in the sand" when they expect to see bad news. Andries and Haddad (2014) provide an alternative explanation. In their model, investors are "disappointment averse" and thus are less attentive in riskier environments. In our case, riskier returns offer less marginal benefit for being attentive because the expected return becomes less responsive to information. Note that the "ostrich effect" commonly refers to *attention to wealth* (Abel, Eberly, and Panageas, 2007, 2013), whereas here we model investors' *attention to news*.
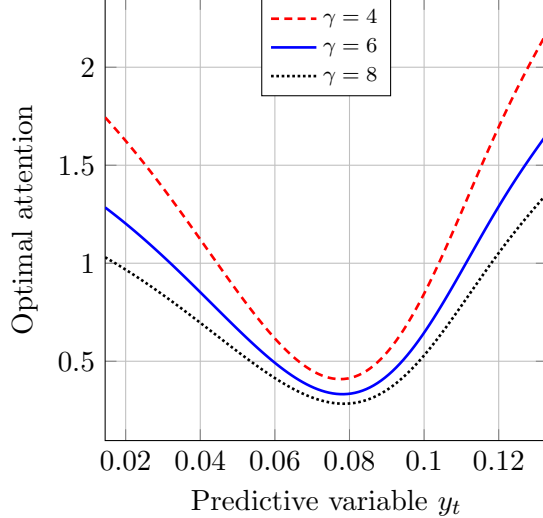
Figure 2: Impact of risk aversion on attention.

The figure depicts the relation between attention and the predictive variable for three different levels of risk aversion. The range of values on the horizontal axis corresponds to the 95% confidence interval for the process y_t. State variables are $\widehat{\beta}_t = \bar{\beta} = 0.59$, $\sqrt{V_t} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_t = \bar{\nu} = 0.28$. Parameter values are provided in Table 1.

We also investigate the robustness of our results by performing a sensitivity analysis of the optimal attention with respect to changes in the dynamics of the predictive coefficient and of the predictive variable. More precisely, we analyze how attention responds to a change in the mean-reversion speed and the volatility of the predictive coefficient and of the predictive variable. Panels (a) and (b) of Figure 3 show that attention decreases when the mean-reversion speed of the predictive coefficient increases and when the volatility of the predictive coefficient decreases. The reason is that, in the limit where either the mean-reversion speed λ_β approaches infinity or the volatility σ_β approaches zero, the predictive coefficient becomes constant (equal to $\bar{\beta}$). In this case, the investor observes the predictive coefficient and therefore does not need to pay any attention to learn about it. Panels (c) and (d) of Figure 3 show that attention decreases as the mean-reversion speed of the predictive variable increases and as the volatility of the predictive variable decreases. By the same logic, when either the mean-reversion speed λ_y converges to infinity or the volatility σ_y converges to zero, the predictive variable becomes constant (equal to $\bar{y}$). The return dynamics in (11) show that, in this situation, the expected stock return becomes equal to $\bar{\mu}$. That is, the investor observes the expected stock return and therefore does not need to pay any attention to learn about it.
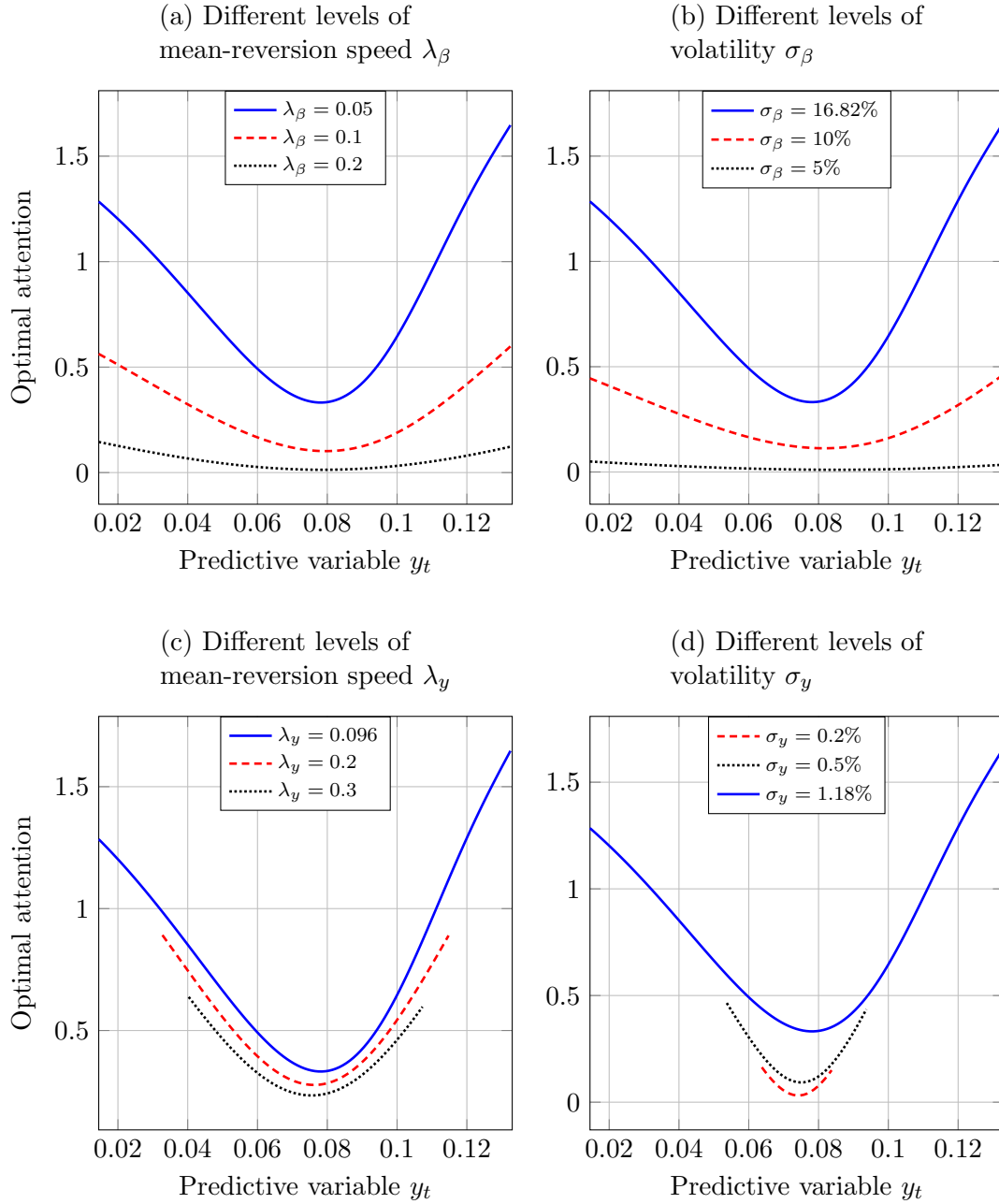
21

Figure 3: Impact of the dynamics of the predictive coefficient and predictive variable on attention.

Panels (a) and (b) depict the relation between attention and the predictive variable for three different levels of the mean-reversion speed of the predictive coefficient and the volatility of the predictive coefficient, respectively. Panels (c) and (d) depict the relation between attention and the predictive variable for three different levels of the mean-reversion speed of the predictive variable and the volatility of the predictive variable, respectively. The range of values on the horizontal axis corresponds to the 95% confidence interval for the process y_t. State variables are $\widehat{\beta}_t = \bar{\beta} = 0.59$, $\sqrt{V_t} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_t = \bar{\nu} = 0.28$. If not stated otherwise, parameter values are provided in Table 1.
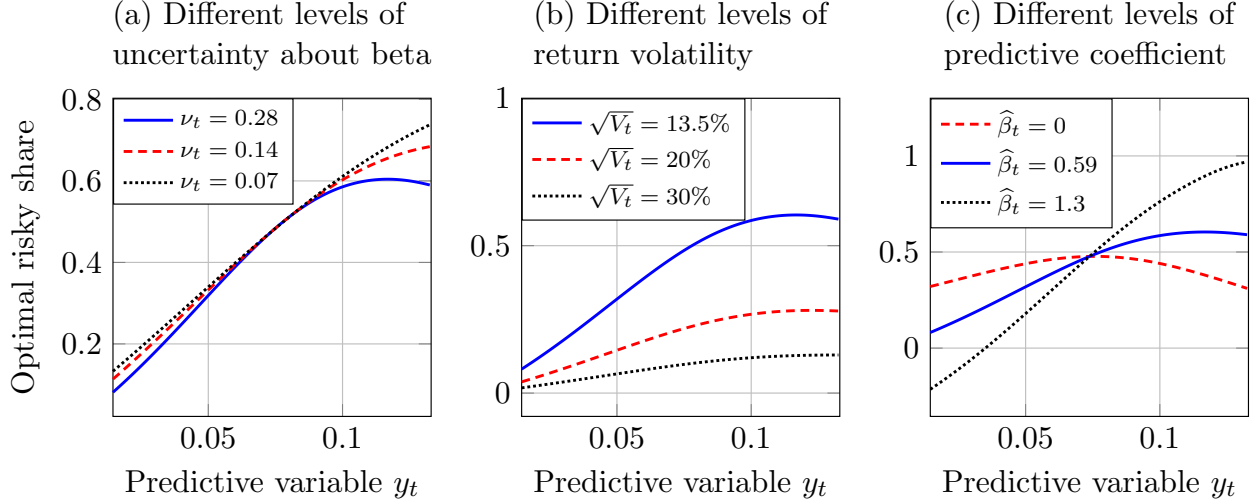
Figure 4: Impact of the predictive variable on the risky investment share.
The relation between the risky investment share and the predictive variable. We plot three curves corresponding to three different levels of uncertainty about beta in panel (a), return volatility in panel (b), and the predictive coefficient in panel (c). The range of values on the horizontal axis corresponds to the 95% confidence interval for the process y_t. If not stated otherwise, state variables are $\widehat{\beta}_t = \bar{\beta} = 0.59$, $\sqrt{V_t} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_t = \bar{\nu} = 0.28$. Parameter values are provided in Table 1.

3.3 Optimal risky investment share

Figure 4 plots the optimal risky investment share against the predictive variable for different values of uncertainty about beta in panel (a), stock-return volatility in panel (b), and the estimated predictive coefficient in panel (c). The benchmark solid blue line is obtained by setting ν_t, V_t, and $\widehat{\beta}_t$ to their long-term levels $\bar{\nu}$, $\sqrt{\bar{V}}$, and $\bar{\beta}$.

To understand the impact of the state variables on the risky investment share, we first have to investigate the properties of the hedging demand

$$H_t^* \equiv w_t^* - \frac{\widehat{\mu}_t - r^f}{\gamma V_t} = \frac{\nu_t (y_t - \bar{y})^2}{\gamma V_t} \frac{\phi_\mu}{\phi} < 0. \tag{35}$$

The hedging demand reflects the investor's willingness to hedge against variations in expected returns. Since returns and expected returns co-move positively (see Equations (11) and (19)), low returns imply low expected returns which triggers a negative hedging demand.[16] Furthermore, the larger the expected return's loading on return shocks, the more negative the hedging demand is. According to Equation (19), this loading increases with both the predictive variable's deviation

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[16]See also Brennan (1998): when $\gamma > 1$, $J_\mu > 0$ and $J < 0$ imply that $\phi_\mu < 0$ and $\phi > 0$.

from its mean and uncertainty about beta. That is, the hedging demand becomes more negative when both the predictive variable's deviation from its mean and uncertainty about beta increase.

The aforementioned properties of the hedging demand imply that the risky investment share is a hump-shaped function of the predictive variable and a decreasing function of uncertainty about beta (see panel (a) of Figure 4). As suggested by Equation (35), the hump is particularly pronounced for high uncertainty about beta and low return volatility.

Panel (b) of Figure 4 shows that the risky investment share decreases with stock-return volatility because the impact of the myopic component in Equation (33) dominates that of the hedging demand. Furthermore, the risky investment share increases with expected returns, which depend on the product of the predictive coefficient and the demeaned predictive variable. That is, the risky investment share increases with the predictive coefficient when the predictive variable is larger than its mean and decreases with it when the predictive variable is smaller than its mean (panel (c)).

3.4 Cost of ignoring news

To quantify the benefits associated with paying attention to news, we compute the wealth certainty equivalent of the optimal strategy relative to that obtained when ignoring news (Xia, 2001; Das and Uppal, 2004; Liu, Peleg, and Subrahmanyam, 2010). That is, the cost of ignoring news is defined as the additional fraction of wealth required by an investor who ignores the news signal—equivalently, an investor who faces an infinite information cost—to reach the expected utility of an investor who optimally pays attention to news.

Table 2 reports the cost of ignoring news for different values of risk aversion γ and information cost parameter k. The cost of ignoring news decreases with both risk aversion and the information cost parameter. A high risk aversion implies a small share invested in the stock and therefore a weak incentive to pay attention to learn about the stock's expected return (see Figure 2). As a result, the optimal attention allocation strategy does not significantly differ from that of ignoring the news signal, i.e., the cost of ignoring information is small. Furthermore, the lower the cost parameter, the higher the optimal attention paid to news, and therefore the larger the cost of ignoring news. The cost of ignoring news can be significant, reaching as much as 3.2% of wealth when risk aversion and the information cost parameter are equal to 2 and 10^{-4}, respectively.

Cost parameter $k \to$			
Risk aversion $\gamma \downarrow$	10^{-4}	10^{-3}	10^{-2}
2	320.51	182.58	62.81
4	168.83	86.79	24.00
6	117.86	55.51	13.57

Table 2: Cost of ignoring news (in bps).
The cost of ignoring news represents the additional fraction of wealth required by an investor who ignores the news signal to reach the expected utility of an investor who optimally pays attention to news. State variables are $\widehat{\beta}_t = \bar{\beta} = 0.59$, $y_t = \bar{y} = 7.4\%$ $\sqrt{V_t} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_t = \bar{\nu} = 0.28$. Parameter values are provided in Table 1.

4   Empirical analysis

We divide our empirical analysis into two parts. In the first part, we show that the model-implied measures of attention, uncertainty, and risky investment share are related to existing empirical measures. In the second part, we test the specific predictions of our model. The data confirm that attention is a U-shaped function of the predictive variable, an increasing function of both the squared predictive coefficient and uncertainty about beta, and a decreasing function of stock-return variance.

4.1   Model-implied quantities and their empirical counterparts

In Section 3.1, we estimated the parameters of the model and the predictive coefficients using S&P 500 monthly returns $\log\left(P_{t+1/12}/P_t\right)$, the S&P 500 earnings-to-price ratio, and the annualized variance of S&P 500 returns V_t. We illustrate in Figure 5 the dynamics of the earnings-to-price ratio y_t, the predictive coefficient $\widehat{\beta}_t$, and the annualized return variance V_t. The gray shaded areas represent NBER recessions.

In order to obtain *model-implied* time series of attention and uncertainty about beta, we first discretize the dynamics of uncertainty about beta in (16) as follows:

$$\nu_{t+\Delta} = \nu_t + \left[-\left(\frac{(y_t - \bar{y})^2}{V_t} + a_t^*(\widehat{\mu}_t, y_t, V_t, \nu_t)\right)\nu_t^2 - 2\lambda_\beta \nu_t + \sigma_\beta^2\right]\Delta, \tag{36}$$

where $\Delta = 1/12 = 1$ month, $\widehat{\mu}_t \equiv \bar{\mu} + \widehat{\beta}_t(y_t - \bar{y})$, the initial value is $\nu_0 = \bar{\nu} = \sigma_\beta^2/(2\lambda_\beta)$, and the optimal attention $a_t^*(\widehat{\mu}_t, y_t, V_t, \nu_t)$ defined in (34) depends on the function $\phi(.)$ defined in (31). The parameter values provided in Table 1 and the solution method described in Appendix E yield the
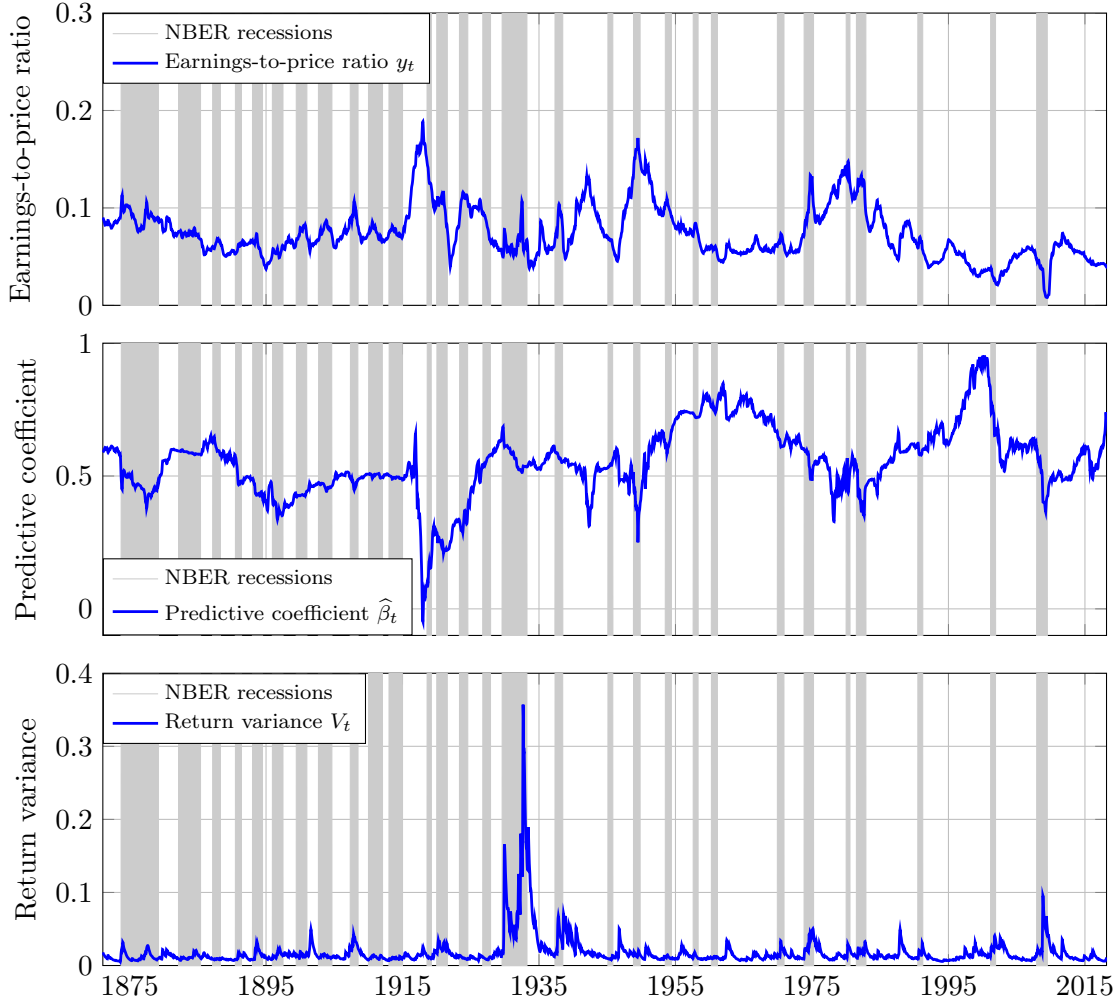
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Figure 5: Time series of the state variables.
The data is at the monthly frequency from 02/1871 to 03/2018.

function $\phi(.)$. Therefore, using the initial value $\nu_0 = \bar{\nu}$ and sequentially substituting the values of the state variables depicted in Figure 5 in Equations (33), (34), and (36) provides model-implied time series for the risky investment share, attention, and uncertainty about beta. These model-implied time series are depicted in Figure 6.

The top panel depicts the model-implied attention, which varies roughly between 0.15 and 1.7. These bounds can be interpreted in terms of the informativeness of the signal acquired by the investor $(1/\sqrt{a_t})$. More precisely, they imply that the informativeness of the signal varies between 0.75 and 2.6. When compared with the drift of the signal (which equals β_t and has a mean of 0.55), the *signal-to-noise* ratio, that is, the drift-to-volatility ratio of the signal, varies between 0.21 and 0.75. This signal-to-noise ratio is similar to that of the S&P 500 return. Indeed, the mean S&P
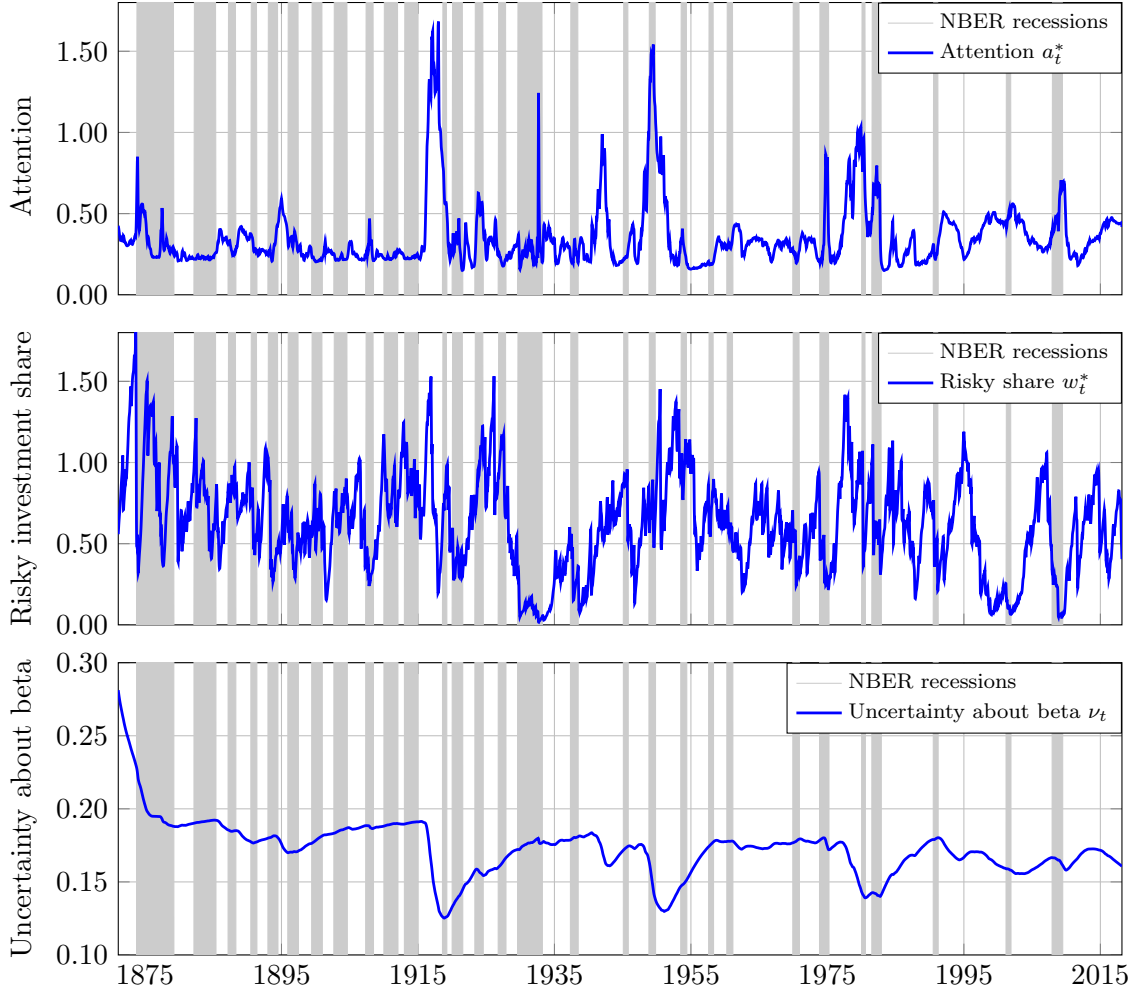
Figure 6: Time series of the model-implied attention, risky investment share, and uncertainty about beta.
The data is at the monthly frequency from 02/1871 to 03/2018.

500 return is about 8.5%, its minimum volatility is about 10%, and its maximum volatility is about 55%, which yields an average signal-to-noise ratio varying between 0.15 and 0.85. The economic interpretation of this is that the informativeness of the news signal on the predictive coefficient β is similar to the informativeness of S&P 500 returns on expected S&P 500 returns.

To provide evidence that these model-implied measures of attention, uncertainty, and risky investment share are realistic, we compare them to their corresponding empirical proxies. Chordia and Swaminathan (2000), Lo and Wang (2000), Gervais, Kaniel, and Mingelgrin (2001), Barber and Odean (2008), and Hou, Peng, and Xiong (2009) argue that trading volume is a good proxy for attention because investors must trade stocks they pay attention to. Most important, Fisher et al. (2016) and Gargano and Rossi (2017) provide empirical evidence that attention to news is highly

correlated to trading volume. Thus, our empirical measure of attention, a_t^E, is the S&P 500 monthly trading volume detrended by its past 36-month moving average (Tetlock, 2007; Fisher et al., 2016).[17] This empirical measure is available at the monthly frequency from 12/1952 to 03/2018.

For the empirical measure of uncertainty about expected returns, η_t^E, we use the 1-month-ahead financial-uncertainty index constructed by Jurado, Ludvigson, and Ng (2015). Note that, by definition, the financial-uncertainty index measures financial risk, whereas in our model uncertainty about beta ν_t measures the inaccuracy of the estimate of the predictive coefficient $\widehat{\beta}_t$ rather than financial uncertainty. In our framework, a measure of financial uncertainty comparable with the index of Jurado et al. (2015) is the *uncertainty about expected returns* η_t, which we define in Equation (20) of Lemma 1. The empirical measure of uncertainty about expected returns η_t^E is available at the monthly frequency from 07/1960 to 12/2017.

Finally, the empirical proxy for the risky investment share, w_t^E, is constructed as follows. We record the stock holdings (in dollar amount) of the institutions belonging to the Thomson-Reuters institutional database (13F). This provides a proxy for the amount of money institutions invest in risky assets. The "Institutional Money Funds" index provides a proxy for the amount of money institutions invest in riskless assets. Both time series are obtained at the quarterly frequency from Q1/1985 to Q2/2016. Since the risky investment share is defined as the risky amount divided by the sum of the risky amount and the riskless amount, we scale the time series of risky amounts and the time series of riskless amounts by their respective initial values. This implies that the empirical risky investment share time series starts at 0.5 and that its average is approximately 0.5, consistent with the average risky investment share obtained in the model (see middle panel of Figure 6).

We compare the model-implied measures of attention, risky share, and uncertainty with their empirical counterparts in Figure 7. We use the empirical sample lengths in each case, and we standardize all time series for ease of interpretation. Overall, comparing the solid and dashed lines shows that the model-implied and empirical measures are fairly well aligned.

To confirm this observation from a statistical point of view, we regress the empirical measures of attention, uncertainty about expected returns, and risky investment share on their model-implied counterparts. Table 3 shows the results. There is a positive relation between the empirical and

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[17]Note that the results presented thereafter are qualitatively the same if the moving window is of size 12 months, 24 months, 48 months, or 60 months.
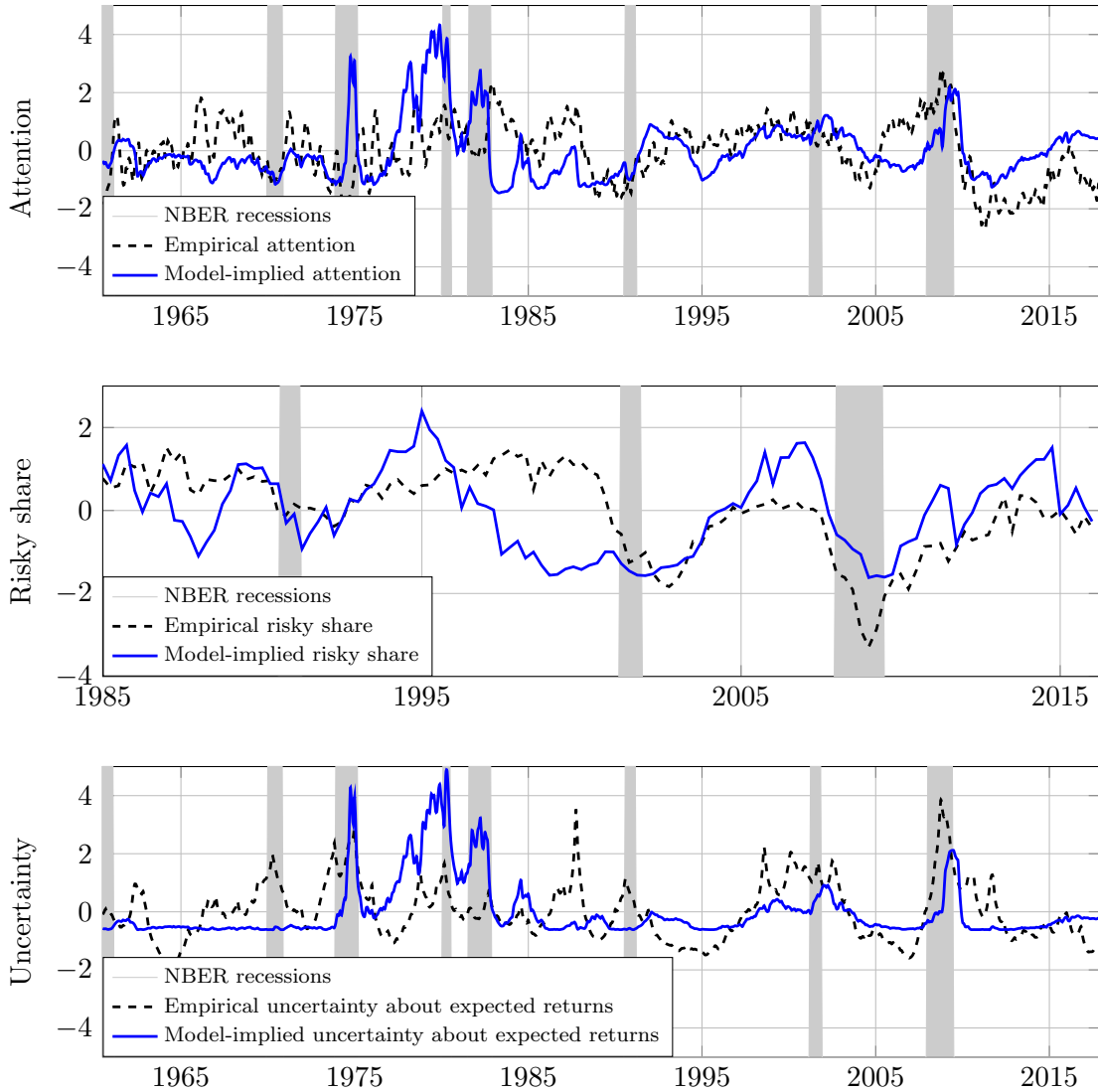
Figure 7: Standardized attention, risky investment share, and uncertainty about expected returns.

All time series are standardized in order to have them on the same scale. The risky share is at the quarterly frequency from Q1/1985 to Q2/2016. Attention and uncertainty about expected returns are at the monthly frequency from 07/1960 to 12/2017.

model-implied measures of attention, even after controlling for the autocorrelation in the empirical measure. This suggests that our model likely provides a realistic description of the dynamic attention allocation problem faced by investors. Furthermore, there is a positive relation between the empirical and model-implied measures of uncertainty about expected returns, although the relation loses its significance when controlling for the autocorrelation in the empirical measure. Finally, the empirical and model-implied measures of the risky investment share are positively related, even after controlling for the autocorrelation in the empirical measure. Overall, Table 3 confirms the

	Attention a_t^E	Attention a_t^E	Expected return uncertainty η_t^E	Expected return uncertainty η_t^E	Risky share w_t^E	Risky share w_t^E
a_t^*	0.277***	0.126***				
	(4.47)	(3.43)				
η_t^*			293.839***	−4.105		
			(6.66)	(−0.39)		
w_t^*					0.108***	0.018**
					(2.95)	(2.18)
Intercept	Yes	Yes	Yes	Yes	Yes	Yes
$a_{t-1/12}^E$		Yes				
$\eta_{t-1/12}^E$				Yes		
$w_{t-1/12}^E$						Yes
R^2	0.042	0.478	0.102	0.964	0.143	0.878
Observations	690	689	690	689	125	124

Table 3: Empirical vs. model-implied attention, uncertainty about expected returns, and risky investment share.
The variables a_t^E, η_t^E, and w_t^E denote the empirical measures of attention, uncertainty about expected returns, and risky investment share, respectively. The variables a_t^*, η_t^*, and w_t^* denote the model-implied counterparts. We report Newey and West (1987) t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***. Attention and uncertainty about expected returns are at the monthly frequency from 07/1960 to 12/2017. The risky investment share is at the quarterly frequency from Q1/1985 to Q2/2016.

message of Figure 7 and shows that our model provides a realistic description of the dynamics of attention, uncertainty, and institutional investors' risky investments.

We finally test whether the model-implied and empirical measures of attention predict the VIX index. As shown in Equation (21) of Lemma 1, attention is a proxy for the variance of expected returns, which should imply a positive relation between current attention and future VIX. Table 4 shows that this is indeed the case. Both the model-implied and empirical measures of attention positively predict the VIX, even after controlling for the autocorrelation in the latter. That is, attention can be interpreted as a measure of future market risk. This is also consistent with recent findings by Fisher et al. (2016), who document that an increase in media attention positively relates to an increase in implied volatility.

4.2 Testing the predictions of the model

We can now test three main predictions of our theoretical model.

Prediction 1. *Equation* (34) *and the results depicted in Figure 1 show that attention can be*

	VIX_t	VIX_t	VIX_t	VIX_t
$a^*_{t-1/12}$	0.250**	0.048**		
	(2.26)	(2.03)		
$a^E_{t-1/12}$			0.122**	0.045***
			(2.14)	(2.70)
Intercept	Yes	Yes	Yes	Yes
$\text{VIX}_{t-1/12}$		Yes		Yes
R^2	0.108	0.722	0.117	0.733
Observations	335	335	335	335

Table 4: The predictive power of attention on the VIX.

The variables a^E_t and a^*_t denote the empirical and model-implied measures of attention, respectively. We report Newey and West (1987) t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***. The data is at the monthly frequency from 01/1990 to 12/2017.

approximated as follows:

$$a_t \approx C_0 + C_1 \nu_t^2 + \tilde{C}_2(\widehat{\beta}_t, V_t)\nu_t^2(y_t - \bar{y})^2 \approx C_0 + C_1\nu_t^2 + C_2\frac{\widehat{\beta}_t^2}{V_t}\nu_t^2(y_t - \bar{y})^2, \tag{37}$$

where the sign of C_1 is ambiguous and $C_2 > 0$.

The first testable prediction is that the curvature of the U-shaped relation between attention and the demeaned predictive variable decreases with the stock-return variance, increases with the squared predictive coefficient, and increases with the squared uncertainty about beta. In addition, the squared uncertainty about beta determines the level of the U-shaped relation.

Prediction 2. *Equation* (33) *and the results depicted in Figure 4 show that the risky investment share can be approximated as follows:*

$$w_t \approx K_0 + K_1\frac{\bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}) - r^f}{V_t} + K_2\frac{\nu_t(y_t - \bar{y})^2}{V_t}, \tag{38}$$

where $K_1 > 0$ and $K_2 < 0$.

The second testable prediction is that the risky investment share increases with the Sharpe ratio of the stock and features a negative hedging demand that is hump-shaped in the demeaned predictive variable. In addition, the curvature of the hedging demand decreases with the stock-return variance and increases with uncertainty about beta.

The coefficients C_0, C_1, C_2, K_0, K_1, and K_2 are estimated by ordinary least squares. Model-implied data is used to quantify the size and significance of these coefficients in our model, while the

	Attention a_t^*	Attention a_t^*	Attention a_t^E	Attention a_t^E
$\frac{\widehat{\beta}_t^2}{V_t}\nu_t^2(y_t - \bar{y})^2$	236.665***	221.491***	78.457***	36.143*
	(16.61)	(14.09)	(4.43)	(1.93)
ν_t^2		−5.737**		−15.998***
		(−2.23)		(−5.32)
Intercept	Yes	Yes	Yes	Yes
R^2	0.547	0.559	0.033	0.084
Observations	690	690	690	690

Table 5: Attention vs. predictive variable, predictive coefficient, return variance, and uncertainty about beta.

The variables a_t^E and a_t^* are the empirical and model-implied measures of attention, respectively. We report Newey and West (1987) t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***. The data is at the monthly frequency from 07/1960 to 12/2017.

empirical proxies are used to test the predictions of the model. The left-hand side of Equations (37) and (38) define the dependent variables, while the right-hand side define the independent variables.

The first and second columns of Table 5 quantify Prediction 1 using model-implied attention data. The coefficients C_1 and C_2 are negative and positive, respectively, and both are highly statistically significant. Comparing the R^2 obtained in these two columns shows that adjusting the level of the quadratic relation between attention and the predictive variable with the squared uncertainty about beta has only a marginal impact on the goodness of fit. The third and fourth columns test Prediction 1 using the empirical measure of attention. Consistent with the prediction of the model, the curvature of the quadratic relation between attention and the predictive variable decreases with stock-return variance and increases with both the squared predictive coefficient and the squared uncertainty about beta. In addition, the fourth column shows that the empirical attention decreases with the squared uncertainty about beta. The coefficient is statistically significant, as its model-implied counterpart obtained in the second column.

The first and second columns of Table 6 quantify Prediction 2 using model-implied data, and confirm that the coefficients K_1 and K_2 are positive and negative, respectively. That is, the risky investment share increases with the Sharpe ratio of the stock and features a negative hedging demand. The second column also shows that the hedging demand is statistically significant in our model. The third and fourth columns of Table 6 test Prediction 2 using the empirical proxy for the risky investment share. Consistent with the model's prediction, the empirical measure of the risky investment share increases with the Sharpe ratio of the stock. While the model predicts a negative

	Risky share w_t^*	Risky share w_t^*	Risky share w_t^E	Risky share w_t^E
$\frac{\bar{\mu}+\widehat{\beta}_t(y_t-\bar{y})-r^f}{V_t}$	0.148***	0.143***	0.017***	0.019***
	(102.60)	(90.20)	(3.31)	(3.75)
$\frac{\nu_t(y_t-\bar{y})^2}{V_t}$		-7.296***		1.892
		(-6.82)		(0.90)
Intercept	Yes	Yes	Yes	Yes
R^2	0.982	0.989	0.169	0.176
Observations	376	376	125	125

Table 6: Risky investment share vs. Sharpe ratio and hedging demand.
The variables w_t^E and w_t^* are the empirical and model-implied measures of risky investment share, respectively. We report Newey and West (1987) t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***. The empirical and model-implied risky investment shares are respectively at the quarterly and monthly frequencies from 01/1985 to 04/2016.

and significant hedging demand, the empirical hedging demand turns out to be insignificant.

It is worth noting that the empirical loadings on $\frac{\widehat{\beta}_t^2}{V_t}\nu_t^2(y_t-\bar{y})^2$ in Table 5 and on $\frac{\bar{\mu}+\widehat{\beta}_t(y_t-\bar{y})-r^f}{V_t}$ in Table 6 are smaller than their model-implied counterparts. This suggests that the coefficient of relative risk aversion of actual investors might potentially be larger than that assumed in the model ($\gamma = 6$ in the model). Indeed, as shown in Proposition 3 and Figure 2, a larger risk aversion implies a weaker curvature for the relation between attention and the predictive variable, as well as a weaker relation between the risky share and the Sharpe ratio.

Finally, the model yields a prediction on the conditional relation between attention and the risky investment share. That is, Figures 1 and 4 suggest that large positive values of the product $\widehat{\beta}_t(y_t-\bar{y})$ imply both high attention and a large *positive* risky investment share. Intuitively, when expected returns are high, investors are highly attentive and increase their risky investment share. In contrast, large negative values of the product $\widehat{\beta}_t(y_t-\bar{y})$ imply high attention and a large *negative* risky investment share (according again to Figures 1 and 4). Together these statements imply the testable Prediction 3 described below.

Prediction 3. *The model predicts a positive relation between the risky investment share and attention when $\widehat{\beta}_t(y_t-\bar{y})$ is highly positive, and a negative relation when $\widehat{\beta}_t(y_t-\bar{y})$ is highly negative. Therefore, the relation between the risky investment share and attention satisfies*

$$w_t = M_0 + M_1\widehat{\beta}_t(y_t-\bar{y})a_t, \tag{39}$$

where $M_1 > 0$.

	Risky share w_t^*	Risky share w_t^E
$\widehat{\beta}_t(y_t - \bar{y})a_t^*$	42.139***	4.335*
	(19.42)	(1.79)
Intercept	Yes	Yes
R^2	0.520	0.070
Observations	376	125

Table 7: Relation between risky investment share and attention.
The variables w_t^E and w_t^* are the empirical and model-implied measures of risky investment share, respectively. The variable a_t^* is the model-implied measures of attention. We report Newey and West (1987) t-statistics in brackets and label statistical significance at the 10%, 5%, and 1% levels with */**/***. The empirical and model-implied risky investment shares are respectively at the quarterly and monthly frequencies from 01/1985 to 04/2016.

The first column of Table 7 quantifies Prediction 3 and shows that the coefficient M_1 is, as expected, highly significant when using the model-implied risky investment share. The second column of Table 7 tests this prediction using the empirical proxy for the risky investment share, and finds empirical support for $M_1 > 0$. Overall, the tests performed in Tables 5, 6, and 7 provide empirical support for dynamics of attention and investment predicted by our theoretical model.

5   Conclusion

This paper aims to understand investors' dynamic attention behavior observed in financial markets. In most of the existing literature, investors acquire information passively in the sense that they do not control the quality of information they collect. In contrast, we consider an investor who can, at each point in time, improve the accuracy of acquired information at a cost.

Our analysis provides several interesting insights. The optimal level of attention paid to news is a U-shaped function of the stock return predictor, an increasing function of uncertainty about predictability, a decreasing function of stock-return volatility, and an increasing function of the squared predictive coefficient. In addition, the risky investment share increases with the Sharpe ratio of the stock and features a negative hedging demand. The hedging demand is a hump-shaped function of the return predictor, an increasing function of stock-return volatility, and a decreasing function of uncertainty about predictability. We show that the data lends support to these theoretical predictions, and conclude that empirically documented fluctuations in investors' attention result from a rational information gathering behavior.

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Appendix

A   Proof of Proposition 1

We apply the following standard theorem.

Theorem A1. *(Theorem 12.7, page 36 of Liptser and Shiryaev, 2001) Consider an unobservable process u_t and an observable process ξ_t with dynamics given by*

$$du_t = [a_0(t, \xi_t) + a_1(t, \xi_t)u_t]\, dt + b_1(t, \xi_t)dZ_t^u + b_2(t, \xi_t)dZ_t^\xi \tag{A.1}$$

$$d\xi_t = [A_0(t, \xi_t) + A_1(t, \xi_t)u_t]\, dt + B_1(t, \xi_t)dZ_t^u + B_2(t, \xi_t)dZ_t^\xi. \tag{A.2}$$

All the parameters can be functions of time and of the observable process. The posterior mean $\widehat{u}_t$ (the filter) and the posterior variance ν_t (the Bayesian uncertainty) evolve according to (we drop the dependence of coefficients on t and ξ_t for notational convenience):

$$d\widehat{u}_t = (a_0 + a_1 \widehat{u}_t)dt + [(b \circ B) + \nu_t A_1^\top](B \circ B)^{-1}[d\xi_t - (A_0 + A_1 \widehat{u}_t)dt] \tag{A.3}$$

$$\frac{d\nu_t}{dt} = a_1 \nu_t + \nu_t a_1^\top + (b \circ b) - [(b \circ B) + \nu_t A_1^\top](B \circ B)^{-1}[(b \circ B) + \nu_t A_1^\top]^\top, \tag{A.4}$$

where

$$b \circ b = b_1 b_1^\top + b_2 b_2^\top \tag{A.5}$$

$$B \circ B = B_1 B_1^\top + B_2 B_2^\top \tag{A.6}$$

$$b \circ B = b_1 B_1^\top + b_2 B_2^\top. \tag{A.7}$$

In our setup, the observable variables are changes in realized returns (2), changes in the predictive variable (4), changes in the instantaneous variance of stock returns (6), and changes in the signal (7). The unobservable variable is β_t. Write the dynamics of the observable variables

$$\begin{bmatrix} d \ln P_t \\ dy_t \\ dV_t \\ ds_t \end{bmatrix} = \left(\begin{bmatrix} \bar{\mu} - V_t/2 \\ \lambda_y(\bar{y} - y_t) \\ \lambda_V(\bar{V} - V_t) \\ 0 \end{bmatrix} + \begin{bmatrix} y_t - \bar{y} \\ 0 \\ 0 \\ 1 \end{bmatrix} \beta_t \right) dt + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dB_{\beta,t} + \begin{bmatrix} \sqrt{V_t} & 0 & 0 & 0 \\ 0 & \sigma_y & 0 & 0 \\ 0 & 0 & \sigma_V \sqrt{V_t} & 0 \\ 0 & 0 & 0 & 1/\sqrt{a_t} \end{bmatrix} \begin{bmatrix} dB_{P,t} \\ dB_{y,t} \\ dB_{V,t} \\ dB_{s,t} \end{bmatrix} \tag{A.8}$$

and of the unobservable variable β_t:

$$d\beta_t = \left[\lambda_\beta \bar{\beta} + (-\lambda_\beta)\beta_t\right] dt + \sigma_\beta dB_{\beta,t} + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dB_{P,t} \\ dB_{y,t} \\ dB_{V,t} \\ dB_{s,t} \end{bmatrix} \tag{A.9}$$

We therefore obtain

$$b \circ b = \sigma_\beta^2, \quad B \circ B = \begin{bmatrix} V_t & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 \\ 0 & 0 & \sigma_V^2 V_t & 0 \\ 0 & 0 & 0 & 1/a_t \end{bmatrix}, \quad b \circ B = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.10}$$

and

$$[(b \circ B) + \nu_t A_1^\top](B \circ B)^{-1} = \begin{bmatrix} \frac{\nu_t(y_t - \bar{y})}{V_t} & 0 & 0 & \nu_t a_t \end{bmatrix} \tag{A.11}$$

Replacing this into (A.3)-(A.4) yields the dynamics of the filter

$$d\widehat{\beta}_t = \lambda_\beta(\bar{\beta} - \widehat{\beta}_t)dt + \begin{bmatrix} \frac{\nu_t(y_t - \bar{y})}{\sqrt{V_t}} & 0 & 0 & \nu_t\sqrt{a_t} \end{bmatrix} d\widehat{B}_t^\perp \tag{A.12}$$

and the dynamics of the posterior variance

$$\frac{d\nu_t}{dt} = -\left(\frac{(y_t - \bar{y})^2}{V_t} + a_t \right)\nu_t^2 - 2\lambda_\beta\nu_t + \sigma_\beta^2, \tag{A.13}$$

which are Equations (15)-(16) in Proposition 1. In Equation (A.12), $\widehat{B}^\perp$ is a 4-dimensional standard Brownian motion defined as

$$d\widehat{B}_t^\perp \equiv \begin{bmatrix} 1/\sqrt{V_t} & 0 & 0 & 0 \\ 0 & 1/\sigma_y & 0 & 0 \\ 0 & 0 & 1/(\sigma_V\sqrt{V_t}) & 0 \\ 0 & 0 & 0 & \sqrt{a_t} \end{bmatrix} \left\{ \begin{bmatrix} d\ln P_t \\ dy_t \\ dV_t \\ ds_t \end{bmatrix} - \left(\begin{bmatrix} \bar{\mu} - V_t/2 \\ \lambda_y(\bar{y} - y_t) \\ \lambda_V(\bar{V} - V_t) \\ 0 \end{bmatrix} + \begin{bmatrix} y_t - \bar{y} \\ 0 \\ 0 \\ 1 \end{bmatrix} \widehat{\beta}_t \right) dt \right\}. \tag{A.14}$$

This completes the proof of Proposition 1. $\qquad\square$

B   Proof of Lemma 1

Equation (19) results from an application of Itô's Lemma on $\widehat{\mu}$, defined in (18):

$$d\widehat{\mu}_t = (y_t - \bar{y})d\widehat{\beta}_t + \widehat{\beta}_t dy_t \tag{B.15}$$

$$= \left[(y_t - \bar{y})\lambda_\beta(\bar{\beta} - \widehat{\beta}_t) + \widehat{\beta}_t\lambda_y(\bar{y} - y_t) \right]dt + \begin{bmatrix} \frac{(y_t - \bar{y})^2\nu_t}{\sqrt{V_t}} & \sigma_y\widehat{\beta}_t & 0 & \nu_t\sqrt{a_t}(y_t - \bar{y}) \end{bmatrix} d\widehat{B}_t^\perp \tag{B.16}$$

Replacing $\widehat{\beta}$ from (18) in (B.16) yields

$$d\widehat{\mu}_t = (\lambda_y + \lambda_\beta)\left(\bar{\mu} + \frac{\bar{\beta}\lambda_\beta(y_t - \bar{y})}{\lambda_y + \lambda_\beta} - \widehat{\mu}_t \right)dt + \begin{bmatrix} \frac{(y_t - \bar{y})^2\nu_t}{\sqrt{V_t}} & \sigma_y\widehat{\beta}_t & 0 & \nu_t\sqrt{a_t}(y_t - \bar{y}) \end{bmatrix} d\widehat{B}_t^\perp \tag{B.17}$$

Furthermore, writing

$$\mu_t = \bar{\mu} + \beta_t(y_t - \bar{y}) \tag{B.18}$$

$$\widehat{\mu}_t = \bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}) \tag{B.19}$$

as in (3) and (18), yields

$$(\mu_t - \widehat{\mu}_t) = (y_t - \bar{y})(\beta_t - \widehat{\beta}_t) \tag{B.20}$$

and thus

$$\mathbb{E}\left[(\mu_t - \widehat{\mu}_t)^2 | \mathcal{F}_t \right] = (y_t - \bar{y})^2 \mathbb{E}[(\beta_t - \widehat{\beta}_t)^2 | \mathcal{F}_t], \tag{B.21}$$

which yields Equation (20) in the text.

The instantaneous variance of the estimated expected return is obtained directly from (B.17):

$$\text{Var}[d\widehat{\mu}_t] = \begin{bmatrix} \frac{(y_t-\bar{y})^2\nu_t}{\sqrt{V_t}} & \sigma_y\widehat{\beta}_t & 0 & \nu_t\sqrt{a_t}(y_t-\bar{y}) \end{bmatrix} \begin{bmatrix} \frac{(y_t-\bar{y})^2\nu_t}{\sqrt{V_t}} & \sigma_y\widehat{\beta}_t & 0 & \nu_t\sqrt{a_t}(y_t-\bar{y}) \end{bmatrix}^\top \tag{B.22}$$

$$= \nu_t^2(y_t-\bar{y})^2\left(a_t + \frac{(y_t-\bar{y})^2}{V_t}\right) + \sigma_y^2\widehat{\beta}_t^2, \tag{B.23}$$

which is Equation (21) in the text. This Equation shows that $\text{Var}[d\widehat{\mu}_t]$ is a monotone increasing function of attention. When $a_t \to \infty$, the agent perfectly observes β_t and thus $\widehat{\beta}_t = \beta_t$. Then, an application of Itô's Lemma on (B.18) yields Equation (22) in the text.

$$\lim_{a_t\to\infty}\text{Var}[d\widehat{\mu}_t] = \sigma_\beta^2(y_t-\bar{y})^2 + \sigma_y^2\widehat{\beta}_t^2. \tag{B.24}$$

This completes the proof of Lemma 1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

C   Proof of Proposition 2

The dynamics of the vector of state variables $Z_t \equiv [\widehat{\mu}_t, y_t, V_t, \nu_t]^\top$ satisfy

$$dZ_t = m_t dt + \Sigma_{1,t}d\widehat{B}^\perp_{1,t} + \Sigma_{2,t}\begin{bmatrix} d\widehat{B}^\perp_{2,t} \\ d\widehat{B}^\perp_{3,t} \\ d\widehat{B}^\perp_{4,t} \end{bmatrix}, \tag{C.25}$$

where the 4-dimensional vector of drift m, the 4-dimensional vector of diffusion Σ_1, and the 4×3 matrix of diffusion Σ_2 satisfy

$$m = \begin{bmatrix} (\lambda_y+\lambda_\beta)\left(\bar{\mu}+\frac{\bar{\beta}\lambda_\beta(y-\bar{y})}{\lambda_y+\lambda_\beta}-\widehat{\mu}\right) \\ \lambda_y(\bar{y}-y) \\ \lambda_V(\bar{V}-V) \\ -\left(\frac{(y-\bar{y})^2}{V}+a\right)\nu^2 - 2\lambda_\beta\nu + \sigma_\beta^2 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} \frac{(y-\bar{y})^2\nu}{\sqrt{V}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \sigma_y\frac{\widehat{\mu}-\bar{\mu}}{y-\bar{y}} & 0 & \nu\sqrt{a}(y-\bar{y}) \\ \sigma_y & 0 & 0 \\ 0 & \sigma_V\sqrt{V} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.26}$$

The HJB equation satisfies

$$0 = -\delta J + \max_{c,a,w}\left(u(c_t) + \mathcal{D}^{W,Z}J\right), \tag{C.27}$$

where $\mathcal{D}^{W,Z}$ is the infinitesimal generator such that[18]

$$\mathcal{D}^{W,Z}J = J'_Z m + J_W\left[(r^f - K(a_t))W_t + w_t W_t\left(\widehat{\mu}_t - r^f\right) - c_t\right] \tag{C.28}$$

$$+ \frac{1}{2}J_{WW}W_t^2 w_t^2 V_t + \frac{1}{2}\text{tr}\left[(\Sigma_1\Sigma_1' + \Sigma_2\Sigma_2')J_{ZZ}\right] \tag{C.29}$$

$$+ W_t w_t\sqrt{V_t}\Sigma_1' J_{WZ}. \tag{C.30}$$

Differentiating (23) partially with respect to the control variables yields the first order conditions:

$$0 = u_c - J_W \tag{C.31}$$

$$0 = J_W W_t(\widehat{\mu}_t - r^f) + J_{WW}W_t^2 V_t w_t + J_{W\mu}W_t\nu_t(y_t-\bar{y})^2 \tag{C.32}$$

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[18]For notational convenience, we drop hats when state variables appear as indices.

$$0 = -K'(a_t)J_W W_t - \nu_t^2 J_\nu + \frac{1}{2}\nu_t^2 (y_t - \bar{y})^2 J_{\mu\mu}. \tag{C.33}$$

Solving the first order conditions yields Proposition 2. $\qquad \square$

D Parameter estimation

The mean of y_t, the variance of y_t, and $\text{cov}(y_{t+\Delta}, y_t)/\text{var}(y_t)$, where $\Delta = 1/12 = 1$ month and y_t is the S&P 500 earnings-to-price ratio, are estimated by OLS. Since the long-term mean of y_t is $\bar{y}$, the long-term variance of y_t is $\sigma_y^2/(2\lambda_y)$, and $\text{cov}(y_{t+\Delta}, y_t)/\text{var}(y_t) = e^{-\lambda_y \Delta}$, solving these three moment conditions yields the parameters $\bar{y}$, σ_y, and λ_y.

The mean of V_t, the variance of V_t, and $\text{cov}(V_{t+\Delta}, V_t)/\text{var}(V_t)$, where $\Delta = 1/12 = 1$ month and V_t is the S&P 500 annualized return variance obtained using a GARCH(1,1) model, are estimated by OLS. Since the long-term mean of V_t is $\bar{V}$, the long-term variance of V_t is $\sigma_V^2 \bar{V}/2\lambda_V$, and $\text{cov}(V_{t+\Delta}, V_t)/\text{var}(V_t) = e^{-\lambda_V \Delta}$, solving these three moment conditions yields the parameters $\bar{V}$, σ_V, and λ_V.

Since the news signal s_t acquired by the investor is unobservable to us, we make the simplifying assumption that the only source of information is the history of S&P 500 returns when jointly estimating the parameters $\bar{\mu}$, $\bar{\beta}$, σ_β, and λ_β and the time-varying predictive coefficients $\widehat{\beta}_t$. Discretizing the dynamics provided in Proposition 1 yields

$$\log\left(P_{t+\Delta}/P_t\right) = \left(\bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}) - \frac{1}{2}V_t\right)\Delta + \sqrt{V_t}\sqrt{\Delta}\epsilon_{t+\Delta} \tag{D.34}$$

$$\equiv \left(\bar{\mu} + \widehat{\beta}_t(y_t - \bar{y}) - \frac{1}{2}V_t\right)\Delta + u_{t+\Delta} \tag{D.35}$$

$$\widehat{\beta}_{t+\Delta} = e^{-\lambda_\beta \Delta}\beta_t + \bar{\beta}\left(1 - e^{-\lambda_\beta \Delta}\right) + \frac{\tilde{\nu}_t(y_t - \bar{y})}{\sqrt{V_t}}\sqrt{\frac{1 - e^{-2\lambda_\beta \Delta}}{2\lambda_\beta}}\epsilon_{t+\Delta} \tag{D.36}$$

$$\tilde{\nu}_{t+\delta} = \tilde{\nu}_t + \left[-\frac{(y_t - \bar{y})^2}{V_t}\tilde{\nu}_t^2 - 2\lambda_\beta \tilde{\nu}_t + \sigma_\beta\right]\Delta \tag{D.37}$$

where $\log\left(P_{t+\Delta}/P_t\right)$ is the S&P 500 monthly return, y_t is the S&P 500 earnings-to-price ratio, V_t is the S&P 500 annualized return variance, $\tilde{\nu}_t$ is the uncertainty about beta when the only source of information is the history of S&P 500 returns, $\Delta = 1/12 = 1$ month, and ϵ_t is a normally distributed random variable with mean equal to zero and variance equal to one. Maximizing the log-likelihood function

$$L(\Theta; u_\Delta, \ldots, u_{N\Delta}) = \sum_{i=1}^{N}\log\left(\frac{1}{(2\pi)^{1/2}\sqrt{|\Sigma_{(i-1)\Delta}|}}\right) - \frac{1}{2}u_{i\Delta}^\top \Sigma_{(i-1)\Delta}^{-1}u_{i\Delta}, \tag{D.38}$$

where $\Theta \equiv (\bar{\mu}, \bar{\beta}, \sigma_\beta, \lambda_\beta)^\top$, N is the number of observations, $\top$ is the transpose operator, $|.|$ is the determinant operator, $u_{t+\Delta}$ is defined in Equation (D.35), $\mathbb{E}_t(u_{t+\Delta}) = 0$, and $\Sigma_t \equiv \text{Var}_t(u_{t+\Delta}) = V_t\Delta$, provides an estimate of the vector of parameters Θ.

E   Numerical solution method

E.1   Chebyshev collocation

Substituting $K(x) = kx^2$, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, and the first order conditions (C.31)-(C.33) in Equation (C.27) yields a partial differential equation (PDE) for the value function $J(W, \widehat{\mu}, y, V, \nu)$. Then, using the conjecture $J(W, \widehat{\mu}, y, V, \nu) = \frac{W^{1-\gamma}}{1-\gamma} \phi(\widehat{\mu}, y, V, \nu)$ provides a PDE for the function $\phi(\widehat{\mu}, y, V, \nu)$.

Since the state variables $\widehat{\mu}_t$ and y_t belong to the real line by definition, they do not imply boundary conditions. In contrast, the dynamics of V_t and ν_t imply that $V_t = 0$ and $\nu_t = 0$ can be approached but not attained, implying two boundary conditions. When either $V_t = 0$ or $\nu_t = 0$, the investor observes the expected return μ_t and the predictive coefficient β_t. This immediately implies that $\widehat{\mu}_t \equiv \mu_t$, $\nu_t = 0$, and $a_t = K(a_t) = 0$.

Writing the HJB equation using the dynamics of W_t, μ_t, y_t, and V_t, substituting the first order conditions, and conjecturing the solution

$$J_{|\nu=0}(W, \mu, y, V) = \frac{W^{1-\gamma}}{1-\gamma} \phi_{|\nu=0}(\mu, y, V), \tag{E.39}$$

yields a 3-dimensional PDE for $\phi_{|\nu=0}(\mu, y, V)$. By definition, the solution $J_{|\nu=0}(W, \mu, y, V)$ is the boundary condition at $\nu = 0$.

Similarly, setting $V_t = 0$ in the dynamics of W_t, writing the HJB equation using the dynamics of W_t, μ_t, and y_t, substituting the first order conditions, and conjecturing the solution

$$J_{|V=0}(W, \mu, y) = \frac{W^{1-\gamma}}{1-\gamma} \phi_{|V=0}(\mu, y), \tag{E.40}$$

yields a 2-dimensional PDE for $\phi_{|V=0}(\mu, y)$. By definition, the solution $J_{|V=0}(W, \mu, y)$ is the boundary condition at $V = 0$.

The PDE for $\phi(\widehat{\mu}, y, V, \nu)$ is solved numerically using the Chebyshev collocation method (Judd, 1998). That is, we approximate the function $\phi(\widehat{\mu}, y, V, \nu)$ as follows:

$$\phi(\widehat{\mu}, y, V, \nu) \approx P(\widehat{\mu}, y, V, \nu) = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \sum_{l=0}^{L} a_{i,j,k,l} T_i(\widehat{\mu}) T_j(y) T_k(V) T_l(\nu),$$

where T_m is the Chebyshev polynomial of order m. The interpolation nodes are obtained by meshing the scaled roots of the Chebyshev polynomials of order $I+1$, $J+1$, $K+1$, and $L+1$. We scale the roots of the Chebyshev polynomials of order $I+1$, $J+1$, $K+1$, and $L+1$ such that they cover the intervals $\widehat{\mu} \in [q_{\widehat{\mu},1\%}, q_{\widehat{\mu},99\%}]$, $y \in [q_{y,1\%}, q_{y,99\%}]$, $V \in [0, q_{V,99\%}]$, and $\nu \in [0, q_{\nu,99\%}]$, respectively. Note that $q_{x,p\%}$ stands for the p percentile of the process x. The polynomial $P(\widehat{\mu}, y, V, \nu)$ and its partial derivatives are then substituted into the PDE, and the resulting expression is evaluated at the interpolation nodes. This yields a system of $(I+1) \times (J+1) \times (K+1) \times (L+1)$ equations with $(I+1) \times (J+1) \times (K+1) \times (L+1)$ unknowns (the coefficients $a_{i,j,k,l}$) that is solved numerically. The mean squared PDE residual computed over the set of 420 interpolation nodes is of order 10^{-30}.

E.2   Transversality, feasibility, and optimality conditions

We define the function $F(c, a, w)$ as follows:

$$F(c, a, w) \equiv -\delta J + u(c_t) + \mathcal{D}^{W,Z} J, \tag{E.41}$$

where $\mathcal{D}^{W,Z}$ is provided in (C.30). Equation (C.27) can therefore be rewritten as

$$\max_{c,a,w} F(c,a,w) = 0. \tag{E.42}$$

As argued in Merton (1998), a solution to problem (23) must satisfy

1. the transversality condition

$$\lim_{t\to\infty} \mathbb{E}\left(e^{-\delta t} J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t)\right) = 0, \tag{E.43}$$

2. the sufficient condition for a maximum

$$\text{Hessian of } F(c,a,w) \equiv \begin{bmatrix} F_{cc} & F_{cw} & F_{ca} \\ F_{wc} & F_{ww} & F_{wa} \\ F_{ac} & F_{aw} & F_{aa} \end{bmatrix} \text{ has only negative eigenvalues,} \tag{E.44}$$

3. and the two feasibility conditions

   - $c_t^* \geq 0$
   - $a_t^* \geq 0.$

The first order conditions of (E.42) are

$$F_c = 0 = u_c - J_W \tag{E.45}$$

$$F_w = 0 = J_W W_t(\widehat{\mu}_t - r^f) + J_{WW} W_t^2 V_t w_t + J_{W\mu} W_t \nu_t (y_t - \bar{y})^2 \tag{E.46}$$

$$F_a = 0 = -K'(a_t) J_W W_t - \nu_t^2 J_\nu + \frac{1}{2}\nu_t^2 (y_t - \bar{y})^2 J_{\mu\mu}. \tag{E.47}$$

From this, we immediately obtain

$$F_{cw} = F_{wc} = F_{ca} = F_{ac} = F_{wa} = F_{aw} = 0, \qquad\qquad F_{cc} = u_{cc} < 0 \tag{E.48}$$

because the utility function $u(.)$ is concave by definition. Furthermore, the value function satisfies

$$J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t) = \frac{W_t^{1-\gamma}}{1-\gamma}\phi(\widehat{\mu}_t, y_t, V_t, \nu_t). \tag{E.49}$$

Therefore, we have

$$F_{ww} = J_{WW} W_t^2 V_t = -\gamma W_t^{1-\gamma}\phi(\widehat{\mu}_t, y_t, V_t, \nu_t) V_t, \tag{E.50}$$

$$F_{aa} = -K''(a_t) J_W W_t = -2k W_t^{1-\gamma}\phi(\widehat{\mu}_t, y_t, V_t, \nu_t). \tag{E.51}$$

Since the Hessian of $F(c,a,w)$ is a diagonal matrix, the diagonal elements are the eigenvalues. This implies that the sufficient condition for a maximum is satisfied if $\phi(\widehat{\mu}_t, y_t, V_t, \nu_t) > 0$ (see Equations (E.50) and (E.51)). $\phi(\widehat{\mu}_t, y_t, V_t, \nu_t) > 0$ also implies that the first feasibility condition is satisfied (see Equation (32)).

Simulations show that both $\phi(\widehat{\mu}_t, y_t, V_t, \nu_t)$ and attention a_t^* are strictly positive, with minimums larger than 10^6 and 10^{-7}, respectively. Therefore, the two feasibility conditions and the sufficient condition for a maximum are satisfied. Figure A1 illustrates the speed of convergence towards zero of the following function of time t: $\mathbb{E}\left(e^{-\delta t} J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t)\right)$. The figure is obtained via
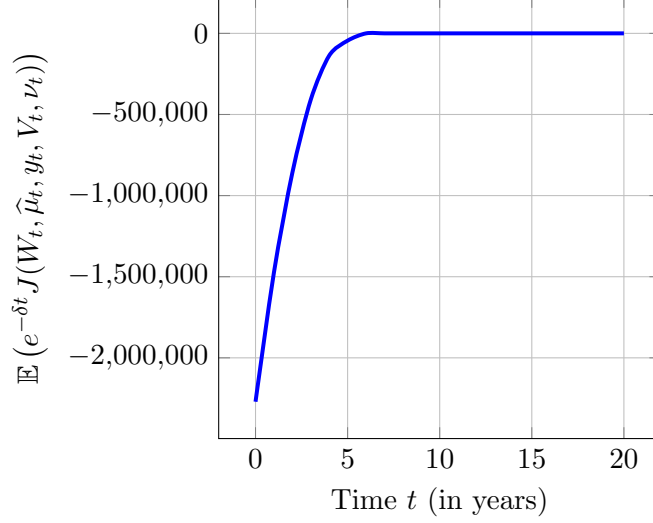
Figure A1: Transversality condition.
Initial state variables are $W_0 = 1$, $\widehat{\beta}_0 = \bar{\beta} = 0.59$, $y_0 = \bar{y} = 7.4\%$, $\sqrt{V_0} = \sqrt{\bar{V}} = 13.5\%$, and $\nu_0 = \bar{\nu} = 0.28$. Parameter values are provided in Table 1.

simulations. It shows that the transversality condition is satisfied, and therefore that our solution indeed solves the investor's maximization problem (23).

F   Alternative information cost specification

Let us assume the following more general information cost function

$$\overline{K}_t = k_1 a_t^{k_2} W_t^{k_3}, \tag{F.52}$$

where $k_1 > 0$ and $k_2 > 1$. Setting $k_1 = k$, $k_2 = 2$, and $k_3 = 1$ yields the quadratic form specification from Section 2.4.

The general cost function (F.52) is *ex ante* independent of wealth if $k_3 = 0$, decreasing with wealth if $k_3 < 0$ (i.e. information gets cheaper as time passes), and increasing with wealth if $k_3 > 0$ (i.e. information gets more expensive as time passes). Note that k_2 has to be larger that 1 in order to obtain a maximum (see first equality in Equation (E.51)).

At the optimum, we obtain

$$a_t^* = \left(\frac{1}{2k_1 k_2}\right)^{1/(k_2-1)} \left(\frac{\nu_t^2\left((y_t - \bar{y})^2 J_{\mu\mu} - 2J_\nu\right)}{J_W}\right)^{1/(k_2-1)} W_t^{-k_3/(k_2-1)}, \tag{F.53}$$

$$\overline{K}_t^* = k_1 \left(\frac{1}{2k_1 k_2}\right)^{k_2/(k_2-1)} \left(\frac{\nu_t^2\left((y_t - \bar{y})^2 J_{\mu\mu} - 2J_\nu\right)}{J_W}\right)^{k_2/(k_2-1)} W_t^{-k_3/(k_2-1)}. \tag{F.54}$$

We are interested in the effect of the parameter k_3 on our qualitative results. In order to do so, we approximate (F.53) by taking a first order Taylor expansion around $k_{11} = 0$, where the

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parameter k_{11} is defined as

$$k_{11} \equiv \left(\frac{1}{2k_1 k_2} \right)^{1/(k_2-1)}, \tag{F.55}$$

and thus does not depend on k_3. The parameter k_{11} converges to zero as $k_1 \to \infty$. In this case, the cost of information is infinite and therefore the investor does not pay attention to news i.e. the news signal s_t has zero precision. The first order Taylor expansion of attention around $k_{11} = 0$ yields

$$a_t^* \approx \left. \left(\frac{\nu_t^2 W_t^{-k_3} \left((y_t - \bar{y})^2 J_{\mu\mu} - 2J_\nu \right)}{J_W} \right)^{1/(k_2-1)} \right|_{k_{11}=0} \times k_{11}. \tag{F.56}$$

When the investor does not pay attention to news ($k_{11} = 0$), we know that the value function is homogeneous in wealth and thus it takes the following functional form

$$J(W_t, \widehat{\mu}_t, y_t, V_t, \nu_t) = \frac{W_t^{1-\gamma}}{1-\gamma} G(\widehat{\mu}_t, y_t, V_t, \nu_t), \tag{F.57}$$

where the function $G(.)$ solves a 4-dimensional PDE. Substituting this in the first term of (F.56), the optimal attention and the *ex post* cost of information are

$$a_t^* = k_{11} H(\widehat{\mu}_t, y_t, V_t, \nu_t)^{1/(k_2-1)} W_t^{(1-k_3)/(k_2-1)}, \tag{F.58}$$

$$\overline{K}_t^* = k_1 k_{11}^{k_2} H(\widehat{\mu}, y, V, \nu)^{k_2/(k_2-1)} W_t^{(k_2-k_3)/(k_2-1)} \tag{F.59}$$

for some function $H(.)$.

This implies that both attention and the *ex post* cost of information increase with wealth if $k_2 > k_3$ and $k_3 < 1$, and decrease with wealth if $k_2 < k_3$. If $k_2 > k_3$ and $k_3 > 1$, then the cost of information increases with wealth, whereas attention decreases with it. The figure below illustrates these three cases.
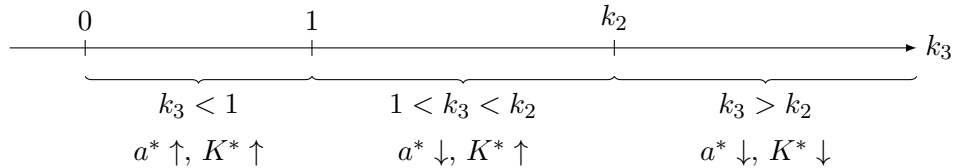
For a fixed amount of wealth, attention described here and attention described in the main body of the paper are driven by the state variables in the same way. The reason is that the signs and sensitivities of the partial derivatives J_W, $J_{\mu\mu}$, and J_ν are qualitatively the same here and in the main body of the paper (refer to the end of Section 2.3 for a discussion on these partial derivatives). We expect therefore that the qualitative implications of our paper remain unchanged (given a fixed amount fo wealth) under a more general cost function specification such as (F.52).