Noname manuscript No.
(will be inserted by the editor)

Financial Maintenance Covenants in Bank Loans

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September 2022

Abstract We develop a model of financial maintenance covenants under moral hazard, adverse selection, and informative signals of varying quality. We explain how public signals can improve the outcome for lenders and borrowers by reducing inefficient risk-taking (in both the pooling and separating equilibrium), and by shielding good firms from the actions of bad (separating) ones. We find that a reduction in signal quality moves the equilibrium from pooling to separating, to no covenants at all, consistent with the recent surge in covenant-lite loans. We also demonstrate that signal quality has a non-monotone effect on covenant strictness.

Keywords Bank Loans · Covenants · Adverse selection equilibrium · Corporate governance

JEL codes: G32, G34

We thank seminar participants at the University of Virginia and Nanyang Technological University for useful suggestions. All mistakes are our own.

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1 Introduction

Bank loans are complicated financial transactions between a borrower and a lender. A typical bank loan contract includes not just key pricing parameters such as interest rate and maturity, but also additional terms that serve to reduce the risk for the lender such as collateral requirements and covenant provisions.

Collateral requirements have been extensively studied in economics. Following the path-breaking work of Kiyotaki and Moore (1997) the role of collateral has justifiably taken center stage in explaining how aggregate economic fluctuations are affected by the financial system.[1] At the same time, covenant provisions have not received as much attention in economic theory, despite being a near-universal feature of bank loans and extensively studied empirically in the finance literature. Covenants are not mere technicalities and have real effects on firm investment (Chava and Roberts (2008)) and on the propagation of the financial system shocks to the rest of the economy (Chodorow-Reich and Falato (2022)). This paper aims to provide a tractable theoretical model of financial covenants.

Financial maintenance covenants are requirements that certain financial indicators (often ratios related to cash flow, income, or leverage) be over or under certain thresholds.[2] They are written on publicly observable information and aim to strike a balance between the need for flexibility for the borrower and protection for the lender. Despite the popular use of maintenance covenants, the extant theoretical literature on covenant design has mainly focused on *negative covenants* that directly prohibit some action by the borrower.[3]

We contribute to this literature by building a model that characterizes the equilibria under which financial maintenance covenants arise. We address three main questions: 1) why are maintenance covenants conditioned on publicly observable accounting variables, 2) what is their role in incentivizing borrowers, and 3) what is the effect of the noise in the contracting variable (due to variation in accounting quality) on optimal debt contracting and covenant design.

Our model for maintenance covenants features both moral hazard and adverse selection. We incorporate asymmetric information along three dimensions: the firm's type, the firm's unobservable action, and the relationship between the firm's action and the signal. In our setting, management can take an unobservable action that brings private benefits to equity (and the manager) but costs to the lender. This unobservable action is socially inefficient - the private gain to the firm is not sufficient to offset the cost to the lender. Financial or accounting variables in our model are useful insofar as they are informative signals of the firm's action. As the bank's screening technology is imperfect, the willingness to take a contract with or without a covenant provides a signal of the firm's type.

[1] Fostel and Geanakoplos (2014) provide a recent review of this expansive literature.

[2] Financial maintenance covenants are an important feature of loan contracts. See Appendix D for an overview.

[3] Financial maintenance covenants in loan contracts require the borrower to meet quarterly thresholds of the contracting variable. The contracting variable, while varying across contracts, is typically a ratio derived from items in the balance sheet or the income statement such as the current ratio, the leverage ratio, or the interest coverage ratio. Negative covenants prohibit (or require the bank's approval for) certain actions such as paying dividends, taking on additional debt, and engaging in a merger and acquisition. Interesting descriptive statistics about the kinds of covenants are provided by Chava et al. (2019). Negative covenants have been studied by Rajan and Winton (1995), Gorton and Kahn (2000), and Gârleanu and Zwiebel (2009) amongst others. A notable exception in the theoretical literature is Gigler et al. (2009) who study financial covenants. Our paper complements the analysis in their paper by elaborating the role of moral hazard and the quality of the signal in contract design.

We adapt the Wilson-Miyazaki equilibrium concept to our setting and solve for the equilibrium contracts.[4] Depending on the parameter values, we find that we can have either a separating or a pooling equilibrium. Our model provides a simple criterion that determines the kind of equilibrium that will occur: the payoff to the firm with the lowest incentive to risk shift is maximized. In both equilibria, the mechanism of the model is the same: since the choice of firm action affects the distribution of the (public) accounting variable, the firm can affect the probability of covenant binding. As loan terms for the borrower worsen after a violation, covenants can provide incentives for the firm's action prior to their violation and renegotiation. However, since firms differ by the benefit of risk-shifting and by the precision of the signal, in equilibrium, some firms still risk-shift.

In the case of a separating equilibrium, the set of firms that do not risk-shift enjoy lower average spreads because the costs to their lender are lower. However, these benefits occur because the contract prevents inefficient risk-taking before the covenant is violated or renegotiated, not because the contract groups together lenders with inherently lower costs. The pooling equilibrium is useful in preventing risk-shifting by firms that are easy to incentivize (to not take the risky action), leading to a lower cost for all firms. We show that the pooling equilibrium is preferred in two cases. First, it may be efficient to prevent all firms from risk-shifting. Second, a separating contract imposes an additional constraint on the maximization problem. If the mass of firms with high-risk shifting incentives is small, the benefits of separation do not outweigh the costs. Thus, our paper shows how signals from accounting statements such as financial ratios can improve the outcome for borrowers and lenders by reducing the extent of inefficient risk-taking (in both the pooling and separating equilibrium), and by shielding good firms from the actions of bad (separating) ones.

However, accounting statements are imperfect proxies for the firm's action. For example, errors in accounting statements introduce noise in the signal used in maintenance covenants. As the firm's compliance with the maintenance covenant depends on the observed value of the accounting signal, its noisiness can directly affect the interaction between lenders and borrowers, and consequently the design of optimal debt contracts. This issue has been a subject of a large empirical literature in both finance and accounting. For example, accounting transparency has been shown to affect loan pricing both empirically in Yu (2005) and theoretically in Duffie and Lando (2001).

Our analysis of Audit Analytics data shows that between 2000 and 2018, an average of 15% of U.S. public firms restated their financial reports every year. Overall, this translates to 39.4% of US public firms restating their financial reports at least once during the same period.[5] The high percentage of restatements understates the true level of noise in accounting statements. Despite this evidence of noise, financial ratios based on public accounting statements continue to be commonly used as signals or triggers in loan contracts.

To examine the effect of the quality of accounting information on maintenance covenant design, we augment our framework by introducing a measure of noise in the contracting variable.[6] Our model, which

[4] We discuss the Wilson-Miyazaki equilibrium in section 3.2.1.

[5] We describe our analysis of the restatement data in Appendix C.

[6] In the spirit of the literature of information economics and auction theory, we define a signal Z_1 as more noisy than signal Z_2 if the payoff to the firm with the lowest incentive to risk-shift is lower in the equilibrium with signal Z_1. We also

incorporates the interaction of signaling (by choice of contract), incentive provision (by the incentive role of covenants) and accounting quality, allows us to derive a nuanced view of the relationship between noise and covenant design. On the one hand, when the accounting signal gets noisier, financial covenants need to be tighter to provide incentives; on the other hand since generating incentives becomes more expensive more firms will either choose the no-covenant contract (in the separating equilibrium) or will risk-shift (in the pooling equilibrium). Thus, at low-to-moderate levels of noise, increasing the degree of noise leads to tighter covenant strictness on contracts with covenants *and* more contracts without covenants. For intermediate levels of noise, the separating equilibrium unravels, so all firms receive contracts with covenants. Lastly, for very high level of noise providing incentives to any firm is inefficient, so all contracts are without financial covenants.[7]

So far, we have treated the accounting signal Z (and its noise) as exogenous. However, the existence of covenants written on the signal creates incentives for manipulation. In our final analysis, we extend our model and allow the firm to manipulate the signal with some probability (as in Laux (2022)); the magnitude of this probability is a stand-in for lax internal controls, regulation, etc. We find that even though manipulation is deliberate while noise is random, they both reduce the 'decision-usefulness' of the accounting signal. As the manipulation probability increases, the observed signal from financial statements becomes less informative, which results in stricter covenants. If the firm can freely manipulate, covenants lose all value and are not used. Thus, we find that, despite having a different mechanism, manipulation has effects on the contract and payoff that are identical to (a particular form) of noise.

1.1 Related Literature

The theoretical literature on the role of financial covenants in debt contracts is large; and as such it is important to describe how we relate and contribute to this literature. In general, covenants in the extant literature are derived either to provide a valuable option to lenders to gain control rights, to exert power over managers following an adverse financial event (Smith and Warner (1979), Berlin and Mester (1992), Aghion and Bolton (1992), Dewatripont and Tirole (1994), Gigler et al. (2009)), or to diminish hold-up problems associated with short term debt as in Rajan (1992). More recently, a pair of important papers have modeled covenants as tripwires to aid in the renegotiation of debt contract terms (Gârleanu and Zwiebel (2009) and Gorton and Kahn (2000)).

Our model is closely related to the work of Gârleanu and Zwiebel (2009). In their model, firms that would be easy to incentivize ex-post are willing to give up rights ex-ante to signal their type. Thus covenants reduce the cost of renegotiation since the lender must control a future action by the firm. As the action is presumed to be observable, a covenant on a financial ratio or an accounting variable is an inefficient tool to assign control rights. We explicitly construct our model to contrast the incentive

show that several ways of introducing noise in the accounting signal (random errors, white noise, etc.) are consistent with our definition of noisiness.

[7] Covenant-lite loans, or loans with no financial maintenance covenants, have become increasingly commonplace in the riskiest sector of the syndicated loan market. At the end of 2018, more than 85% of all leveraged loans in the United States were covenant-lite (Edwards (2019)). For a detailed discussion on covenant-lite loans and related agency issues, see Billett et al. (2016).

and signaling role of financial covenants. We show that when covenants are written and tested against publicly observed accounting variables, the contract can serve a signaling purpose only if the covenant also serves a direct incentive role. We find that this conclusion is maintained even in the presence of noise in the accounting signal. Furthermore, we also demonstrate in a nested specification of our model that the signaling role of covenants in Gârleanu and Zwiebel (2009) is not a necessary condition for the existence of maintenance covenants.

Our paper is also related to Rajan and Winton (1995). In their study, the incentive problem is on the side of the financial intermediary. If performing socially beneficial monitoring and exerting control over the borrower is costly, it is efficient to delegate these functions to one of the lenders (the bank). However, this creates free-riding problems. In their model, the variable on which the covenant is written is observable only after costly monitoring. The right to demand early repayment if a covenant is breached gives the bank enough renegotiation power that obtaining information becomes efficient. As the lender must be able to determine if the covenant has been broken, information is acquired as a by-product. While their work describes bank incentives and covenants well, their approach leaves financial maintenance covenants unexplained. First, these covenants are written on public (and freely available) accounting information. Second, the Rajan-Winton model precludes the lender (bank) from conditioning its action on public information, which is the central contracting variable for financial maintenance covenants.

The rest of the paper is organized as follows. We describe the main model in Section 2. In Section 3, we solve for the equilibrium contract and explore its properties. Section 4 is devoted to exploring the effect of variation in accounting quality on the optimal contract. Section 5 explicitly models riskshifting. Section 6 models the effects of performance manipulation. Section 7 concludes. All the proofs are presented in Appendix A. Appendix B has a more realistic model of renegotiation. Appendix C details the computation of the restatement frequency for US public firms, while Appendix D provides estimates of the frequency of financial maintenance covenants in loan contracts.

2 Environment

There are two parties to a relationship, a firm and a lender (bank). The firm needs a loan of size I, and in the following period will have a positive cash flow W if the loan is given. We assume that at most $R \leq W$ of the firm's resources are available to repay the loan. For simplicity of exposition we assume that the interest rate is zero, or equivalently that all sums have been appropriately discounted. Both parties in the relationship are risk-neutral and there is a mass of perfectly competitive banks.

The firm's management can conduct business in a safe (s) or risky (r) manner. This action is unobservable. If the action taken is (r), at the end of the period (after repayment to the bank), the firm gets an additional payoff of x, while the bank suffers a loss of y.

Assuming some fixed benefit/cost of action r that is independent of the contract is a convenient shortcut that we utilize in most of this paper. Section 5 presents a model that explicitly derives x and y from risk-shifting. The action r leads to a mean-preserving spread of the cash flow. Limited liability implies that higher variability of cash flow shifts value from lender to borrower, and the increased probability of costly default implies that the cost to the lender is higher than benefit to the borrower.

We assume that all the relevant firm characteristics are public information, except for the firm's benefit from taking the risky action - x. This assumption is motivated by the fact that the bank's screening technology is imperfect, which implies that even after conditioning on observable variables there is some uncertainty about the permanent characteristics of the firm. In the context of the model, the true benefit of risky action to the firm x is known to itself, but unknown to the bank. Let $M(x)$ be the bank's subjective probability distribution over possible x. Alternatively, we can think that there is a mass of firms with different values of x and the bank cannot distinguish among them. The two interpretations are equivalent, but we will adopt the latter. Let $[x_a, x_b]$ be the support of that distribution. We assume that $M(x)$ is continuous. The firm's x will also serve as its label.

If the private benefit of risk-taking always exceeds the cost to the bank, then it will be efficient to have a simple debt contract with the cost to the bank y priced in. In what follows, we assume that for at least some of the firms the private benefit of risk-taking is lower than the cost to the bank, that is, total surplus is maximized by taking the safe action. However, since the firm still has a private benefit of risk-shifting, incentives must be given to achieve the efficient outcome.

Assumption 1 *$M(y) > 0$ and $x_a > 0$.*

Assumption 1 implies that for a positive mass of firms, the private benefit of risk-taking x is smaller than the cost y borne by the bank.

With competitive banking, the face value of the loan is $D = I$ if it is anticipated that the firm will play s and $D = I + y$ otherwise. The efficient solution is not be consistent with incentives if a firm's value of risk-taking is higher than the value of the safe action, or $x > 0$. The second part of assumption 1 implies that all firms have an incentive to perform the risky action. Under these conditions, the face value of the repayment of the loan is simply $D = I + y$.

We assume that there is some random variable Z that is correlated with the firm's action. The random variable Z has a conditional cumulative distribution function $F(z|a)$, a compact support $[z_a, z_b]$, and a conditional probability density function $f(z|a), a = r, s$ that is continuous and strictly positive on the support. Following Milgrom (1981), we make the assumption that the signal satisfies the monotone likelihood ratio property (MLRP).

Assumption 2 *The likelihood ratio*

$$g(z) \equiv \frac{f(z|r)}{f(z|s)} \tag{1}$$

is strictly decreasing in z.[8]

This condition ensures that a higher value of z is a signal that the firm took the safe action (for any non-degenerate prior on the action, a higher z increases the posterior probability that the firm took the safe action). Intuitively, a low value of z is more likely if the firm took the risky action. So, if some

[8] This is not a serious restriction. Let x be a finite-dimensional vector of all the variables correlated with the firm's action. Then if we define $z \equiv f_x(x|s)/f_x(x|r)$, the one-dimensional variable z satisfies the assumption above. Moreover, the Neyman-Pearson Lemma implies that, up to a strictly monotone transformation, z is the most informative signal.

Table 1 Symbols

I	Investment (size of the loan).
W	Cash flow from the funded project.
R	Cash flow available for repayment.
a	Firm's unobservable action. $a \in \{r, s\}$.
x	Private benefit of the risky action ($a = r$).
y	Cost to the lenders from the risky action ($a = r$).
$M(x)$	Distribution of firm types.
z	Publicly observable signal z, correlated with a.
$F(z\|a)$	Conditional CDF of the signal z.
$H(z)$	$F(z\|r) - F(z\|s)$.
$\mathbf{Z} = [z_a, z_b]$	Support of the signal z.
c	Renegotiation cost
D	Base payment
$A \subseteq \mathbf{Z}$	Set of z-s for which the lenders can demand early repayment.

outcome is tied to the value of the random variable z, the firm might have an incentive to change its action.

We can think of z as financial or accounting information generated by the firm. This signal is produced at no additional cost and is observed by all parties costlessly and without error.

Payments can be conditioned on the outcome of the (costlessly observed) signal z; we will call invoking this option of the contract as covenant renegotiation. There is a renegotiation cost c. The split of costs borne by the firms and the bank does not affect the results (as long as any costs borne for the firm reduce cash flow available for repayment), so for concreteness we assume it is incurred by the bank.[9]

Definition 1 A debt contract is a pair of base payment D, and a Borel-measurable set $A \subseteq [z_a, z_b]$ of realizations of z at which there is a renegotiation.

We assume that the banking sector is competitive. The banks post offered contracts and they are committed to their offers.

The sequence of events is summarized in the timeline. The timeline is also useful for understanding the differences between our model and the extant theoretical models on covenant design. In the Gârleanu and Zwiebel (2009) model everything is observable (at a cost) and the contract is renegotiated before an additional action is taken. The covenant is then the right of the bank to block the (observable) action. In the Rajan and Winton (1995) model, the bank acquires the signal z and depending on its value, it may have the option to forbid an action. Finally, Gorton and Kahn (2000) assume that the bank always has the right to renegotiate; this right the authors call a covenant.

In our work, we assume that the firm has an informational advantage over the lender, but that the nature of its action has consequences for information reflected in accounting measures, z. So the signal z is informative of the past rather than the future. Renegotiation of the contract based on the signal cannot affect the action going forward (since it has already happened). However, since the renegotiation is anticipated, it can have incentive effects, as we shall see.

[9] We interpret c as legal, administrative, and monitoring costs. The cost c can also include the opportunity cost of disrupted future investment or future higher financing costs for the firm. Chava and Roberts (2008) and Roberts and Sufi (2009*a*) document this effect.

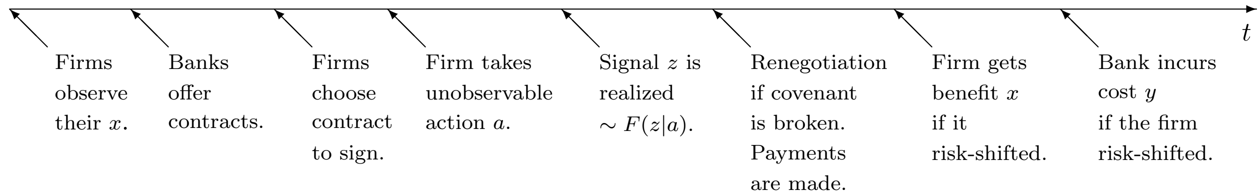

Fig. 1 Timeline in the model

Interpretation of the signal z We interpret the signal z as the set of publicly reported variables, such as financial ratios in accounting statements, generated by the borrower. Access to z can be more or less valuable in the debt contract (in an ex-ante sense) for two reasons. First, the firm may make more or fewer mistakes in preparing the report; second, even if correctly measured, the accounting signal z may be more or less informative of the firm's private information. We interpret both (1) and (2) as noise. We consider the impact of noise on the equilibrium contract formally in section 4.

Covenant violation In our model, when the signal z falls in the set of prohibited values, the lender receives additional payment $(D'(z) - D)$ (which we discuss later on). This modeling choice is consistent with payments made in practice. When a maintenance covenant is violated the loan is considered legally in technical default, which gives the lender the right to demand immediate repayment of the principal (i.e., accelerate the loan date). Even in cases where the lender waives the covenant violation, a combination of a one-time waiver fee, an increase in the interest rate, or additional collateral is often demanded by the lender (Roberts and Sufi (2009*b*), Sufi (2009a), and Vance (2005)). The details of the new contract terms are determined via renegotiation.[10]

3 Optimal contracts

3.1 Incentive constraints

In order to solve the model, we start with analyzing the action of a firm facing a set of contracts. The firm has no external source of funds, so all payments to the bank must be financed by the firm's cash flow. This implies the constraint $D \leq R, D'(z) \leq R$.

Suppose that a firm has signed a debt contract. It would choose to perform the safe action if and only if, given the expected repayment function, the following condition is satisfied:

$$\text{Prob}(A^c|s)D + \int_A D'(z)f(z|s)dz \leq \text{Prob}(A^c|r)D + \int_A D'(z)f(z|r)dz - x \tag{2}$$

We call this condition the incentive constraint.

The repayment function cannot be arbitrary: it has to be consistent with the bargaining process between the firm and the bank after the covenant is violated. We assume a simple bargaining process in which the bank makes a take it or leave it offer to the firm, which, at this point, has no outside option.

[10] For details see Roberts and Sufi (2009*a*), pg. 1688 -1689. According to S&P Leveraged Commentary and Data, covenant renegotiations are costly, with the average covenant waiver fee amounting to 2.4% of the loan amount for US corporations Hyde (2008)).

Then, clearly, $D'(z) = R$ for all $z \in A$. Then a contract is simply a pair (A, D), where A is a Borel set and D is a real number. This implies that the condition for the firm to take the safe action will be:

$$[R - D][\text{Prob}(A|r) - \text{Prob}(A|s)] \geq x. \tag{3}$$

Given a finite set of contracts, the management will choose contract i and action a that maximize its payoff:

$$(i, a) \in \underset{i\in\{1,\dots n\}, a\in\{r,s\}}{\arg\max} W - \text{Prob}(A^i|a)R - (1 - \text{Prob}(A^i|a))D^i + \chi_r(a)x,$$

where $\chi_r(.)$ is an indicator function, which is 1 if $a = r$ and 0 otherwise.

The importance of renegotiation The potential worsening of loan terms following a renegotiation is an important tool in providing incentives to the firm. If the firm can refinance the loan with outside funds at the original terms, the mechanism in our model will not operate. However, the literature on relationship banking shows that there can be substantial costs when switching lenders. Thus, we can interpret the change $R - D$ as constrained by the (inferior) outside option that firms have.

The mechanism of our paper still operates even if incumbent banks have no cost advantage to outsiders. In Appendix B we explicitly model the effect of outside banks on renegotiation. In this case it is impossible to prevent risk-shifting completely (if the firm always plays s, then the outside banks will refinance the loan at the original terms, so no incentives can be provided), but the probability of risk-shifting can be reduced by covenants. As the ex-post probability that the firm risk-shifted depends on z, the renegotiated amount increases with the severity of the covenant violation.

3.2 Adverse selection

We begin our analysis by solving the model for a given (and known) conditional distribution of the signal z. We propose our measure of noise in section 4.1 and show how it can be derived from the distribution of z. Therefore, we first solve the model for a fixed level of noise. In section 4.2 we consider how varying the degree of noise affects the equilibrium.

Earlier, we discussed the behavior of firms who are committed to a specific loan contract. In practice, however, firms choose the contracts they sign in a competitive loan market. Next, we model a setting where banks offer a variety of contracts (face value - covenant pairs) and firms are free to select the contract that best suits them. Since the covenant affects the firm's choice of a contract, it performs both incentive and screening functions.

3.2.1 Equilibrium

Ever since the seminal work of Rothschild and Stiglitz (1976), it has been well known that competitive equilibria with adverse selection and screening might not exist. The intuition is simple. Suppose that in equilibrium the high-risk and low-risk firms are separated. If the number of high-risk firms is small, the cost to low-risk firms of being pooled will be small, so a pooling contract will be preferred by everyone.

On the other hand, if the contract is pooling it will be possible to offer a contract that is preferred only by low-risk firms. As a result no equilibrium set of contracts exists.

The pooling contract is not sustainable, since it is profitable to "steal" low-risk customers and split the savings between the bank and the firms. However, the cost savings are contingent on the high-risk firm remaining on the original contract. If the original contract was pooling risks and breaking even, then under the new circumstances it would not be breaking even, and consequently it would be withdrawn. Thus, the pooling contract is fragile due to the myopia of the party offering a new contract.

Recognizing this, Wilson (1977) and Miyazaki (1977) introduce an equilibrium concept (sometimes called anticipatory equilibrium) in which every party correctly predicts the consequences of its offered contract on all other parties. We adapt the Wilson-Miyazaki equilibrium to our environment: the crucial additional restriction is that when introducing new contracts every bank takes into account that money-losing contracts will be withdrawn from the market.[11]

The equilibrium consists of the set of contracts on offer and the actions of the firms. The firms choose which contract to sign and what action to take. The first choice is denoted by the contract assignment function $b : [x_a, x_b] \to \{1, \ldots, n\}$. Given that, the set of firms that take up contract i is $B_i = \{x : b(x) = i\}$. The second choice is the action and is denoted by the action recommendation $a(x)$. The first restriction on the equilibrium is that all firms choose a contract and an action that maximize their profits.

Definition 2 Given an arbitrary finite set of contracts, $\{(A^i, D^i)\}, i = 1 \ldots n$, a contract assignment $b : [x_a, x_b] \to \{1, \ldots, n\}$ and an action recommendation $a : [x_a, x_b] \to \{r, s\}$ are *consistent with individual rationality* if

$$(b(x), a(x)) \in \underset{b \in \{1,\ldots n\}, a \in \{r,s\}}{\arg\max} W - \text{Prob}(A^b|a)R - (1 - \text{Prob}(A^b|a))D^b + \chi_r(a)x, \quad \forall x \in [x_a, x_b].$$

Notice that for a given set of contracts the restriction above uniquely determines the equilibrium contract and action for all but a measure zero of firms. Also, if $b(x)$ (and hence B_i) and $a(x)$ are consistent with individual rationality, they are also Borel-measurable.

The second restriction on the equilibrium concerns the way banks form expectations about the profit related to offering a new contract. In a traditional screening equilibrium banks 1) assume that all other contracts will remain on offer, and 2) correctly anticipate which firms will take the new contract, given assumption 1. In our equilibrium we add an additional demand on a bank's rationality: a bank is able to take into account its effect on the profitability of other banks and hence anticipate that contracts that start losing money will be withdrawn.

Definition 3 Given a finite set of contracts, a *surviving contracts set* is a subset of the original set such that (i) given the new contract assignment of firms and action recommendations consistent with individual rationality, all the contracts are making nonzero profits, (ii) the collection is maximal by inclusion amongst all collections with property (i).

[11] The equilibrium is static, even though we use dynamic language to describe it. Netzer and Scheuer (2014) give a game-theoretic foundation of the equilibrium concept. They show that Wilson contracts are the unique robust equilibrium of the game.

We can think of arriving at a surviving contracts set as the outcome of the following procedure. Pick an arbitrary money-losing contract and eliminate it. Then let the firms reoptimize and take up the contract that maximizes their payoffs. Given the new distribution of firms along contracts, recompute expected bank profits. Then continue the procedure until there are no money-losing contracts left. Since the order of elimination matters, there will be a collection of surviving contracts sets.

At this stage we are ready to define an equilibrium.

Definition 4 A competitive equilibrium in an economy with adverse selection consists of a finite set of contracts $\mathbf{S} = (A^i, D^i), i = 1 \dots n$, a contract assignment function $b(x)$, and an action recommendation $a(x)$ such that:

1. The partition of firms along contracts and the action recommendation are consistent with individual rationality.
2. Banks break even on each contract:

$$\int_{B_i} \left[\text{Prob}(A^i|a(x))(R-c) + (1 - \text{Prob}(A^i|a(x))D^i - I - \chi_r(a(x))y\right] dM(x) \geq 0.$$

3. There does not exist a finite set of alternative contracts $\mathbf{S}' = \{(A^j, D^j)\}$, different from the existing contracts, such that for one of the surviving contract sets A for $\mathbf{S} \cup \mathbf{S}'$, $\mathbf{S}' \subseteq A$ and for some $s \in \mathbf{S}'$, profits are strictly positive.

Condition (3) requires that it is impossible to add a contract that will survive the iterated elimination of money-losing contracts and make a positive profit. By our definition, we have made the additional restriction as loose as possible - the bank holds the most optimistic view about the profitability of a new contract. On the other hand, we have imposed the restriction that all the introduced contracts are in the surviving contracts set. This prevents introduction of a pair of contracts, one of which is deliberately losing money in order to eliminate an existing contract.

The equilibrium as described appears to be complicated and its existence does not appear to be guaranteed. However, we are able to show that an equilibrium *always* exists. Second, we show that the equilibrium contracts have a simple structure and are the optimal policy in a constrained maximization problem. In the rest of the section, we simplify the equilibrium analysis and show that the equilibrium will be one of three types: separating, risk-taking, and pooling.

We can simplify the analysis with three observations. First, any contract can be transformed into a contract of the type $([z_a, \tilde{z}], D)$ in such a way that the firm's incentives for action and expected returns are unchanged and the bank's costs are lowered. This does not change the endogenous distribution of firms along contracts. Second, if two firms choose the same action, they choose the same contract - the one that minimizes expected payment, conditional on the action. Therefore, there can be at most two types of contracts taken up. Third, if a firm x optimally plays r, then all firms with $x' > x$ will take the same contract and play r. Similarly, if firm x plays s, a firm with $x' < x$ will also play s. Then here are the possibilities:

Firm type	$x_a \leq x \leq \hat{x}$	$\hat{x} \leq x \leq x_b$
Action	s	r

The cutoff point $\hat{x}$ summarizes the possibilities: if $\hat{x} = x_a$, then in equilibrium all firms will choose the risky action, which implies that the contract will be without covenants; if $x_a < \hat{x} < x_b$, in equilibrium some firms will play risky and some will play safe; finally if $\hat{x} = x_b$ then all firms will play safe, so all observed contracts will have covenants. The contract with covenants is $([z_a, \tilde{z}], D)$ and the contract without a covenant is simply a flat payment of D.

Summarizing the discussion above, we have the following:

Proposition 1 *Suppose an equilibrium exists. Define* $\hat{x} \equiv \sup\{x \in [x_a, x_b] :$ *Firm* x *plays* $s\}$*; if the set is empty set* $\hat{x} \equiv x_a$*.*

Then:

1. *All firms with* $x \in [x_a, \hat{x})$ *play* s *and all firms with* $x \in (\hat{x}, x_b]$ *play* r*.*
2. *If* $\hat{x} = x_a$*, then in equilibrium all firms with* $x \in (x_a, x_b]$ *will choose the contract* $(\emptyset, I + y)$*.*
3. *If* $\hat{x} = x_b$*, then in equilibrium all firms with* $x \in [x_a, x_b)$ *will choose the contract* $([z_a, \tilde{z}], D)$ *with binding incentive and break-even constraints.*

Proof In Appendix A.

We note that the only possibility for two contracts to coexist is if $\hat{x} \in (x_a, x_b)$ and all firms with $x \in [x_a, \hat{x})$ take one contract, while the firms $x \in (\hat{x}, x_b]$ take the other. Somewhat abusing the standard terminology, we call this a separating equilibrium. In all other cases, there is only one contract on the market. If all firms take action $a = r$, we call this a risk-taking equilibrium; all other equilibria we call pooling.

Definition 5 An equilibrium is *separating* if there exist two contracts, each taken up by a nonzero mass of firms, such that all firms in the first contract take action $a = s$ and all firms in the second contract take action $a = r$. An equilibrium such that all firms choose $a = r$ is *risk-taking.* An equilibrium that is not separating or risk-taking is *pooling.*

In the following sections, we consider briefly the different types of equilibria.

3.2.2 Separating equilibria

First, we conjecture that the equilibrium is separating. For any contract, (almost) all firms that choose it will take the same action.

Let's guess that there are (a positive mass of) firms playing s and r. First, look at some contract (A, D) taken up by firms playing r. The break-even constraint and the assumption of separation implies that $\text{Prob}(A|r)R + (1 - \text{Prob}(A|r))D \geq I + y$. If the inequality is strict, it would be feasible to add the contract $(\emptyset, D')$ such that $I + y < D' < \text{Prob}(A|r)R + (1 - \text{Prob}(A|r))D$. The new contract would make positive profits, which is a contradiction. So the contract taken up by the firms playing r is simply $(\emptyset, I+y)$.

Now consider the contract taken up by the 'safe' firms. By the assumption of separation, 'risky' firms must weakly prefer the 'risky' contract. So, the contract for safe firms must satisfy:

$$\text{Prob}(A|r)R + (1 - \text{Prob}(A|r))D \geq I + y.$$

Also the bank must break even on the contract. Since all firms that take up the contract play s, then the break-even constraint is as follows:

$$\text{Prob}(A|s)R + (1 - \text{Prob}(A|s))D \geq I + \text{Prob}(A|s)c.$$

The equilibrium contract must minimize expected repayment from that contract subject to the two constraints. If it does not, a bank can offer a new contract that attracts a mass of firms playing s and makes a positive profit. Then the payoff for the 'safe' firms from a separating contract is given by the problem P1 below:

$$v_s^* = \sup_{A,D} W - \text{Prob}(A|s)R - \text{Prob}(A^c|s)D \quad (4)$$

$$\text{subject to } D \leq R, \quad A \subseteq [z_a, z_b]$$

$$\text{Prob}(A|s)R + \text{Prob}(A^c|s)D \geq I + c\,\text{Prob}(A|s) \quad (5)$$

$$\text{Prob}(A|r)R + \text{Prob}(A^c|r)D \geq I + y, \quad (6)$$

where by convention $v_s^* = -\infty$ if the constraint set is empty.

Lemma 1 *If the constraint set for problem P1 is nonempty, then the optimum is attained, $A = [z_a, \tilde{z}]$, (5) and (6) are binding, and the solution is unique.*

3.2.3 Risk-taking equilibria

The case when all firms take action $a = r$ is easy to analyze. Competitive forces lead to the contract $(\emptyset, I+y)$ being offered. The payoff to the lowest-x firm from this contract is $v_r^* = W + x_a - I - y$.

3.2.4 Pooling equilibria

We call all non-separating and non-risktaking equilibria pooling. In a pooling equilibrium, either all firms take the same action $a = s$, or the actions s and r coexist for some contract. As a consequence, conditional on the chosen action, the expected payment is the same for all contracts. This implies the following Lemma.

Lemma 2 *Suppose there exists an equilibrium that is pooling. Then there exists only one contract taken up by a positive mass of firms.*

Proof In Appendix A.

Given the unique equilibrium contract, firms will choose action r if $x \geq \hat{x} \equiv (\text{Prob}(A|r) - \text{Prob}(A|s))(R - D)$. Thus we see the two possible cases for the pooling equilibrium: if $\hat{x} \in (x_a, x_b)$, some firms take action $a = r$ and others choose $a = s$; if $\hat{x} \geq x_b$ then almost all firms choose the safe action.

The key to characterizing the equilibrium pooling contract is the fact that it maximizes the profit of firm x_a, subject to appropriate constraints. Suppose that the contract on offer does not do that. Then it will be possible to offer a contract that is preferred by low-risk firms with $x \in [x_a, x')$ for some x'. Then the existing contract is saddled with firms that are more likely to risk-shift, so average costs rise and the original contract is withdrawn. Finally, all firms take up the new contract, which by construction breaks even.

Therefore, the equilibrium pooling contract can be derived with a contract theory approach. In the contract design problem, we specify the actions of all firms directly. The contract specifies $\hat{x}$, z and D and must satisfy incentive constraints and break even constraints. The value of the low-x firm when the cutoff firm is $\hat{x}$ is derived by the following problem P2:

$$v_p^*(\hat{x}) = \sup_{A \subseteq [z_a, z_b], D} W - \text{Prob}(A|s)R - \text{Prob}(A^c|s)D \tag{7}$$

$$\text{subject to } (\text{Prob}(A|r) - \text{Prob}(A|s))(R - D) = \hat{x} \tag{8}$$

$$\text{Prob}(A|s)R + \text{Prob}(A^c|s)D + (1 - M(\hat{x}))\hat{x} \geq$$

$$I + y[1 - M(\hat{x})] + c[M(\hat{x}) \text{Prob}(A|s) + (1 - M(\hat{x})) \text{Prob}(A|r)]. \tag{9}$$

Equation (8) is the incentive constraint and condition (9) is the break-even constraint.[12] Let the value of the best pooling contract be $v_p^* = \sup\{v_p^*(\hat{x})\}$.

Below we show that with an additional assumption, problem P2 either has a well-defined maximum, or no pooling contract is feasible (so $v_p^* = -\infty$).

Assumption 3 *Let $A(\hat{x})$ be the optimal A in the pooling contract if the cutoff firm is $\hat{x}$; if no contract is feasible for $\hat{x}$, define $A(\hat{x}) = \emptyset$. Then* $\sup \text{Prob}(A|s) < 1$.

Lemma 3 *If $x_a > 0$, assumption 3 holds, $M(x)$ is continuous and the constraint set of P2 is nonempty, then the optimum for problem P2 is attained, the constraint (9) is binding and $A = [z_a, \tilde{z}]$.*

Proof In Appendix A.

3.2.5 Equilibrium existence

In Sections 3.2.2, 3.2.3, and 3.2.4, we showed that banks seek to offer contracts that are attractive to low-x firms. As a result, we showed that if a separating equilibrium contract exists, it must be a solution to P1; if a pooling separating contract exists, it must be a solution to P2 and if in equilibrium all firms act $a = r$, then the equilibrium contract is $(\emptyset, I + y)$. The technical Lemmas 1 and 3 ensure that the problems P1 and P2 either have solutions or are infeasible. Then, if an equilibrium exists, it maximizes

[12] It is possible that the optimal pooling contract induces expected payment by the risky firms of more than $I + y$; in this case some bank will offer the contract $(\emptyset, I + y + \epsilon)$ and unravel the pooling equilibrium. This will never happen given the parameter assumptions we have made.

the value of the x_a firm among the best separating, risk-taking or pooling contract. Formally, we have the following theorem:

Theorem 1 *An equilibrium exists. The equilibrium contract maximizes the payoff to firm x_a, which is given by $v^* = \max\{v_s^*, v_r^*, v_p^*\}$.*

Proof In Appendix A.

Our environment always has an equilibrium outcome of the Wilson-Miyazaki type. Moreover, it has a clear structure and can be derived as the maximum of three auxiliary problems.

What ensures the existence of equilibrium? In the classic adverse selection setting, an equilibrium fails to exist if there are too few high-risk agents. In this case, a pooling contract is preferred by everyone, but the pooling equilibrium will itself be unravelled by a contract that appeals only to low-risk agents. In our equilibrium, pooling survives because when a bank offers a new contract, it takes into account its effect on other debt contracts. The bank cannot steal the good risks and leave the rest, since it knows that contracts saddled with bad risks will be withdrawn.

As a result, in the pooling case, banks compete by offering contracts that are designed to break even when all the firms take them up. Thus, the Wilson-Miyazaki equilibrium always exists and can be formulated as a contract theory problem.

3.3 Analysis of the adverse selection model

By Lemmas 1 and 3, the covenant set A is just an interval $A = [z_a, \tilde{z}]$. Since breaking the covenant is costly, it is optimal to minimize the probability that it will be in effect. The MLRP property implies that the signal is most informative of risk-taking for a low value of z, so it is optimal to choose a set A with the smallest possible values for z, and hence an interval.

This result conforms to contracts that we observe in practice and allows us to identify the "strictness" or "tightness" of a covenant with the threshold value $\tilde{z}$.

3.3.1 Analysis of the separating equilibrium

In this section, we use the fact that the contract in the separating equilibrium is the solution to problem P1. This allows us to find a simple necessary condition for the existence of a separating equilibrium.

Proposition 2 *A necessary condition for a separating equilibrium is that the problem P1 has a solution (denoted $(\tilde{z}, \tilde{D})$) and $\hat{x} \in (x_a, x_b)$, where $\hat{x} = y - cF(\tilde{z}|s)$.*

If the equilibrium is separating, then all firms with $x \in [x_a, \hat{x})$ will take the contract with covenants $(\tilde{z}, \tilde{D})$ and all firms with $x \in (\hat{x}, x_b]$ will take a contract without a covenant with $D = I + y$.

Proof In Appendix A.

In problem P1, we find the optimal separating contract for the firms taking action $a = s$. In other words, the proposition above states that a necessary condition for the equilibrium to be separating is that a positive measure of firms take action $a = s$ when faced with the separating contract. If $\hat{x} \leq x_a$, then the simple risk-taking contract will be preferred by all firms. When $\hat{x} \geq x_b$, then a pooling contract in which all firms are incentivized to choose $a = s$ is preferable.

Next, we turn to comparative statics on the equilibrium contract.

Proposition 3 *The optimal covenant strictness $\tilde{z}$ in the separating contract is increasing in y and I and decreasing in c, while the threshold firm $\hat{x}$ is decreasing in c and I.*

Proof In Appendix A.

The effect of renegotiation cost If the renegotiation cost c increases, the contract with covenants becomes less attractive, so the marginal firm will shift towards the no-covenant contract. Both the strictness $\tilde{z}$ and the prevalence $\hat{x}$ of the contract go down; however, the total renegotiation cost $F(\tilde{z}|s)c$ increases.

It is interesting to contrast the separating contract to the contract with known type. In the latter case, an increase in the cost of renegotiation leads to stricter covenants. The reason for the different conclusion is that with adverse selection the marginal firm to be incentivized is endogenous – when the covenant contract becomes less attractive, more firms will switch over to the no-covenant contract; then the covenant necessary to provide incentives is looser. In general, this implies we see that a model with known propensity to risk-shift may be misleading.

The effect of debt amount If the size of the investment increases, the bank needs to collect more revenue to break even; this requires an increase in the covenant strictness. This effect is partially offset by the fact that the covenant contract has now become (relatively) less attractive to good risk-shifters, so $\hat{x}$ goes down.

The effect of the cost of risk-shifting y The effect of a change in y is more ambiguous. The no-covenant contract becomes more expensive since the additional cost must be factored into the flat repayment. This in turn requires the covenant contract to be stricter, even though none of the firms that take it up actually inflict the cost y on the bank. The effect on the set of firms that take up each contract is indeterminate and depends on the exact parameters of the model, in particular the cost of renegotiation and the correlation of the signal z with the action.

3.3.2 Analysis of the pooling equilibrium

The pooling equilibrium is slightly more complicated to characterize since the optimal policies are not necessarily continuous. However, the decomposition of the problem into two steps allows us to analyze comparative statics using numerical methods.

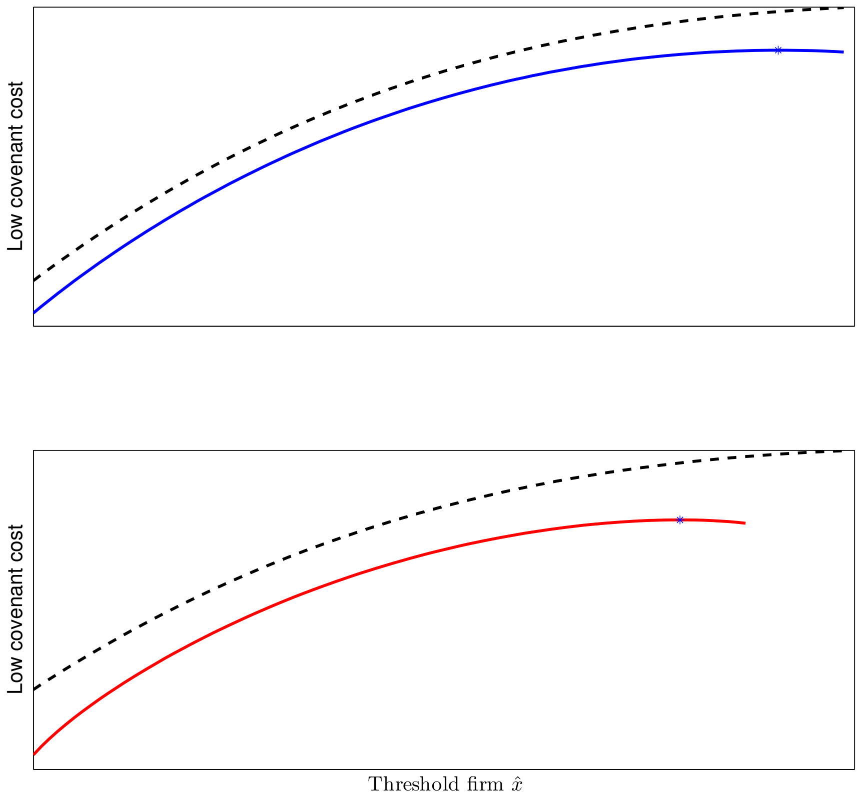

Fig. 2 Pooling equilibrium analysis

Proposition 4 *The equilibrium pooling contract is of the form* $A = [z_a, \tilde{z}], \tilde{D}$ *and it is the solution of the problem:*

$$\min_{z,D,x} F(z|s)R + (1 - F(z|s))D \tag{10}$$

$$(F(z|r) - F(z|s))(R - D) = \hat{x} \tag{11}$$

$$F(z|s)R + (1 - F(z|s))D + M(\hat{x})\hat{x} \geq$$

$$I + y[1 - M(\hat{x})] + c[M(\hat{x})F(z|s) + (1 - M(\hat{x}))F(z|r)] \tag{12}$$

Suppose the equilibrium involves pooling and the cost renegotiation c goes up. Then the cutoff $\hat{x}$ and the covenant strictness $\tilde{z}$ both fall. If the cost to the bank y goes up, then the cutoff and the covenant strictness $\tilde{z}$ will increase.

The intuition behind these results can be demonstrated in Figure 2. We plot (with a black dotted line) the value of a pooling contract with perfect information that instructs all firms above the cutoff to risk-shift. Since risk-shifting is inefficient, clearly this curve is upward-sloping. Next, we plot (with a red solid line) the value of the contract with a covenant. The difference between them is the renegotiation cost. The optimal threshold is the point that maximizes the black curve. In the lower panel, we have the same graph with higher renegotiation costs. Since the increase in renegotiation costs is higher at higher thresholds, the value of the contract with covenants falls *more* at higher threshold values. Therefore the optimal threshold firm is reduced.

3.3.3 Discussion

This section demonstrated that we can analyze a model in which both adverse selection and moral hazard are present. In particular, the firm's unobserved type is their willingness to engage in moral hazard.

The key to analyzing this general model is that the firm with the lowest incentive to risk-shift is the most desirable customer for the banks. So the only contract that is immune from "client-poaching" is the one that maximizes that firm's profit subject to information, incentive, and break-even constraints.

In the separating equilibrium, covenants add value to low-x firms in two ways. First, they keep firms with high incentive to risk-shift away from the low-risk contract. So they are a communication mechanism. However, this signaling role is inextricably linked to their second incentive role - the low-risk contract is more attractive to firms with low-x precisely because it lowers costs by *preventing ex-ante* inefficient behavior of the firms *taking up that contract.*

In contrast, we show that covenants exist in the pooling equilibrium only for their incentive role. In the pooling equilibrium risk-shifting firms pay on average more than other firms. However, the additional costs they bring to the pool are larger than the additional revenues, so they are subsidized by the non-risk-shifting firms. The benefit of covenants is that they reduce the mass of firms in the pool that risk-shift; this eliminates some of the inefficient risk-taking and lowers costs for everybody.

Finally, what determines the choice between the two kinds of equilibria? In the separating equilibrium, the optimal contract does not depend on the distribution of firms M. Since all the constraints in problem P1 are binding, the optimal solution is strictly bounded away from the full-information case. However, for the firms with low x the benefits of separation from risk-shifters depend on the number of firms with high risk. If the right tail of the x-distribution is thin, it is optimal to prevent only firms with moderate x (but the large mass of firms) from risk-taking, since this distorts the optimal contract less.

4 Signal quality and covenant design

The role of borrower accounting quality in debt contracting is an important topic in accounting research (Armstrong et al. (2010)). What is the effect of noise in the accounting signal on covenant design? How does the equilibrium type change when the amount of noise in the signal increases? To answer these questions, we need first to simplify the signal structure and to introduce our measure of noise.[13]

4.1 Measures of noise

The distribution of the signal z can be complicated. However, since any monotone transformation preserves the information in the signal, the signal can be easily normalized as follows.

Lemma 4 *Without loss of generality, we can assume that* $z_a = 0, z_b = 1$ *and* $Z|s \sim Unif(0, 1)$.

[13] We use the terms signal quality or precision, and its inverse 'noise', interchangeably as appropriate.

Proof In Appendix A.

With this normalization, we just need to compare the distribution of the signal when the action $a = r$. The question of informativeness of signals has been extensively studied in Statistics and in Auction Theory with some classic results by Blackwell (1953) and more recent work by Ganuza and Penalva (2010). In our problem, since the signal can be easily normalized, variance or dispersion do not make a signal more or less desirable. Thus, in our model, the signal is only relevant in the information it contains about the action of the firm. This brings us to the following definition of informativeness:

Definition 6 A signal Z_1 is more informative than Z_2 if for all allowable primitives of the model, the payoff of the firm with the lowest x is higher with signal Z_1.

Proposition 5 *A signal Z_1 is more informative than Z_2 if and only if $F_{Z_1}(x|r) \geq F_{Z_2}(x|r)$ for all $x \in [0, 1]$.*

Proof In Appendix A.

Are there economically plausible mechanisms that can generate noise, consistent with our definition? Here we consider two concrete examples in which noise can be introduced to a signal and show that they agree with our definition. In both

4.1.1 White noise

First, suppose that (the correctly measured) signal Z reflects some useful underlying conditions and some temporary and irrelevant information. Specifically, suppose that Y is a random variable that satisfies the assumptions of the model (MLRP) and that W is a random variable that is independent of the action a. Then let the signal Z_α be defined as $Z_\alpha = Y + \alpha W$, $\alpha \geq 0$. W is irrelevant noise that needs to be filtered out. A larger α "drowns out" the valuable signal Y. To simplify the exposition of the proofs, we will assume that Y and W are positive and have continuous densities.[14]

Proposition 6 *Let $\hat{Z}_\alpha$ be the normalized signal Z_α. Suppose that $\hat{Z}_\alpha$ satisfies MLRP. If $\alpha_1 < \alpha_2$, then $\hat{Z}_{\alpha_1}$ is more informative than $\hat{Z}_{\alpha_2}$.*

4.1.2 Random errors

Another mechanism to introduce noise is to assume that with some probability an error occurs and a completely uninformative signal is reported. Let Z_α take the value of Z with probability $1 - \alpha$ and of W with probability α, where Z satisfies the assumptions of the model and W is independent of a. We assume that Z and W have a common support and that $F_W(z) = F_Z(z|s)$.

Proposition 7 *Let $\hat{Z}_\alpha$ be the normalized signal Z_α. Then $\hat{Z}_\alpha$ has a continuous pdf, support $[0, 1]$ and satisfies MLRP. If $\alpha_1 < \alpha_2$, then $\hat{Z}_{\alpha_1}$ is more informative than $\hat{Z}_{\alpha_2}$.*

[14] We assume that $\hat{Z}_\alpha$ satisfies MLRP. For sufficient conditions to ensure that MLRP is preserved, see Shanthikumar and Yao (1986).

4.2 Analysis

What is the effect of noise on the kind of equilibrium that prevails and on covenant strictness? We will consider a family of signals that can be compared in terms of noise (the reverse of informativeness). Let $\alpha \in [0, 1]$ index the family of distributions, which have CDFs $F(z, \alpha|a)$. Higher values of α correspond to more noise. Then Lemma 4 and Proposition 5 imply that we can normalize $F(z, \alpha|s) = z$ and $\frac{\partial}{\partial \alpha} F(z, \alpha|r) < 0$ for all $z \in (0, 1)$. The extreme case $\alpha = 1$ corresponds to no information – $F(z, 1|r) = F(z, 1|s),\ \forall z$.

4.2.1 The contract with known x

First, we consider the case of a known x. We look at the interaction of adverse selection and noise later on in this section. We use Lemma A.2 to characterize the relationship between noise and the contract.

Proposition 8 *Let $z(\alpha)$ denote the optimal covenant strictness. Suppose that $z(0) > 0$. There exists a cutoff signal precision $\bar{\alpha} \in (0, 1)$ such that $z(\alpha)$ is strictly decreasing on $[0, \bar{\alpha})$ and the optimal contract is without covenants for $\alpha \in (\bar{\alpha}, 1]$.*

Proof In Appendix A.

As the noise in the signal increases, covenants need to get increasingly stricter in order to provide the correct incentives. As a result, the expected costs of covenant violations increase (since covenants will be binding even though firms choose action $a = s$). Finally, for sufficiently high level of noise, the no-covenant contract dominates.

4.2.2 Separating equilibrium

We next turn to the effect of noise on the separating equilibrium. As we shall see, noise affects the kind of equilibrium that prevails. Let B be the set of α-s (possibly empty) such that the equilibrium is separating. We consider how noise affects the contract on that set. Then we have the following result, similar to Proposition 8.

Proposition 9 *The optimal covenant strictness $z(\alpha)$ is strictly increasing on B and the cutoff firm $\hat{x}(\alpha)$ is decreasing in α.*

Proof We show that $z(\alpha)$ is strictly increasing in Appendix A. Since $\hat{x} = y - F(z|s)c$, then $\hat{x}(\alpha) = y - z(\alpha)c$ is strictly decreasing since $z(\alpha)$ is strictly increasing.

As before, the covenant needs to become stricter to provide incentives for the firm to choose $a = s$. However, since the value of the no-covenant contract is independent of the noise level, more and more firms will prefer the no-covenant contract, and consequently the number of firms that play s will shrink. Since the risky action is socially inefficient, an increase in the level of noise not only redistributes resources to risk-shifting firms, but also increases the prevalence of value-destroying risky activities.

4.2.3 Separating, pooling, and the no-covenant equilibrium

Finally, what can we say about the equilibrium type as a function of the level of noise? The equilibrium contract selects from the optimal pooling, separating, or the no-covenant contract to maximize the payoff of the x_a firm. In general, the payoff of the pooling contract will be continuous and decreasing, but can be highly non-concave, depending on the distribution of firm types. Thus we employ numerical analysis to study the equilibria. We experiment with a variety of functional forms and parameters and find that our results are robust.[15]

We find that two parameters are crucial: (1) y, the cost of risk-shifting, relative to the distribution of firms x, and (2) the cost c of breaking the covenant. We consider four possible cases.

We present our results in Figure 3. In all cases, the level of noise is on the X-axis. First, we plot the equilibrium type, where 1, 2, and 3 stand for the no-covenant, pooling, and separating equilibria respectively. Underneath we plot the share of firms that choose action $a = s$. If the equilibrium is pooling, then this is the fraction of firms that undertake the safe action. If the equilibrium is separating, this is the fraction of firms that take the 'safe' contract. Next we show covenant strictness for firms who take up contracts with covenants, and the payoff for the safest firm.

Low y, low c In this case, when the noise level is low, the separating equilibrium is optimal. This is because the constraint to separate the bad risks out is easy to satisfy. Then as signal quality gets worse, we need stricter covenants. Since the payoff of the outside option is fixed at $R-I-y$ when the separating contract gets worse, more and more firms choose the risk-taking contract. Finally, as the quality of the signal becomes very bad, there is a switch to the pooling equilibrium as allowing some risk-taking is cheaper than keeping all the bad risks out. The covenant keeps getting stricter, but as signal quality goes down, we see more and more risk-taking. In the end, we get contracts without covenants.

Low y, high c The results are similar to the case above due to the same intuition. However, since the renegotiation cost c is high, the equilibrium switches directly to the no-covenant case, without going through the pooling contract.

High y, low c In this scenario risk-taking is very costly (y is high), but preventing it is relatively nondistorting. As a result, over a large set of signal precision, *all* firms will be incentivized to play $a = s$.

High y, high c We get the same logic as above, but now covenants are more costly. As a result, the switch to the no-covenant occurs at a lower level of noise. Also, the share of firms in the pooled contract that risk-shift is higher.

Ultimately, the relationship between noise and the type of equilibrium is driven by the differential effect of noise on the payoffs of the different kind of equilibria. The separating contract is *all or nothing* in that it isolates the x_a firm completely from the effects of risk-shifting. In other words, it will be preferred when the noise level is low and when the costs of enforcing the covenant are low. As noise increases

[15] In our numerical experiments we focus on single-peaked distributions of firm types. In the example we present, $M(x) = \left(\frac{x-x_b}{x_b-x_a}\right)^\beta$, and the signal z has a linear density $f(z, \alpha|r) = 1 - m(\alpha)/2 + m(\alpha)z$.

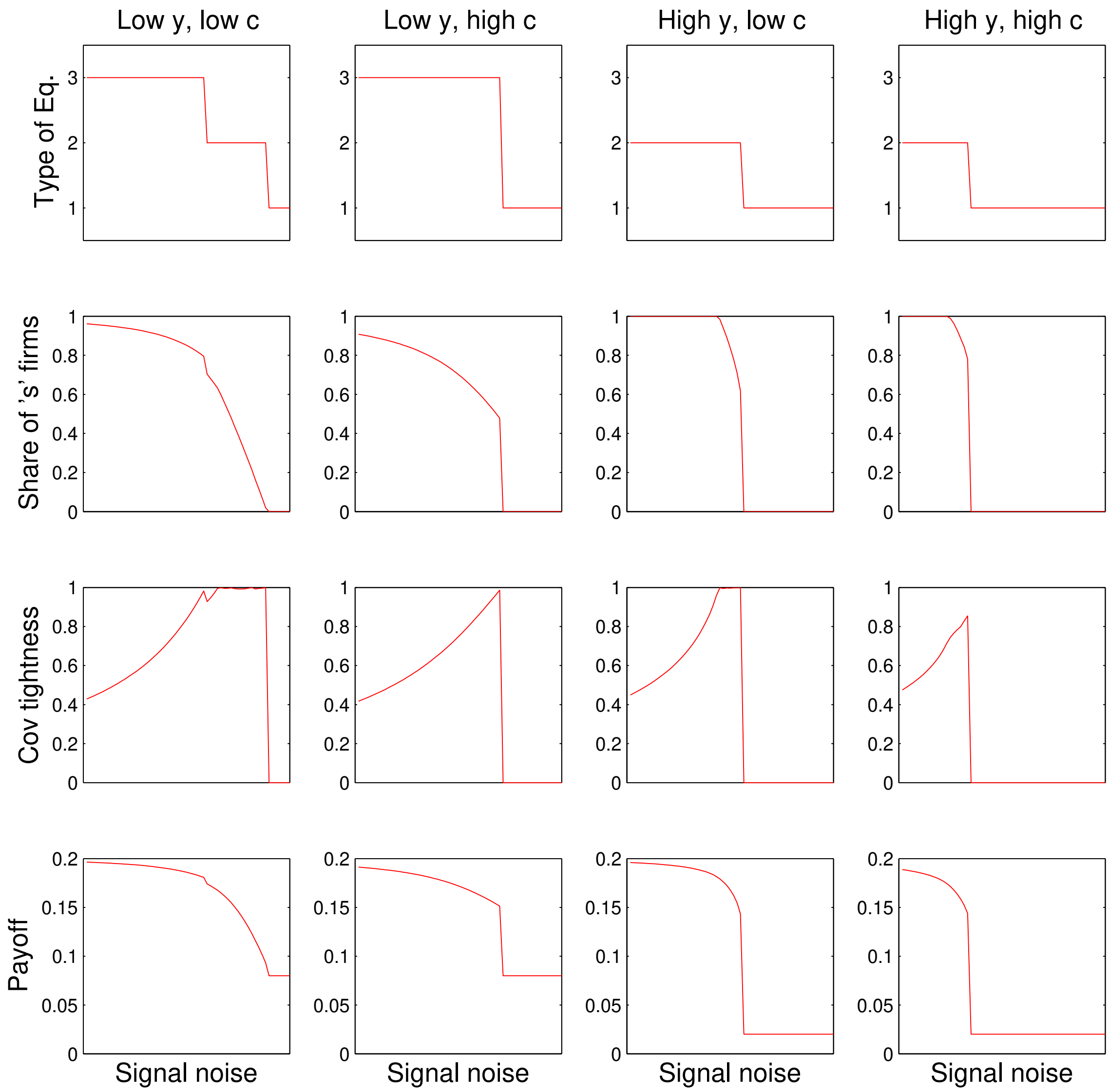

Fig. 3 Equilibrium analysis with respect to noise

the optimal contract shifts to pooling since there are benefits from preventing risk-taking by some of the pooled firms. Finally, the expected costs of covenant violation are so large that the no-covenant contract dominates. Interestingly, we find that contracts without covenants exist for high and low – but not intermediate – levels of signal precision, in the first case since separation of different types of firms is very effective, in the second case since the noise level is so high that no firm benefits from covenants.

4.3 Analytical solution

In this section, we explore a simple special case of the model that allows us to find an explicit solution to the contract. First, we assume that the firm's x is known (either $M(x)$ is degenerate or the equilibrium is pooling and all firms act $a = s$.) Then, the firm chooses the action s if:

$$E[W] - F(\tilde{z}|s)R - (1 - F(\tilde{z}|s))D \geq E[W] - F(\tilde{z}|s)R - (1 - F(\tilde{z}|s))D + x,$$

which is simplified to

$$(F(\tilde{z}|r) - F(\tilde{z}|s))(R - D) \geq x. \tag{13}$$

The bank's break-even constraint now looks like

$$F(\tilde{z}|s)(R - c) + (1 - F(\tilde{z}|s))D \geq I. \tag{14}$$

By the same arguments as in Lemma 7 we can show that all the constraints will be binding.

The second additional assumption that we make is regarding the distribution of the signal Z. As we showed in Lemma 4, we can assume without loss of generality that the signal Z is defined on $[0, 1]$ and $F(z|s) = z$. We assume that $F(z|r)$ is quadratic. In our model $F(z|r) - F(z|s)$ has an inverted U-shape with $F(0|r) - F(0|s) = 0$ and $F(1|r) - F(1|s) = 0$. Finally $F(z|r)$ is increasing with $F(0|r) = 0$, $F(1|r) = 0$. These constraints imply that the CDF $F(z|r)$ has the form:

$$F(z|r) = z + (1 - b)z(1 - z)$$

with $b \in [0, 1)$. Since $F(z|r) - F(z|s) = (1 - b)z(1 - z)$, b corresponds directly to the noisiness of the accounting signal. In other words, lower values of b indicate better accounting quality, while higher values of b indicate relatively poorer accounting quality.

Substituting the functional form for $F(z|a)$ in (13), we obtain

$$D = R - \frac{x}{(1 - b)\tilde{z}(1 - \tilde{z})}.$$

Using the expression above, we rewrite the break-even constraint (14) as:

$$R - \tilde{z}c - \frac{x}{(1 - b)\tilde{z}} \geq I. \tag{15}$$

Since the break-even constraint is binding, the firm's payoff in equilibrium turns out to be $\pi_f = E[W] - I - \tilde{z}^* c)$. Then clearly it is optimal to set $\tilde{z}$ as small as possible, so solving (15) with equality implies:

$$\tilde{z}^* = \frac{R - I - \sqrt{(R - I)^2 - \frac{4xc}{1-b}}}{2c},$$

and the firm's payoff is:

$$\pi_f = E[W] - I - \frac{R - I - \sqrt{(R - I)^2 - \frac{4xc}{1-b}}}{2}.$$

Given this, we can easily calculate the loss from poor accounting quality (higher b):

$$\frac{d\pi_f}{db} = -\left[(R - I)^2 - \frac{4xc}{1 - b}\right]^{-\frac{1}{2}} \frac{xc}{(1 - b)^2} < 0$$

and
$$\frac{d^2\pi_f}{dbdx} = -\frac{c}{(1-b)^2}\left[(R-I)^2 - \frac{4xc}{1-b}\right]^{-\frac{3}{2}}\left((R-I)^2 - \frac{2xc}{1-b}\right) < 0.$$

As expected, improving the quality of the signal (lower b) is beneficial for the firm since providing incentives is cheaper. This benefit is greatest for firms with high x (incentive to risk-shift). Prior research has documented that firm characteristics such as high leverage are associated with poor accounting quality Dechow et al. (2010)). However, firms with high leverage also have greater incentives to risk-shift Jensen and Meckling (1976)). This implication of our model suggests that it is precisely these firms that would benefit most from committing to high accounting quality.

Next, we can turn to the relationship between noise and covenant tightness.

$$\frac{d\tilde{z}^*}{db} = \frac{x}{(1-b)^2}\left[(R-I)^2 - \frac{4xc}{1-b}\right]^{-\frac{1}{2}} > 0$$

$$\frac{d^2\tilde{z}^*}{dbdc} = \frac{2x^2}{(1-b)^3}\left[(R-I)^2 - \frac{4xc}{1-b}\right]^{-\frac{3}{2}} > 0$$

$$\frac{d^2\tilde{z}^*}{db^2} = \frac{2x}{(1-b)^3}\left[(R-I)^2 - \frac{4xc}{1-b}\right]^{-\frac{3}{2}}\left[(R-I)^2 - \frac{3xc}{1-b}\right] > 0.$$

All else being equal, firms with higher signal noise have tighter covenants. The intuitive argument behind this result is that when the signal is more accurate at indicating action $a = r$, less strictness is required to prevent risktaking. The effects of noise on optimal covenants increase with higher bank covenant violation costs (and though unmodeled here, bankruptcy cost).

5 Model of Risk-shifting

In this section we present a richer model that derives the private benefit x of action $a = r$ and the lender cost y from primitives. To maintain tractability here we assume that there is no ex ante private information, that is all firm characteristics, except the action it takes, are known.

The firm's management takes an unobservable action $a \in \{r, s\}$ which is the choice whether to conduct business in a safe (s) or risky (r) manner. The effect of the action is to affect the distribution of the cash flow W, which is described by the density function $h(W|a)$ and the CDF $H(W|a)$ and has support $[\underline{W}, \overline{W}]$. We focus on a traditional notion of risk-shifting – the expected value of cash flows ($EW|a$) is the same for any choice of action, but the risky action increases volatility (in a manner to be made precise below).[16] Given this setup, as Merton (1974) points out, the equity claim is a call option with a strike price equal to the face value of the debt, and thus it increases in the volatility of the underlying cash flows. To make the risky action $a = r$ socially undesirable, we allow for costly bankruptcy in the event of a default. In what follows we model the incentives for risk-shifting more explicitly.

Let $v(W) \equiv h(W|r)/h(W|s)$ be the ratio of the two densities. We assume that v is strictly decreasing on $[\underline{W}, W^*)$ and strictly increasing on $(W^*, \overline{W}]$. Thus, the risky action shifts the probability towards the

[16] The value of preventing the risky action is strengthened if the risky action lowers the mean of cash flows in addition to the increase in volatility.

extremes. Then $\int_{\underline{W}}^{\overline{W}} Wh(W|r)dW = \int_{\underline{W}}^{\overline{W}} Wh(W|s)dW$ implies that $v(\underline{W}) > 1, v(W^*) < 1$ and $V(\overline{W}) > 1$. Next, we collect some technical assumptions on the distribution $W|a$:

Assumption 4 *The probability density function $h(W|a)$ is continuously differentiable, $h(W|a) > 0$ for all $W \in [\underline{W}, \overline{W}], a \in \{r, s\}$ and the hazard rate $h(W|a)/(1 - H(W|a))$ is strictly increasing in the cash flow W.*

We assume that the financial contract between the firm and the lender is in the form of debt. The justification for this assumption is that it is costly to observe the cash flows (Townsend (1979)) or that debt is necessary to provide incentives for the manager (Jensen and Meckling (1976), Cole (2013) and others).

In our framework bankruptcy is triggered if the random cash flow W falls below the debt value, D. Bankruptcy is costly; for tractability we allow for fixed distress costs to the firm γ_f and to the bank γ_b. In our model, we don't consider covenant renegotiation costs directly and all the losses when covenants are violated emerge from higher bankruptcy probability. Adding renegotiation costs will not affect, but actually strengthen our results.

The last element of the model is the signal Z that is informative of the firm's action. We maintain all the assumptions made in Section 2.

5.1 Equilibrium without covenants

We start with exploring the incentives of the firm to risk-shift, and the net cost of risk-shifting. This outcome will serve as a benchmark to examine the value added by covenants in the contract.

Let's define $EP(D, a)$ to be the expected payment from the firm to the bank. Since the cash flow has a pdf $h(W|a)$ and support $[\underline{W}, \overline{W}]$,

$$EP(D, a) = \int_{\underline{W}}^{D} Wh(W|a)dW + (1 - H(D|a))D.$$

The expected payment is affected by the debt level and the firm's action. The firm's payoff is:

$$\pi_f(D, a) = EW - EP(D, a) - H(D|a)\gamma_f.$$

The bank's net revenue is

$$\pi_b(D, a) = EP(D, a) - H(D, a)\gamma_b.$$

What are the firm's incentives for risk-shifting? The change of the firm's payoff if it changes action from s to r is given by

$$\Delta\pi(D) \equiv (EP(D, s) - EP(D, r)) - (H(D|r) - H(D|s))\gamma_f.$$

The first term represents the gain of the option value coming from the increase in volatility and hence is always positive; the second term corresponds to the increased probability of costly bankruptcy. The optimal action for the firm, risky or safe, depends on which effect is stronger.

As we show in the following lemma, the second term (an increase in bankruptcy costs) dominates the first one when the face value of the debt D is small.

Lemma 5 *There exist some cutoffs, D_1 and D_2, such that $\underline{W} < D_1 < D_2$ and $\Delta\pi(D) < 0$ if $D \in (\underline{W}, D_1)$ and $\Delta\pi(D) > 0$ if $D \in (D_2, \overline{W}]$.*

Proof In Appendix A.

Under some additional conditions, the two cutoffs are the same: $D_1 = D_2 \equiv D_r$. The banks can predict the behavior of the firm, so given the debt level the firm's action is known. Let $\hat{D}(a)$ be the minimum face value of the debt that is sufficient to ensure the bank breaks even, conditional on action a: $\hat{D}(a) = \min\{D : \pi_b(D, a) \geq I\}$.

If the break-even payment conditional on the safe action is below the risk-shifting threshold, the action $a = s$ can be attained; otherwise risk-shifting will occur.

Proposition 10 *The optimal contract is given by:*

$$(a^*, D^*) = \begin{cases} (s, \hat{D}(s)) \text{ if } \hat{D}(s) \leq D_r \\ (r, \hat{D}(r)) \text{ Otherwise.} \end{cases}$$

Proof In Appendix A.

What are the costs and benefits of risk-shifting? Let's denote the firm's gain from shifting from s to r by $x \equiv \Delta\pi(D^*) = (EP(D^*, s) - EP(D^*, r)) - (H(D^*|r) - H(D^*|s))\gamma_f$. Similarly, we can define the cost to the bank by $y \equiv \pi_b(D, s) - \pi_b(D, r) = EP(D^*, s) - EP(D^*, r) + (H(D^*|r) - H(D^*|s))\gamma_b$.

The net cost of the risky action is then $y - x = (H(D^*|r) - H(D^*|s))(\gamma_f + \gamma_b)$, which is the increase in expected cost of distress. For some parameter values $y < x$, implying that the action r is efficient. When the level of debt is high and bankruptcy is almost inevitable, the risky action *reduces* the likelihood of bankruptcy since it increases the probability that the revenues will be high enough to avoid bankruptcy.

It is easy to show that if $y - x < 0$, then $x > 0$. In other words whenever the action r is efficient, the firm will have an incentive to take it. In the rest of this article, we focus on the case that the action s is efficient, but simple debt cannot attain it: $y - x > 0$, but $x > 0$.

5.2 Contract with financial maintenance covenants

In the earlier section we explored the details of risk-shifting with a debt contract without covenants. Next, we introduce covenants in the same fashion as in Section 2.

Definition 7 A *debt contract* is a pair of base payment D and a covenant trigger level $\tilde{z}$. If the realization of the signal Z falls in the interval $[z_a, \tilde{z}]$, the loan is renegotiated with the bank making a take-it-or-leave-it offer.

In our model, in the post-violation renegotiation the bank demands a new level of debt repayment that maximizes its payoff. At this point, the bank's first-order condition is given by

$$\frac{d}{dD}\pi_b(a,D) = -\gamma_b h(D|a) + (1 - H(D|a)) = (1 - H(D|a))\left[1 - \gamma_b \frac{h(D|a)}{1 - H(D|a)}\right] = 0,$$

where $h(D|a)/(1 - H(D|a))$ is the hazard rate of W. We restrict our analysis to cases where the hazard rate is increasing, which is satisfied for most commonly used distributions (uniform, normal, truncated normal, gamma). Hence the bank's profit is strictly concave and is maximized at some level $D(a)^* < \overline{W}$, where a refers to the action that the firm took. Importantly, since the action a is unobservable, the bank uses its *belief* of what the firm action was when renegotiating.[17]

Given this setup, the payoff of the firm with a contract $(\tilde{z}, D)$, action a, and bank's belief of the firm's action $\hat{a}$ is

$$F(\tilde{z}|a)\pi_f(D^*(\hat{a}), a) + (1 - F(\tilde{z}|a))\pi_f(D, a)$$

In a pure strategies equilibrium, the firm's action and the bank's belief coincide, so we can write the incentive constraint as

$$(F(\tilde{z}|r) - F(\tilde{z}|s))(\pi_f(D, s) - \pi_f(D^*(s), s)) \geq F(\tilde{z}|r)\Delta\pi(D^*(s)) + (1 - F(\tilde{z}|r))\Delta\pi(D). \tag{16}$$

The left-hand side is the cost of the action r to the firm: an increase in the probability of covenant violation leads to a higher loan spread (cost of debt). The right-hand side is the weighted sum of the benefits of risk-shifting, conditional on the debt level.

Next, we consider the bank's break-even constraint. It is given by:

$$F(\tilde{z}|a)\pi_b(D^*(a), a) + (1 - F(\tilde{z}|a))\pi_b(D, a) \geq I, \tag{17}$$

where a is the equilibrium firm's action (which coincides with the bank's belief).

In what follows, we look at cases when we want to induce behavior $a = s$. The loan contract maximizes the value of the firm subject to the incentive and break-even constraints.

$$\max_{\tilde{z}, D} F(\tilde{z}|s)\pi_f(D^*(s), s) + (1 - F(\tilde{z}|s))\pi_f(D, s), \quad s.t.(16), (17).$$

Note that even though the "punitive" payment $D^*(s)$, the new value of the debt if the covenant is broken, affects the problem both directly and through the constraints, it cannot be chosen in advance: it must be consistent with the bank's incentives and beliefs. For some parameter values the firm's value is increased if the punitive payment is $\overline{W}$, but this is not attainable.

Thus the financial maintenance covenant is beneficial in that it prevents inefficient risk-taking *ex-ante*. Expectation of higher debt payment provides the incentive for the firm to take the safe action. However, the increase in debt payment also leads to a higher probability of bankruptcy in the event of a covenant violation. Thus, the covenant contract adds value when the first effect dominates the second.

[17] We can also impose the restriction $D \leq D(s)^*$, because the bank will unilaterally reduce D, that is a contract with $D > D(s)^*$ is not renegotiation proof. As we will show, if $D \geq D^*(s)$, the incentive constraint can't be satisfied, so renegotiation-proofness follows from incentive compatibility.

Lemma 6 *Suppose that $\hat{D}(s) \leq D_r$. Then the optimal contract is without covenants: $(\emptyset, \hat{D}(s))$.*

Proof In Appendix A.

Simply put, if the firm has an incentive to take the safe action, then even without any cost to writing or enforcing covenants, they will not be used.

Next, we make two additional assumptions that greatly simplify the analysis, by implying that both constraints are binding at the optimum.

Assumption 5 *The cost of bankruptcy for the firm $\gamma_f = 0$.*

The face value of the debt that satisfies break-even condition if there are no covenants, $\hat{D}(r)$ satisfies $\hat{D}(r) \leq W^*$.

The first part of the assumption states that all the costs of bankruptcy fall on the holders of debt (the bank). Intuitively, this assumes that the previous owners of the firm get wiped out, the bank becomes the new owner and any costs of bankruptcy reduce the bank's payoff. The second part of the assumption implies that the face value of the debt without covenants will not be so high so that reducing the debt level *increases* the incentives for risk shifting.

Lemma 7 *Suppose that $\hat{D}(s) > D_r$ and assumption 5 holds. Then the optimal contract that induces the action $a = s$ is with a covenant and the constraints* (16) *and* (17) *are binding.*

Proof In Appendix A.

The following Lemma 7 simplifies the constraint set and is our main tool to providing comparative statics results for the contract. Before doing so, we need the following technical assumption.

So far we have implicitly imposed the assumption that the financial maintenance covenant is broken if the realization of the signal z falls below a certain threshold $\tilde{z}$. With this assumption, we can prove that the optimal form of the covenant set is an interval: $[z_a, \tilde{z}]$. The proof is an extension of the proof of the lemma above and is also in Appendix A.

Proposition 11 *The optimal covenant tightness $\tilde{z}^*$ increases if (i) the loan amount I increases; (ii) the bank's cost of bankruptcy γ_b increases; (iii) there is a mean-preserving spread in the distribution of $W|r$.*

Proof In Appendix A.

In cases (i) and (iii), the benefit of riskshifting increases, which necessitates a tighter covenant. In case (ii) the maximum amount of extra spread $D^*(s)$ the bank is willing to ask, is reduced because the bank internalizes the extra cost of bankruptcy. As a result the incentives are weaker, requiring a tighter covenant.

6 Performance manipulation

If the firm has some control over the signal z then a bank loan with covenants written on z creates an obvious incentive to manipulate the accounting signal. How does the optimal contract change when we consider the possibility of manipulation? In this section, we extend our model to allow for strategic misreporting.

The extension uses the setting with known firm type x introduced in Section 4.3. We make one additional assumption. With probability m, after the signal z is realized, the firm can choose to manipulate, that is report a value of the signal z of its own choosing.[18]

The firm's management may be unable to manipulate due to the strength of internal controls; similarly it may be unwilling to manipulate due to legal concerns or the fear of being found out and suffering reputation damage, or enforcement action by a regulator. Intuitively, $1 - m$ is an index of the strength of these forces.

Let's suppose that the contract has a covenant. If the signal realization is above the cutoff $\tilde{z}$, the firm has no incentive to misreport. On the other hand, if the signal z falls below the cutoff, the firm will want to misreport, given the opportunity, *even if it took the safe action.* In this case, the exact value of the report z is not pinned down, but the report will be equal to or exceed $\tilde{z}$.

From the firm's optimal reporting strategy it follows that the incentive constraint now looks like:

$$E[W] - F(\tilde{z}|s)(1-m)R - (1 - F(\tilde{z}|s))D - mF(\tilde{z}|s)D \geq$$
$$E[W] - F(\tilde{z}|r)(1-m)R - (1 - F(\tilde{z}|r))D - mF(\tilde{z}|r)D + x,$$

Straightforward rearranging of the constraint above yields:

$$(1-m)(F(\tilde{z}|r) - F(\tilde{z}|s))(R - D) \geq x, \tag{18}$$

while the lender's break-even constraint is:

$$D + (1-m)F(\tilde{z}|s)(R - D - c) \geq I. \tag{19}$$

As we discussed above, if the firm manipulates, it is indifferent between reports on the interval $[\hat{z}, z_b]$. Inspecting the constraints above, we can see that without loss of generality we can assume that the report is z_b. In this case, manipulation induces a distribution of reported z exactly the same as from the random errors model of noise from Section 4.1.2. Thus the ability to manipulate is isomorphic to a particular kind of noise.

Proposition 12 *Suppose that for some $m > 0$ the optimal contract is with a covenant. There exists a cutoff level of manipulation probability $\bar{m} \in (0, 1)$, such that the optimal contract is with covenants for*

[18] Laux (2022) uses a similar information structure. An alternative approach is to allow manipulation, but make it costly. This is explored by Guttman and Marinovic (2018). In principle, we can also consider the size of the possible reporting manipulation (nudging z just a little bit over the cutoff value may be easier than manipulating a lot). The model can be extended so that, in addition to m, the firm is characterized by the parameter a, which is the maximum amount the firm is able and willing to manipulate the signal z. The results, in this case, will be substantially similar, so we abstract from this complication.

$m \in [0, \bar{m})$ and without covenants for $m \in (\bar{m}, 1]$. Over $[0, \bar{m})$, the optimal $\tilde{z}$ and $(1-m)\tilde{z}$ are increasing in m.

Proof In Appendix A.

Overall, our findings in this section confirm the similarity of the manipulation channel to noise. In particular, when the manipulation probability is large, the observed signal becomes so uninformative that covenants lose value. However, just as noise is not infinite, manipulation is neither unlimited nor cost-free. Covenants add value (and will be used) for sufficiently low values of m. The cutoff value $\tilde{z}$ increases with m (since providing incentives is now harder). Moreover, this cutoff value is increased enough so that the probability that the covenant will be binding increases, despite the fact that the firm can manipulate the signal, in parallel to the result on noise (proposition 8).

7 Conclusion

The existing theoretical literature on the design of debt covenants has focused to a large extent on their role in signaling borrower's type and the nature of the ex-post investment, allowing control rights, and providing incentives to the financial intermediary to monitor the borrower. In this article, we focus on the design of financial maintenance covenants in debt contracts under moral hazard and adverse selection. We incorporate asymmetric information along three dimensions: the firm's type, the firm's unobservable action, and the relationship between the accounting signal and the firm's action.

We show that financial maintenance covenants based on accounting ratios help decrease moral hazard, and that their signaling role is not a necessary condition for their existence. Our model can characterize optimal covenants as a function of the firm's incentive to risk shift, the debt amount, the cost of renegotiation, and more importantly, and accounting quality (or noise in the public signal).

In our analysis, we pay particular attention to this latter variable, as technical default of the financial maintenance covenant is based on the actual value of the accounting signal generated by the firm. In an environment where accounting signals (financial ratios) are often estimated with error and subject to manipulation, we find that the extent of noise in the accounting signal changes the nature of the equilibrium contract and has a profound impact on covenant design. Specifically, we find that increasing the level of noise moves the equilibrium from pooling to separating, and has a non-monotone effect on covenant strictness. As accounting signals become noisier (i.e. accounting quality reduces), covenant strictness increases to maintain the correct incentives for the borrower. However, for a large enough level of noise there is an abrupt reversal and the optimal contract has no covenants.

We conclude by discussing the role of the simplifying assumptions in our model. The first issue is with respect to risk-neutrality. If the firm faces future credit constraints, then the value it will place on current cash flows will be nonlinear. Moreover, the value of risk-shifting x will depend on the firm's net cash flows. Formally, this model would be isomorphic to a costly state verification model with errors. Additionally, there will be a trade-off between risk-sharing (which would minimize the variance of payments) and incentives (which would maximize them). The optimal contract will therefore imply more renegotiation and smaller changes in terms of the contract. However, risk aversion does not eliminate the necessity of providing incentives, hence the mechanism central to our study is still in action.

A second question concerns the renegotiation process. In our model the constraints on the bank's renegotiation strategy are encapsulated by R - the maximum amount that can be extracted after a covenant violation. Since we do not consider repeated interaction, the bank's best response is to extract all it can. Ultimately R is derived from the option of the firm to refinance its debt with a new lender. So R depends on the cost of switching lenders and will be affected by size of the firm, the relative disadvantage of new lenders (due to the incumbent lender's familiarity with the firm), and the degree of competition between banks. Higher ex-post competition will reduce the power of the contract to provide incentives. Similar results have been derived in other contracting environments, such as by Krueger and Uhlig (2006). We explore this issue in Appendix B.

Third, if we explicitly model the duration of the loan, then some of the details of incentive provision will change. In particular, the need for continuing financing may moderate the firm's incentive to risk-shift. In addition, profits from a continuing relationship may also benefit the bank which implies that they will be more lenient in renegotiation, worsening incentives ex-ante. Since these issues do not change the basic mechanism of financial covenants that we explore in this paper, we leave them for future research.

Lastly, the terms of the contract may also be conditioned on public information about conditions affecting the firm's industry or the macroeconomy. In economic downturns firm-level variables become more volatile, so they are probably less informative of its private information. Hence, an optimal state-contingent covenant should be less tight during recessions. On the other hand, the lender may optimally want to be more conservative in recessions. We consider the question why loan covenants do not have stage contingency an interesting open question for additional research.

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A Proofs

We prove 2 technical results first.

Lemma A.1 *Let A be any Borel-measurable set $A \subseteq [z_a, z_b]$. Define set $A' = [z_a, z']$, where z' is the unique solution of the equation $F(z'|s) = Prob(A|s)$. Then for all positive constants k_1, k_2, $k_1 Prob(A'|r) - k_2 Prob(A'|s) \geq k_1 Prob(A|s) - k_2 Prob(A|r)$. Moreover, if $Prob(A \triangle A'|s) > 0$, then the inequality is strict.*

Proof Consider the problem:

$$\sup_{m(z)} \int_{z_a}^{z_b} m(z)[k_1 f(z|r) - k_2 f(z|s)]dz$$
$$m(z) \in [0,1], m(z) \text{ is a measurable function}$$

A necessary condition for this problem is that the Gateaux derivative satisfies:

$$k_1 f(z|r) - k_2 f(z|s)] \begin{cases} \geq 0 & \text{if } m(z) = 1 \\ = 0 & \text{if } m(z) \in (0,1) \\ \leq 0 & \text{if } m(z) = 0. \end{cases}$$

Then $m(z) = 1$ if $g(z) > k_2/k_1$ and $m(z) = 0$ if $g(z) < k_2/k_1$. MLRP implies that $m(z) = 1$ for $z \in [z_a, \hat{z})$ and $m(z) = 0$ for $z \in (\hat{z}, z_b]$, where $g(\hat{z}) = k_2/k_1$ is unique.

There will be three cases to consider, depending on the relationship of Prob$(A|s)$ and $F(\hat{z}|s)$.

First, assume that $Prob(A|s) < F(\hat{z}|s)$. We consider the following problem:

$$\sup_{m(z)} \int_{z_a}^{z_b} m(z)[k_1 f(z|r) - k_2 f(z|s)]dz$$
$$\text{subject to } \int_{z_a}^{z_b} m(z) f(z|s)dz \leq Prob(A|s)$$
$$m(z) \in [0,1], m(z) \text{ is a measurable function.}$$

If we add the constraint that $m(z) \in \{0,1\}$, we will be looking at the set A' that maximizes $k_1 Prob(A'|r) - k_2 Prob(A'|s)$, subject to the constraint. We show that at the optimum $m(z)$ is either 1 or 0.

This is a convex programming problem with a nonempty, open constraint set, so by Theorem 1 in Luenberger (1969), page 217, there exists some $\lambda \geq 0$ such that the optimal solution maximizes:

$$L(m(z), \lambda) = \int_{z_a}^{z_b} m(z)[k_1 f(z|r) - k_2 f(z|s)]dz - \lambda \int_{z_a}^{z_b} m(z) f(z|s)dz.$$

$\lambda = 0$ is impossible, since it would be then optimal to set $m(z) = 1$ a.s. if $z \leq \hat{z}$ and zero otherwise, so the constraint cannot be satisfied.

We take Gateaux derivatives, and we know that necessary conditions for optimality are the following:

$$D_{m(z)} L(m(z), \lambda) = k_1 f(z|r) - k_2 f(z|s) - \lambda f(z|s) \begin{cases} \geq 0 & \text{if } m(z) = 1 \\ = 0 & \text{if } m(z) \in (0,1) \\ \leq 0 & \text{if } m(z) = 0. \end{cases}$$

MLRP implies that either $D_{m(z)}L > 0$ for all $z \in [z_a, z_b)$, $D_{m(z)}L < 0$ for all $z \in (z_a, z_b]$, or $D_{m(z)}L > 0$ for all $z \in [z_a, z')$ and $D_{m(z)}L < 0$ for all $z \in (z', z_b]$. The first case is inconsistent with the constraint; the second case contradicts the assumption for $Prob(A)$. So at the optimum $m(z) = 1$ for $z \in [z_a, z')$ and $m(z) = 0$ for $z \in (z', z_b]$. The condition that $F(z'|s) = Prob(A|s)$ follows from the binding constraint. This proves the statement for all sets A such that $Prob(A'|s) < Prob([z_a, \hat{z}]|s)$.

When Prob$(A|s) > F(\hat{z}|s)$, the proof is handled similarly, but the constraint is $\int_{z_a}^{z_b} m(z) f(z|s)dz \geq Prob(A|s)$. Finally, in the case of inequality, we solve the unconstrained problem.

The last part of the proof follows from the fact that a necessary condition for the maximization is that the condition above holds a.s.

Lemma A.2 *Suppose that the firm type x is known and that the action recommendation is $a = s$. If the constraint set is nonempty, there exists an optimal contract. Moreover, at the optimum, $A = [z_a, z^*]$, $D = R - x/[F(z^*|r) - F(z^*|s)]$, and z^* is the smallest z such that $R - (1 - F(z|s))x/[F(z^*|r) - F(z^*|s)] \geq I + F(z|s)c$. At the optimum, the inequality binds.*

Proof The proof is in 4 steps.

1. Without loss of generality, we impose the additional restriction that $A = [z_a, \tilde{z}]$ for some $\tilde{z}$.
   Take any contract that satisfies incentive compatibility and bank's break-even condition. Then by lemma A.1, if we set $A' = [z_a, z']$ where $F(z'|s) = Prob(A|s)$, the expected payment by the firm is the same as before, the bank still breaks even, and the incentive constraint is (weakly) strengthened. Rewriting the incentive constraint for $A = [z_a, \tilde{z}]$, we get the following form of the incentive constraint:

$$(D^* - D)[F(z|r) - F(z|s)] - x \geq 0.$$

2. The incentive constraint is binding.
   Take an arbitrary contract $([z_a, \tilde{z}], D)$ such that the incentive constraint is not binding. Define $D(z)$ implicitly by:

$$F(z|s)D^* + (1 - F(z|s))D(z) = F(\hat{z}|s)D^* + (1 - F(\hat{z}|s)D.$$

   $D(z)$ is well-defined and strictly decreasing in z. Define $v(z)$ by:

$$v(z) = (D^* - D(z))[F(z|r) - F(z|s)] - x.$$

   The contract $([z_a, z], D(z))$ will be incentive-compatible if and only if $v(z) \geq 0$. By assumption $v(\hat{z}) > 0$ and $v(0) = -x < 0$. v is continuous, so $\tilde{z}' = \min\{z : v(z) \geq 0\}$ is well-defined. The contract $([z_a, \tilde{z}'], D(\tilde{z}'))$ satisfies incentive compatibility, keeps the firm as well off as before and relaxes the break-even constraint.
3. The break-even constraint is binding. From 1 and 2, we know that $A = [z_a, \hat{z}]$ and $D = D^* - x/[F(\tilde{z}|r) - F(\tilde{z}|s)]$. Let $v_2(z)$ be the expected payment by the firm, and $v_3(z)$ be the expected profit by the bank:

$$v_2(z) = D^* - (1 - F(\tilde{z}|s))\frac{x}{F(\tilde{z}|r) - F(\tilde{z}|s)}$$

$$v_3(z) = v_2(z) - I - F(\tilde{z}|s)c.$$

   Assume that for some contract $v_3(\hat{z}) > 0$. Clearly, $\lim_{z \to z_a} v_2(z) = -\infty$, therefore $\lim_{z \to z_a} v_3(z) = -\infty$. Then from continuity of v_3 for some $z' \in (z_a, \hat{z})$, $v_3(z') = 0$. If $v_2(z') \geq v_2(\tilde{z})$, then $v_3(z') > v_3(\tilde{z}) > 0$; so $v_2(z') < v_2(\tilde{z})$. Thus we reduced the firm's expected payment and kept all the constraints, which is a contradiction.
4. There exists some $\hat{z} > z_a$ such that $R - x(1 - F(z|s))/[F(z|r) - F(z|s)] < I + F(z|s)c$ for all $z \in (z_a, \hat{z}]$.
   Clearly $\lim_{z \to z_a}(1 - F(z|s))/[F(z|r) - F(z|s)] = \infty$, so there exists some $\hat{z}$ such that for all $z_a < z \leq \tilde{z}$, $(1 - F(z|s))/[F(z|r) - F(z|s)] > (R - I)/x$. Then clearly for all $z_a < z \leq \hat{z}$, $R - x(1 - F(z|s))/[F(z|r) - F(z|s)] < I < I + F(z|s)c$.
5. An optimum contract exists if the constraint set is nonempty.
   In part 3 of the proof, we show that if a contract does not satisfy the break-even condition, then we can strictly reduce the firm's expected payment. If a contract does not satisfy 1 or 2, then we can find a variation that relaxes the break-even condition and keeps the firm's payment constant; since this implies that further modification of the contract will reduce the firm's expected payment this is a contradiction. Therefore 1 and 2 are necessary conditions. Then 1, 2, and 3 imply that $A = [z_a, \tilde{z}]$, $D = R - x/[F(\tilde{z}|r) - F(\tilde{z}|s)]$, and $\tilde{z}$ is such that $v_3(z) = 0$.
   Condition 3 implies that expected payment by the firm is $I + F(\tilde{z}|s)c$, so it is optimal to choose the lowest $\tilde{z}$ such that $v_3(\tilde{z}) = 0$. Condition 4 implies that such smallest $\tilde{z}$ exists.

Lemma A.3 *For any* $z \in [z_a, z_b]$,

$$\frac{1 - F(z|s)}{f(z|s)} \geq \frac{1 - F(z|r)}{f(z|r)}$$

and the inequality is strict if $z \neq z_b$.

Proof For any z, we have that

$$1 - F(z|s) = \int_z^{z_b} f(w|s)dw = \int_w^{z_b} f(w|r)\frac{f(w|s)}{f(w|r)}dw \geq \int_z^{z_b} f(w|r)\frac{f(z|s)}{f(z|r)}dw = (1 - F(z|r))\frac{f(z|s)}{f(z|r)}.$$

(The inequality follows from the fact that z satisfies MLRP.) Strict MLRP implies that the inequality is strict if $z < z_b$. Rearranging, we get the desired result.

Proof (Proof of Proposition 1)

Suppose that $\hat{x} > x_a$. Let $x < \hat{x}$. Then by definition of supremum there exists $x' \in (x, \hat{x}]$ such that firm with x' (weakly) prefers playing s. Let $pf(x, a)$ be the optimal payoff for a firm of type x with action a. Since

$$pf(x, r) = x - x' + pf(x', r) < pf(x', r) \leq pf(x', s) = pf(x, s),$$

firm x strictly prefers action s. So all firms with $x \in [x_a, \hat{x})$ prefer action s.

On the other hand, suppose that $\hat{x} < x_b$. By definition no firm with $x > \hat{x}$ prefers s weakly, therefore all firms with $x \in (\hat{x}, x_b]$ prefer action r. This proves claim 1 of the proposition.

Suppose $\hat{x} = x_a$ and an equilibrium exists. If we add the contract $(\emptyset, I + y)$, it will break even no matter what kind of firms take it up. For any contract, almost all firms that take it up will risk-shift. Then the expected payment from the contract must be greater or equal to $I + y$. Then the only contract taken up by a positive mass of firms will have expected payment of $I + y$. Finally, such a contract will break even only if there is no covenant.

Now suppose that $\hat{x} = x_b$. By individual rationality, expected payment when playing s must be equal among all contracts taken up by firms. Suppose that for one of those contracts and some $x^* < x_b$ playing r is weakly preferred. Then all firms with $x \in (x^*, x_b]$ will strictly prefer to take up this contract and play r. This contradicts the assumption that $\hat{x} = x_b$. So all equilibrium contracts provide incentives for firms with x_b to play s and break even. The optimal contract that satisfies those restrictions is derived in Lemma A.2.

Proof (Proof of lemma 1) Suppose (A, D) satisfies the constraints. Define $\tilde{z}$ by $F(\tilde{z}|s) = Prob(A|s)$. By lemma A.1, switching to contract $([z_a, \tilde{z}], D)$ does not affect the objective function and (5) and (weakly) strengthens (6).

Next, we show that if one or both of the constraints are slack, there is a variation that will increase the objective function.

1. $\tilde{z} < z_b$.
   If, on the contrary, $\tilde{z} = z_b$, the contract $([z_a, z], 0)$ will satisfy all the constraints for some z sufficiently close to z_b and it will increase the objective function.
2. Assume both constraints are slack. Then we can reduce D until some constraint binds and lower the objective function.
3. Assume that (6) is slack and (5) is binding. $\tilde{z} = z_a$ will be impossible, since then (5) will be slack. Then it will be possible to reduce $\tilde{z}$ and change D in a way to keep both the constraints satisfied. From the binding (5) constraint, it follows that the objective function is $R - I - F(\tilde{z}|s)c$. Then the objective function will be increased by the variation.
4. Finally, assume that (5) is slack and (6) is binding. Consider increasing z and decreasing D. From the implicit function theorem applied to (6), we get that:

$$\frac{dD}{d\tilde{z}} = -\frac{f(\tilde{z}|r)(D^* - D)}{1 - F(\tilde{z}|r)}.$$

   Then the change in expected payment from this variation is given by:

$$(D^* - D)\left[f(\tilde{z}|s) - \frac{1 - F(\tilde{z}|s)}{1 - F(\tilde{z}|r)}f(\tilde{z}|r)\right].$$

We know that $D \leq D^*$ and that $D = D^*$ is impossible (both constraints would be slack), therefore $D^* - D > 0$. Lemma A.3 implies that the expression in the parenthesis is negative. Therefore this variation reduces the expected payment.

The objective function is clearly continuous in D and $\tilde{z}$. To show the existence of an optimum it is sufficient to show that, without loss of generality, we can restrict z, D to a compact set. From (5) and (6), we can ensure that

$$D \geq h(\tilde{z}) = \max\left\{\frac{I + F(\tilde{z}|s)c - F(\tilde{z}|s)D^*}{1 - F(\tilde{z}|s)},\ \frac{I + y}{1 - F(\tilde{z}|s)}\right\}.$$

From (6), and plugging in the objective function, we get that:

$$F(\tilde{z}|s)D^* + (1 - F(\tilde{z}|s))D \geq R[F(\tilde{z}|s) - F(\tilde{z}|r)] + \frac{1 - F(\tilde{z}|s)}{1 - F(\tilde{z}|r)}(I + y) \geq D^*[F(\tilde{z}|s) - F(\tilde{z}|r)] + \frac{f(\tilde{z}|s)}{f(\tilde{z}|r)}(I + y).$$

It follows that for $\tilde{z}$ sufficiently large, call it z_1, (5) will not be binding. However, we showed that if (5) is not binding, we can find a new allocation in which the constraints are binding and the objective function is reduced. Then without loss of generality, we can set the constraint set to be

$$M = \{(\tilde{z}, D) : \tilde{z} \in [z_a, z_1], D \in [h(\tilde{z}), D^*], (5), (6) \text{ are satisfied.}\}$$

M is clearly compact, so P1 has a minimum. Moreover, at the minimum, the constraints cannot be slack, because we would be able to reduce the objective function if they were not.

Finally, assume that $(\tilde{z}, D)$ and $(\tilde{z}', D')$ are both solutions of the problem. Since all the constraints are binding, the value of the objective function is $I - F(\tilde{z}|s)c$, so $\tilde{z} = \tilde{z}'$. Since all the constraints are binding and strictly monotone in D, then $D = D'$. Therefore the solution of the problem is unique.

Now we introduce two technical lemmas for the proof of lemmas 2 and 3.

Lemma A.4 *Consider the problem P2′:*

$$\inf_{m(z)} D + (D^* - D)\int_{z_a}^{z_b} m(z)f(z|s)dz \tag{A.20}$$

$$\textit{subject to} \int_{z_a}^{z_b} m(z)(f(z|r) - f(z|s)) = \frac{\hat{x}}{D^* - D} \tag{A.21}$$

$$-D - (D^* - D)\int_{z_a}^{z_b} m(z)f(z|s)dz - a_2\hat{x} \leq$$

$$-I - ya_2 - c\int_{z_a}^{z_b} m(z)[a_1 f(z|s) + a_2 f(z|r)]dz - K \tag{A.22}$$

$$0 \leq m(z) \leq 1. \tag{A.23}$$

where, a_1, a_2, D, D^* and K are some constants such that $a_1 \geq 0, a_2 \geq 0, D^* - D > 0$ and the problem is feasible. Then at the optimum $m(z) = 1$ for $z \in [z_a, \bar{z})$ for some $\bar{z}$ and $m(z) = 0$ for $z \in (\bar{z}, z_b]$ for some $\bar{z}$.

Proof As in lemma A.1, we can form the Lagrangian and take the first-order condition. If λ is the multiplier to the first constraint and μ to the second, we have:

$$D^* - D - \mu(D^* - D - ca_1) + (c\mu a_2 + \lambda)g(z) - 1 \begin{cases} \leq 0 & \text{if } m(z) = 1 \\ = 0 & \text{if } m(z) \in (0, 1) \\ \geq 0 & \text{if } m(z) = 0. \end{cases}$$

If $c\mu a_2 + \lambda \leq 0$, then $m(z) = 1$ for $z \in (\bar{z}, z_b]$ (for some $\bar{z}$) and $m(z) = 0$ for $z \in [z_a, \bar{z})$, which contradicts the incentive compatibility constraint. Then $c\mu a_2 + \lambda > 0$, so $m(z) = 1$ for $z \in [z_a, \bar{z})$ and $m(z) = 0$ for $z \in (\bar{z}, z_b]$.

Lemma A.5 *For the auxiliary problem P2X:*

$$p_2(\hat{x}, K) = \inf_{A,D} Prob(A^c|s)D + Prob(A|s)D^*$$
$$\textit{subject to } [Prob(A|r) - Prob(A|s)](D^* - D) = \hat{x}$$
$$D + [M(\hat{x})Prob(A|s) + (1 - M(\hat{x}))Prob(A|r)](D^* - D - c) \geq$$
$$I + y[1 - M(\hat{x})] + K$$

the solution is of the form $A = [z_a, \tilde{z}]$, $D = D^* - \hat{x}/[F(\tilde{z}z|r) - F(\tilde{z}|s)]$ and $\tilde{z}$ is the smallest z solving the equation:

$$D^* - \hat{x}\frac{1 - F(\tilde{z}|s)}{F(\tilde{z}|r) - F(\tilde{z}|s)} = I + (y - \hat{x})[1 - M(\hat{x})] + c[M(\hat{x})F(\tilde{z}|s) + (1 - M(\hat{x}))F(\tilde{z}|r)] + K,$$

the solution contract minimizes the payment of the safe-playing firm, subject to the incentive constraint and the condition that the bank makes a profit of at least K.

Proof If we set $a_1 = M(\hat{x})$ and $a_2 = 1 - M(\hat{x})$, then lemma A.4 shows that for any feasible D it is optimal to set $A = [z_a, \tilde{z}]$. Plugging in the incentive constraint and rearranging:

$$\inf_{\tilde{z},D}(1 - F(\tilde{z}|s))D + F(\tilde{z}|s)D^*$$
$$\text{subject to } [F(\tilde{z}|r) - F(\tilde{z}|s)](D^* - D) = \hat{x}$$
$$D + F(\tilde{z}|s)(D^* - D) \geq$$
$$I + (y - \hat{x})[1 - M(\hat{x})] + c[M(\hat{x})F(\tilde{z}|s) + (1 - M(\hat{x}))F(\tilde{z}|r)] + K$$

Then from the incentive constraint we immediately see that $D = D^* - \hat{x}/[F(\tilde{z}|r) - F(\tilde{z}|s)]$. Plugging this in the objective function:

$$v(\tilde{z}) = D^* - \hat{x}\frac{1 - F(\tilde{z}|s)}{F(\tilde{z}|r) - F(\tilde{z}|s)}$$
$$v'(\tilde{z}) = \hat{x}\frac{f(z|r)[1 - F(z|s)] - f(z|s)[1 - F(z|r)]}{[F(\tilde{z}|r) - F(\tilde{z}|s)]^2} > 0$$

where the last inequality follows from lemma A.3 . So, the objective function is strictly decreasing if we decrease $\tilde{z}$. It is easy to show that for $\tilde{z}$ low enough the last constraint will not be satisfied, so it must be binding at the optimum. Finally, from the continuity of the constraint function, it follows that the set of $\tilde{z}$ such that the constraint is satisfied is compact, so the minimum is attained.

Proof (Proof of Lemma 2) Assume that there are n contracts in equilibrium, taken up by a positive mass of firms. From individual rationality it follows that $Prob(A_i|s)R + Prob(A^c|s)D^i$ is the same for all $i = 1, 2, \ldots n$ and similarly for $Prob(A_i|r)R + Prob(A^c|r)D^i$. Again individual rationality implies that all firms with $x < \hat{x} = [Prob(A^i|r) - Prob(A^i|s)](R - D^i)$ will choose the action s and all firms with $x > \hat{x}$ will choose r.

Suppose that $Prob(A_i|s)R + Prob(A_i^c|s)D^i > p_2(\hat{x}, 0)$. By continuity, for small enough $\epsilon > 0$, $p_2^*(\hat{x}, \epsilon) < Prob(A_i|s)R + Prob(A_i^c|s)D^i$. Moreover, expected payment for firms who play r in the alternative contract will be: $p_2(\hat{x}, \epsilon) + \hat{x} < Prob(A_i|r)R + Prob(A_i^c|r)D^i$. So the new contract is strictly preferred by all firms and gives profit $\epsilon > 0$ to the bank, which is a contradiction. Therefore $Prob(A_i|s)R + Prob(A_i^c|s)D^i \leq p_2(\hat{x}, 0)$.

Define $\mu_i = Prob(B_i \cap (\hat{x}, x_b])/Prob(B_i)$. μ_i is the fraction of risk-taking firms to total number of firms that take up contract i. Let μ_j be the largest μ_i. Clearly $\mu_j \geq 1 - M(\hat{x})$. If $\mu_j > 1 - M(\hat{x})$, then contract j satisfies the constraints for P2 with $K = 0$, but the unique optimal solution for P2 with $K = 0$ does not satisfy the break-even conditions for contract j, so we get that $p_3(\hat{x}, 0) < Prob(A_i|s)R + Prob(A_i^c|s)D^i$, which as we showed is impossible. Then it must be that $\mu_i = 1 - M(\hat{x})$ for all i. If for some i the contract differs from the solution to P2, since the solution to P2 is unique, we must have $p_3(\hat{x}, 0) < Prob(A_i|s)R + Prob(A_i^c|s)D^i$, so again we reach a contradiction.

Therefore the only contract on offer is the solution to P2X with $K = 0$.

Proof (Proof of Lemma 3) The claim that (9) binds and $A = [z_a, \tilde{z}]$ follow from Lemma A.5.

If $x_a > 0$, there exists some z_1 such that if $z < z_1$, $R - x_a(1 - F(z|s))/H(z) < I$. Then for all $x \in [x_a, x_b]$, if $z < z_1$:

$$R - x\frac{1 - F(z|s)}{H(z)} \leq R - x_a\frac{1 - F(z|s)}{H(z)} < I + (y - x)[1 - M(x)] + c[M(x)F(z|s) + (1 - M(x))F(z|r)]$$

Then for all x such that $z(x)$ exists, $z(x) \geq z_1$. Let $M = \{x \in [x_a, x_b] : \text{ Pooling contract exists for } x\}$. For brevity let's denote $z_2 = \sup\{z(x) : x \in M\}$.

By hypothesis, $M \neq \emptyset$. Let $x_i, i = 1, 2, \ldots, x_i \in D$ be an arbitrary sequence in M. Then the sequence of tu-ples $(x_i, z(x_i)) \subseteq [x_a, x_b] \times [z_1, z_2]$ must have a convergent subsequence with limit $(x^*, z^*) \in [x_a, x_b] \times [z_1, z_2]$. If we show that z^* satisfies the constraint for x^* , then $x^* \in M$ and therefore M is compact. But this follows from the fact that the constraint function is continuous and $z_2 < z_b$. The payoff function $p_2(x, 0)$ is continuous by the maximum theorem. Then by Weierstrass extreme value theorem, a maximum exists.

Proof (Proof of Theorem 1) Suppose that $v_s^* \geq v_p^*, v_s^* \geq v_r^*$ and that the optimal separating contract is on offer. Suppose that it is possible to add a finite number of contracts that all remain in the surviving contracts set. By construction, the old contracts still break even, so they are in the surviving contracts set. Suppose that the new allocation is separating. Since the old contract maximizes the utility of the firms playing s and is still on offer, the new contract must be taken only by firms playing r. Then break-even constraint implies that the new contract is $D = I + y$. However, this contract already exists. Suppose that the new allocation is pooling. If this is the optimal pooling contract, then the bank cannot make a profit on it, by construction. If it is not the optimal pooling contract, then all the firms in $[x_a, \hat{x}]$ will strictly prefer the old contract, which contradicts the assumption.

Now suppose that $v_p^* \geq v_r^*, v_p^* \geq v_r^*$ and that the optimal pooling contract is on offer. Suppose that it is possible to add a finite number of contracts that all remain in the surviving contracts set. The new allocation cannot be separating for the same reason as in the case of separating equilibrium. Suppose that the new contract is pooling. All firms $[x_a, \hat{x})$ will strictly prefer the old contract. Then as shown in Lemma 2, no other contract can be taken up by a positive mass of firms.

The case when the equilibrium is risk-taking and $v_r^* \geq v_p^*, v_r^* \geq v_s^*$ is similar to the two cases above.

Proof (Proof of Proposition 2) Let $(\tilde{z}^*, D^*)$ be the solution to P1 and $\hat{x}^* = y - F(\tilde{z}^*|s)c$. We will prove the first statement by contrapositive; that is, we will show that if $\hat{x}^* \notin [x_a, x_b]$, then the equilibrium is not separating. First, suppose that $\hat{x}^* < x_a$. In this case all firms strictly prefer the contract $(\emptyset, I + y)$, so the contract is pooling.

Second, assume that $\hat{x}^* > x_b$. Let $\tilde{z}(x)$ be the optimal covenant in the case when there is no adverse selection. We know that $\tilde{z}(x)$ is strictly increasing. Since the contract $(\tilde{z}^*, R, D^*)$ satisfies all the constraints of the no-adverse selection contract for x^*, we have that $\tilde{z}^* \geq \tilde{z}(x^*) > \tilde{z}(x_b)$. Then a contract derived from the no-adverse selection case with $x = x_b$ will be preferred by all firms, who will take the action s.

Finally, suppose that $\hat{x}^* \in [x_a, x_b]$ and the equilibrium is separating. We showed that the firms taking action s must have contract $(\emptyset, R, I + y)$. Suppose that the contract $(\tilde{z}, R, D)$ differs from $(\tilde{z}^*, R, D^*)$. The contract $(\tilde{z}, R, D)$ satisfies the constraints to P1 - (5) and (6). Since problem P1 has a unique solution, this implies that the contract $(\tilde{z}, R, D)$ gives strictly lower payoff for firms $x \in [x_a, \hat{x}^*)$. So for some $\epsilon > 0$ small enough, the contract $(\tilde{z}^*, R, D^* + \epsilon)$ will be taken up by a positive mass of firms and is strictly profitable, which is a contradiction.

Proof (Proof of Proposition 3)

Define $f(y, I, c, z) = R - \frac{1-F(z|s)}{H(z)}[y - F(z|s)c] - I - F(z|s)c$. The separating contract will be feasible if $f(y, I, c, z) \geq 0$ for some $z \in (z_a, z_b)$. f is strictly decreasing in I and y and strictly increasing in c and for any fixed y, I, c, such that $y > 0$, $\lim_{z \to z_a} f(y, I, c, z) = -\infty$.

Fix $y > 0, c$ and I and suppose that $I' > I$ and $\tilde{z}'$ and $\tilde{z}$ are the respective solutions of problem P1. We want to show that $\tilde{z}' > \tilde{z}$. Suppose not, $\tilde{z}' \leq \tilde{z}$. Then $f(I', \tilde{z}) > f(I, \tilde{z}) \geq 0$. Then by continuity there exists some $\tilde{z}'' < \tilde{z} \leq \tilde{z}'$ such that $f(I', \tilde{z}'') \geq 0$, which contradicts the assumption that $\tilde{z}'$ is optimal. Therefore, $\tilde{z}' > \tilde{z}$. Then $\hat{x}' = y - F(\tilde{z}|s)c < y - F(\tilde{z}|s)c = \hat{x}$. The proof that $\tilde{z}$ is increasing in y is the same.

Now consider c. Suppose that $c' > c$. Since f is strictly decreasing in c, the argument above shows that $\tilde{z}' < \tilde{z}$ A reduction in c, relaxes the constraints of problem P1 and therefore will lower the objective function. Since all the constraints are binding at the optimum, the objective function is lowered strictly, so $I + F(\tilde{z'}|s)c' > I + F(\tilde{z}|s)c$, which immediately implies that $\hat{x}' = y - F(\tilde{z}'|s)c' < y - F(\tilde{z}|s)c = \hat{x}$.

Proof (Proof of Lemma 4) First, we prove the following claim.

Suppose that $h(x)$ is strictly increasing and continuously differentiable function, Z satisfies the assumptions in the paper. Then $Y = h(Z)$ satisfies the assumptions in the paper. Moreover, the equilibria in economies with signals Z and Y are equivalent.

Proof of claim The last statement follows from the fact that $Prob(z \in A|a) = Prob(y \in h(A)|a)$ and $Prob(y \in A|a) = Prob(z \in h^{-1}(A)|a)$ for all Borel sets A. Then any contract (A, D) with signal X has identical payoffs to contract $(h(A), D)$ and signal Y.

$F_Y(y|a) = Prob(Y \leq y|a) = Prob(Z \leq h^{-1}(y)|a) = F_Z(h^{-1}(y)|a)$. Then the support of the signal Y is $[h(x_a), h(x_b)]$. F_Y is differentiable with derivative

$$f_Y(y|a) = f_Z(h^{-1}(y)|a)\frac{1}{h'(h^{-1}(y))}$$

Finally,

$$\frac{f_Y(y|r)}{f_Y(y|s)} = \frac{f_Z(h^{-1}(y)|r)}{f_Z(h^{-1}(y)|s)}$$

is decreasing in y. Thus the claim is proven.

Then define $h(x) = F_Z(x|s)$. h satisfies the assumptions in the claim. Therefore, we can replace Z with $Y = F_z(Z|s)$. Then by a basic theorem of mathematical statistics, $Y|s \sim Unif(0, 1)$. The support of Y is $[h(x_a), h(x_b)] = [0, 1]$.

Proof (Proof of Proposition 5)

We start with the necessity condition. The proof is by contrapositive. Suppose that $F_{Z_1}(z^*|r) < F_{z_2}(z^*|r)$ for some $z^* \in (0, 1)$. Then we will show that for some feasible parameter values, the payoff of the firm with lowest x from the signal Z_1 is strictly lower.

We will assume that the distribution of x is degenerate, so x is known. Let $R > I > 0$ be arbitrary and set $c = (R - I)/(2z^*)$. Next, set $x = \frac{(R-I-z^*c)(F_{Z_2}(z^*|r)-z^*)}{1-z^*}$. Finally set y arbitrarily such that $y > x + z^*c$. Let $D^* = R - x/(F_{Z_2}(z^*|r) - z^*)$. Then it is immediate that (z^*, D^*) satisfies both constraints with equality.

$$\frac{\partial}{\partial z}\left[-\frac{1 - F(z|s)}{F(z|r) - F(z|s)}\right] = \frac{f(z|r)[1 - F(z|s)] - f(z|s)[1 - F(z|r)]}{(F(z|r) - F(z|s))^2} > 0,$$

where the inequality is shown in the proof of proposition 1. Then $-\frac{1-F(z|s)}{F(z|r)-F(z|s)}$ is strictly increasing, so z^* is the smallest z such that both constraints bind. Then by proposition 1 (z^*, D^*) is optimal for signal Z_2.

By assumption, $F_{Z_1}(z^*|r) < F_{Z_2}(z^*|r)$. This implies that $R - \frac{1-z^*}{F_{Z_1}(z^*|r)-z^*}x < I + z^*c$. By Lemma A.2 and the fact that $-\frac{1-z^*}{F_{Z_1}(z^*|r)-z^*}$ is increasing it implies that the optimal contract for Z_1, (z', D') satisfies $z' > z^*$. Then, $R - I - z^*c > R - I - z'c$ and $R - I - z^*c > R - I + x - y$, so the payoff from signal Z_1 is higher.

Next, we turn to sufficient conditions. Suppose that Z_1 satisfies the hypothesis of the proposition with respect to Z_2. Note that due to the normalization, the payoff to the firm from (z, D) is the same for signals Z_1 and Z_2.

Then since $F_{Z_1}(z|r) \geq F_{Z_2}(z|r), \ \forall z \in [0, 1]$, any (z, D) feasible given Z_2 is feasible given Z_1. Then the best separating contract with signal Z_1 has weakly higher payoff than for signal Z_2. Similarly, the best pooling contract with signal Z_1 has weakly higher payoff than for signal Z_2. By theorem 1, this implies that the payoff of signal Z_1 is weakly higher, which concludes the proof.

Proof (Proof of Proposition 6) Suppose that Z_{α_i} satisfy MLRP. Next, we show that informativeness falls with α. I will use proposition 5. Then it will be sufficient to show that if $\alpha_1 < \alpha_2$, $\hat{Z}_{\alpha_2}|r$ FOSDs $\hat{Z}_{\alpha_1}|r$. I will show that for any q,

$H(\alpha) \equiv F_{\hat{Z}_\alpha}(q|r)$ is differentiable and $H'(\alpha) \leq 0$.

$$H(\alpha) = \int_0^{z(\alpha)} f_Z(v|r) F_w \left(\frac{z(\alpha) - v}{\alpha} \right) dv,$$

where

$$\int_0^{z(\alpha)} f(v|s) F_w \left(\frac{z(\alpha) - v}{\alpha} \right) dv = q$$

Then the assumptions on Z ensure that $z(\alpha)$ is differentiable in α (by the implicit function theorem) and

$$z'(\alpha) = \frac{1}{\alpha^2} \frac{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) (z(\alpha) - v) dv}{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) dv}$$

$$H'(\alpha) = -\frac{1}{\alpha^2} \int_0^{z(\alpha)} f_Z(v|r) f_W \left(\frac{z(\alpha) - v}{\alpha} \right) (z(\alpha) - v) dv + z'(\alpha) \int_0^{z(\alpha)} f_Z(v|r) f_W \left(\frac{z(\alpha) - v}{\alpha} \right) dv$$

Then showing that $H'(\alpha) \leq 0$ is equivalent to

$$\frac{\int_0^{z(\alpha)} f_Z(v|r) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) (1 - \frac{v}{z(\alpha)}) dv}{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) (1 - \frac{v}{z(\alpha)}) dv} \geq \frac{\int_0^{z(\alpha)} f_Z(v|r) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) dv}{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) dv}$$

which is equivalent to:

$$\int_0^{z(\alpha)} \frac{f_Z(v|r)}{f_Z(v|s)} f_1(v) dv \geq \int_0^{z(\alpha)} \frac{f_Z(v|r)}{f_Z(v|s)} f_2(v) dv,$$

where f_1 and f_2 are p.d.f.s given by:

$$f_1(v) = \frac{f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) (1 - \frac{v}{z(\alpha)})}{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) (1 - \frac{v}{z(\alpha)}) dv}$$

and

$$f_2(v) = \frac{f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right)}{\int_0^{z(\alpha)} f_Z(v|s) f_W \left(\frac{z(\alpha)-v}{\alpha} \right) dv}$$

Since the function $\frac{f_Z(v|r)}{f_Z(v|s)}$ is decreasing, it is sufficient to show that f_2 FOSDs f_1. This is implied by the fact that $f_2(v)/f_1(v)$ is increasing.

Proof (Proof of Proposition 7)

$$f_{Z_\alpha}(z|a) = \alpha f_Z(z|s) + (1 - \alpha) f_Z(z|a)$$

$$\frac{f_{Z_\alpha}(z|r)}{f_{Z_\alpha}(z|s)} = \alpha + (1 - \alpha) \frac{f_Z(z|r)}{f_Z(z|s)},$$

which is decreasing.

We follow the proof in proposition 6 and define $H(\alpha) \equiv F_{\hat{Z}_\alpha}(q|r)$. We show that H is differentiable and $H'(\alpha) \leq 0$ for all q, which is sufficient to show that informativeness fall with α.

$$H(\alpha) = \alpha F(z(\alpha)|s) + (1 - \alpha) F(z(\alpha)|r)$$

where

$$F(z(\alpha)|s) = q$$

Then clearly, $z'(\alpha) = 0$. So $H'(\alpha) = F(z(\alpha)|s) - F(z(\alpha)|r) \leq 0$.

Proof (Proof of Proposition 8) Let $\alpha_1 < \alpha_2$ be such that there is a feasible contract with $a = s$ for α_2. Direct inspection shows that the contract is still feasible for α_1. This implies that the set of α-s such that a feasible contract exists is $[0, \tilde{\alpha}]$

for some $\tilde{\alpha}$. Next, we show that $\tilde{\alpha}$. Using the properties of MLRP, we see that

$$\frac{d}{dz}\left[R - x\frac{1 - F(z,\alpha|s)}{H(z,\alpha)}\right] > 0.$$

Then for all z,

$$R - x\frac{1 - F(z,\alpha|s)}{H(z,\alpha)} < \lim_{z\to z_b}\left[R - x\frac{1 - F(z,\alpha|s)}{H(z,\alpha)}\right] = R - \frac{x}{1 - f(1,\alpha|r)},$$

where we used L'Hospital's rule in the last step. Since $\lim_{\alpha\to 1} f(1,\alpha|r) = 1$,

$$\lim_{\alpha\to 1}\max_{z}\left\{R - x\frac{1 - F(z,\alpha|s)}{H(z,\alpha)}\right\} = -\infty,$$

which implies that for all α sufficiently large, there is no feasible contract.

Next we show that $z(\alpha)$ is strictly increasing. Let $0 \le \alpha_1 < \alpha_2 \le \tilde{\alpha}$.

$$R - x\frac{1 - z(\alpha_2)}{H(z(\alpha_2),\alpha_2)} > R - x\frac{1 - z(\alpha_2)}{H(z(\alpha_2),\alpha_2)} = I + z(\alpha_2)c,$$

where the equality follows from Lemma A.2. Then Lemma A.2 implies that $z(\alpha_1) < z(\alpha_2)$.

The payoff to the firm from the contract with covenant is $R - I - z(\alpha)c$ and it is strictly decreasing and u.h.s. in z. Then the set $B = \{\alpha \in [0,\tilde{\alpha}] : R - I - z(\alpha)c \ge R - I - y + x\}$ is compact, nonempty (since $0 \in B$) and an interval (since $z(\alpha)$ is increasing). Then a contract with covenants will be weakly (strictly on the interior) preferred on $[0, \max B]$.

Proof (Proof of Proposition 9) We know that (6) is binding, so

$$R - D = \frac{R - I - y}{1 - F(z,\alpha|r)}.$$

Then the optimal z is the smallest that satisfies the break even constraint (5), which can be expressed as:

$$R - (1 - z)\frac{R - I - y}{1 - F(z,\alpha|r)} \ge I + zc.$$

Then showing that $z(\alpha)$ is increasing is identical to the proof in proposition 8.

A.1 Proofs of results in Section 5

We need introduce some additional results and notation, necessary for the proofs of Section 5.

Since $\int_{\underline{W}}^{\overline{W}} g(W)h(W|s) = 1$, there exist $W_1 < W^* < W_2$ such that $g(W) < 1$ for $W \in (W_1, W_2)$ and $g(W) > 1$ on $[\underline{W}, W_1)$ and $(W_2, \overline{W}]$. Thus the risky action shifts the probability towards the extremes.

Let $t(W) = H(W|r) - H(W|s)$. Since $t(\underline{W}) = t(\overline{W}) = 0$ and t is strictly increasing on $[\underline{W}, W_1)$ and $(W_2, \overline{W}]$ and strictly decreasing on (W_1, W_2), there exists some unique $\hat{W} \in (W_1, W_2)$ such that $t(\hat{W}) = 0$. This implies that $H(W|r) > H(W|s)$ on $(\underline{W}, \hat{W})$ and $H(W|r) < H(W|s)$ if $W \in (\hat{W}, \overline{W})$.

Next, we need to consider the bank's payoff. Taking derivatives,

$$\frac{d}{dD}\pi_b(D,a) = -\gamma_b h(D|a) + (1 - h(D|a)) = (1 - H(D|a))\left[1 - \gamma_b\frac{h(D|a)}{1 - H(D|a)}\right].$$

$h(D|a)/(1 - H(D|a))$ is the hazard rate of R, which by assumption is increasing everywhere. Then the bank's profit is maximized at some level $D^*(a) < \overline{R}$, where a refers to the action that the firm took. Also, by definition $\hat{D}(a) \le D^*(a)$.

Proof (Proof of Lemma 5) Let $p(D) = \int_{D}^{\overline{W}}(W - D)(h(W|r) - h(W|s))dW$. From the definition it follows that $\Delta\pi(D) = p(D) - (H(D|r) - H(D|s))\gamma_f$. It is clear that $p(D)$ is continuous, $p(\underline{W}) = p(\overline{W}) = 0$.

First, we prove $p(D) > 0$ for all $W \in (\underline{W}, \overline{W})$. $p'(D) = -\int_D^{\overline{W}} h(W|r) - h(W|s)dW$ and $p''(D) = h(W|r) - h(W|s)$. This implies that p' is increasing on $[\underline{W}, W_1)$, decreasing on (W_1, W_2) and increasing again on $(W_2, \overline{W}]$. The fact that $p'(\underline{W}) = p'(\overline{W}) = 0$ implies that $p'(D) > 0$ on $(\underline{W}, W_1]$ and $p'(D) < 0$ on $(W_2, \overline{W}]$. Since p' is strictly decreasing on (W_1, W_2), there exist some W_c such that $p'(D) > 0$ on $(\underline{W}, W_c)$ and $p'(D) < 0$ on $(W_c, \overline{W})$. Then if $D \in (\underline{W}, W_c)$, $p(D) > p(\underline{W}) = 0$; if $W \in [R_c, \overline{W})$, then $p(D) > p(\overline{W}) = 0$. Therefore $p(D) > 0$ for all $D \in (\underline{W}, \overline{W})$ and $p(\underline{W}) = p(\overline{W}) = 0$.

Then for all $D \in [\hat{W}, \overline{W})$, $\Delta\pi(D) = -(H(W|r) - H(W|s))\gamma_f + p(D) \geq p(D) > 0$, where the inequality follows from the fact that $H(W|r) \leq H(W|s)$ if $W \geq \hat{W}$. Since $\Delta\pi(D)$ is continuous, for some $\epsilon > 0$, $\Delta\pi(D) > 0$ for all $D > \hat{D} - \epsilon$. This establishes the existence of the cutoff D_2.

Next, we prove the existence of D_1. $\Delta\pi'(D) = -(h(D|r) - h(D|s))\gamma_f + p'(D)$. Taking limits $\lim_{D\to\underline{W}} \Delta\pi'(D) = -(h(\underline{W}|s) - h(\underline{W}|r))\gamma_f < 0$. This and the fact that $\Delta\pi(\underline{W}) = 0$ establishes the existence of D_1.

Before proving proposition 10 we need a technical lemma.

Lemma A.6 *Suppose that $\hat{D}(s) < \hat{W}$. Then $\hat{D}(s) < \hat{D}(r)$.*

Proof (Proof of Lemma A.6) In the proof of lemma 5 we showed that $EP(D, s) > EP(D, r)$ for all $D \in (\underline{W}, \overline{W})$. Then for all $D \in (\underline{W}, \hat{D}(s)]$, we have:

$$\pi_b(D, r) = EP(D, r) - \gamma_b M(D|r) < EP(D, s) - \gamma_b H(D|s) = \pi_b(D, s) \leq \pi_b(\hat{D}(s), s) = I.$$

The first inequality uses the assumption that $\hat{D}(s) < \hat{W}$, so for all $D < \hat{D}(s), H(D|r) > H(D|s)$. The second inequality follows from the fact that $\pi_b(D, s)$ is increasing from $\underline{W}$ to $D^*(a)$ and that $\hat{D}(a) \leq D^*(a)$. Then the statement follows from the definition of $\hat{D}(r)$.

Proof (Proof of Proposition 10) Consider the first case.

Let A be the set of face values of the debt such that for all values $D \in A$, the incentive and bank-even constraint hold. Since $\hat{D}(s) \leq D_r$, $\hat{D}(s) \in A$. If $D < \hat{D}(s)$, then the bank's break-even constraint is not satisfied. Then $\hat{D}(s) = \min A$. Since $\pi_f(D, a)$ is strictly decreasing in D, $\hat{D}(s)$ is the best debt value that induces action $a = s$.

Next, we show that this contract is preferred to any contract that induces $a = r$. Let B be the set of debt values D that induce action $a = r$ and satisfy the bank's break-even constraint. By definition, $D \in B$ implies that $D \geq \hat{D}(r)$. Since $D_r \leq \hat{R}$, Lemma A.6 implies that $\hat{D}(r) > \hat{D}(s)$. Then for all $D \in B$, we have

$$\begin{aligned}
\pi_f(D, r) \leq \pi_f(\hat{D}(r), r) &= ER - EP(\hat{D}, r) - H(\hat{D}(r), r)\gamma_f \\
&= ER - I - H(\hat{D}(r), r)(\gamma_f + \gamma_b) \\
&< ER - I - H(\hat{D}(s), r)(\gamma_f + \gamma_b) \\
&< ER - I - H(\hat{D}(s), s)(\gamma_f + \gamma_b) = \pi_f(\hat{D}(s), s),
\end{aligned}$$

which concludes the proof for the first case.

On the other hand, consider the case when $\hat{D}(r) > D_r$. Then no contract can induce action $a = s$ and break-even for the bank. Then all contracts must induce $a = r$ and break-even for the bank, conditional on action $a = r$. Since the firm's payoff is strictly decreasing in D, the optimal contract that induces $a = r$ and break-even is $\hat{D}(r)$.

Lemma A.7 *Define $\xi(x, a) = H(EP^{-1}(x, a)|a)$. Then the function $\xi(x, a)$ is convex on $(-\infty, EW)$ and strictly convex on $(\underline{W}, EW)$ a.*

Proof (Proof of Lemma A.7) We prove the second statement first.

$$\frac{d}{dx}\xi(x, a) = \frac{h(EP^{-1}(x, a)|a)}{1 - H(EP^{-1}(x, a)|a)},$$

where we used the definition of ξ and EP. The fact that EP^{-1} is increasing and the fact that the hazard rate is strictly increasing implies the result. For every $x < \underline{R}$, $\frac{d}{dx}\xi(x, a) = 0$. Finally, the fact that $\frac{d}{dx}\xi(x, a) > 0$ for $x > \underline{R}$, establishes the global convexity.

Lemma A.8 *Let $A > X > 0$. Then the function $\zeta(z, A, X, a) \equiv z\xi(A, a)+(1-z)\xi[(X-zA)/(1-z), a]$ is strictly increasing in $z \in [0, 1)$*

Proof $\zeta(z, A, X, a)$ is differentiable in z everywhere, except at $z = (X - \underline{W})/(A - \underline{W})$. At the points of differentiability,

$$\frac{d}{dz}\zeta(z, A, X, a) = \xi(A, a) - \xi\left(\frac{X - zA}{1 - z}, a\right) - \frac{A - X}{1 - z}\xi\left(\frac{X - zA}{1 - z}, a\right)$$
$$> (A - (I - zA)/(1 - z))\xi'\left(\frac{X - zA}{1 - z}, a\right) - \frac{A - I}{1 - z}\xi'\left(\frac{X - zA}{1 - z}, a\right) = 0,$$

where we used the convexity of ξ established in Lemma A.7.

The function $\xi()$ denotes the probability of default as a function of the expected payment to the bank. We show that this probability is a convex function of the expected payment. The function ζ is similar, but here the expected payment is A with probability z and B with probability $1 - z$, where $zA + (1 - z)B = X$.

Proof (Proof of Lemma 6) First, we find the optimal contract when the action is observable: the action a can be specified in the contract. First we show that conditional on choosing any a, it is optimal to set the contract $(\emptyset, \hat{D}(a))$.

Clearly, we can assume without loss of generality that the break-even constraint 17 is binding. Then for any covenant tightness z, the corresponding value of the debt $D(z)$ satisfies

$$F(z|s)\pi_b(D^*(a), a) + (1 - F(z|s))\pi_b(D(z), s) = I.$$

Then the probability of bankruptcy is given by $\zeta(F(z|s), \pi_b(D^*(a), a), a)$. Lemma A.8 shows that this probability is minimized by setting $z = z_a$, or equivalently, to a contract with no covenants.

Then consider an alternative contract without a covenant and value D' defined by $\pi_b(D', a) = I$. The new contract reduces the probability of bankruptcy (strictly if $z > z_a$), while it keeps the break-even constraint for the bank. This increases the payoff of the firm and the bank.

Next, we need to find out what action is optimal to induce. From the definition and the fact that the bank's break-even constraint is binding,

$$\pi_f(\hat{D}(a), a) = ER - I - H(\hat{D}(a)|a)(\gamma_f + \gamma_b).$$

Since $\hat{D}(s) < D_r$, by lemma A.6, $\hat{D}(s) < \hat{D}(r)$. Then $H(\hat{D}(r)|r) > H(\hat{D}(s)|r)$. Since $\hat{D}(s) < D_r$, it follows that $H(\hat{D}(s)|r) > H(\hat{D}(s)|s)$, which implies that $\pi_f(\hat{D}(s), s) > \pi_f(\hat{D}(r), r)$.

The contract $(\emptyset, \hat{D}(s))$ induces the action $a = s$ since $\hat{D}(s) < D_r$. Since this contract maximizes the firm's payoff in the relaxed problem, it is optimal in the fully constrained problem.

Proof (Proof of Lemma 7) Recall that $\hat{D}(r)$ is defined implicitly by $\pi_b(\hat{D}_a, r) = I$. By assumption $\hat{D}(r) \leq W^*$.

1. Constraint (17) binds.
   Let (z^*, D^*) be an optimal contract that satisfies (17) and (16) and the inequality in constraint (17) is strict. Incentive compatibility (16) implies that $z^* \in (0, 1)$. If $D \geq \hat{D}(r)$, then the payoff from this contract is:
   $$F(z^*|s)\pi_f(D^*(s), s) + (1 - F(z^*|s))\pi_f(D, s) < \pi_f(\hat{D}(r), s) < \pi_f(\hat{D}(r), b),$$
   which is a contradiction since $(\emptyset, \hat{D}(r))$ is a feasible contract. Therefore, $D^* < D$.
   Since $F(z^*|s)\pi_b(D^*(s), s) + (1 - F(z^*|s))\pi_b(D, s)$ is continuous, increasing and unbounded from below in D, there exists D' such that (17) binds with equality. Since $D' < D < \hat{D}(r) \leq W^*$, $\Delta\pi(D') < \Delta\pi(D)$, so the constraint (16) is still satisfied. The firm strictly prefers the new contract.
2. Optimal contract exists.
   If there is no contract that induces $a = s$ and dominates $(\emptyset, \hat{D}(r))$, then $(\emptyset, \hat{D}(r))$ is the optimal contract,
   Suppose that there exists a contract (z_1, D_1) that dominates $(\emptyset, \hat{D}(r))$. By part (i) we can assume that (17) holds with equality. Let (z', D') be any contract such that $z' > z_1$ and it satisfies (17) with equality. By lemma A.8, the

probability of bankruptcy $Prob(Default)$ is higher. Since the firm's payoff is $EW - I - Prob(Default)\gamma_b$, the firm's payoff is strictly lower. So without loss of generality, we can impose the constraint $z \in [0, z_1]$.
The incentive compatibility constraint (16) cannot be satisfies by $D > D^*(s)$, so again without loss of generality we can impose constraint $D \leq D^*(s)$. Next, (17) implies

$$\pi_b(D, s) = \frac{1 - F(z|s)\pi_b(D^*(s), s)}{1 - F(z|s)}$$

Define $\underline{D} \equiv \min z \in [0, z_1] \frac{1-F(z|s)\pi_b(D^*(s),s)}{1-F(z|s)}$. Then using the fact that $D \geq \pi_b(D, s)$, we obtain the constraint that $D \in [\underline{D}, D^*(s)]$.
So we showed that we can impose the additional constraints, $D \in [\underline{D}, D^*(s)],\ z \in [0, z_1]$. Since the constraints (16) and (17) are continuous in z, D, the constraint is closed and bounded, hence compact. Then since the firm's payoff is continuous, an optimal contract exists.

3. If constraint (16) doesn't bind, then the contract is not optimal.
Let (z^*, D^*) be a contract that satisfies (17) and (16) and the inequality in constraint (16) is strict. We will show that there exists an alternative contract (z', D') that gives strictly higher payoff for the firm and satisfies the constraints (16) and (17). Then the existence of optimal contract implies the result.
Let $X = F(z^*|s)EP(D^*(s), s) + (1 - F(z^*|s))EP(D^*, s)$. Define $D(z)$ implicitly by

$$F(z|s)EP(D^*(s), s) + (1 - F(z|s))EP(D(z), s) = X.$$

$D(z)$ is uniquely defined for all $z \leq z^*$. We next show that $(z^* - \epsilon, D(z^* - \epsilon)$ keeps the firm's payoff the same and satisfies all the constraints for $\epsilon > 0$ sufficiently small.
Continuity implies that (16) will be satisfied for all $(z^* - \epsilon, D(z^* - \epsilon)$ if $\epsilon > 0$ is sufficiently small.
Direct inspection reveals that $F(z|s)H(D^*(s)|s) + (1 - F(z|s))H(D(z)|s) = \zeta(F(z|s), EP(D^*(s), s), X, a)$. Lemma A.8 shows that $F(z|s)H(D^*(s)|s) + (1 - F(z|s))H(D(z)|s)$ is strictly increasing in z. Then by construction

$$F(z|s)\pi_b(D^*(s), s) + (1 - F(z|s))\pi_b(D(z), s) = X - [F(z|s)H(D^*(s)|s) + (1 - F(z|s))H(D(z)|s)]\gamma_f,$$

which implies that (17) is satisfied for $(z, D(z)$ with strict inequality if $z < z^*$. The firm's payoff is

$$F(z|s)\pi_f(D^*(s), s) + (1 - F(z|s))\pi_f(D(z), s) = EW - X,$$

so it is not changed.
Lastly let $z < z^*$ be such that $(z, D(z))$ satisfies both constraints with strict inequality. Then for $\eta > 0$, $(z, D(z) - \eta)$ still satisfies all the constraints and is strictly preferred by the firm.

4. The constraint (16) is binding at the optimum.
This follows from point 2 (optimal contract exists) and (3) if (16) is slack, the contract is not optimal.

Proof (Proof of Proposition 11) We prove the three statements in turn.

1. Suppose that $0 < I_1 < I_2$ are two possible face values of the debt; let $(\tilde{z}_1, D_1)$ and $(\tilde{z}_2, D_2)$ be the corresponding optimal contracts. Since the optimal contract minimizes $\tilde{z}$ subject to the incentive and promise-keeping constraints, for all $(\tilde{z}, D)$ such that $\tilde{z} \leq \tilde{z}_1$ and (16) holds we have:

$$F(\tilde{z}_1|s)\pi_b(D^*(s)|s) + (1 - F(\tilde{z}|s))\pi_f(D|s) \leq I_1 < I_2.$$

Therefore in the case of I_2, there is no feasible contract with covenant trigger $z \leq \tilde{z}_1$ that satisfies (17) and (16). This implies that $\tilde{z}_2 > \tilde{z}_1$.

2. Let $\gamma_{b1} < \gamma_{b2}$. Let (z_i, D_i) be the respective optimal contracts. Recall that $D_i^*(s)$ is the corresponding value that maximizes $\pi_f(D, s; \gamma_{bi})$. It is immediate that $D_1^*(s) > D_2^*(s)$. Then it is immediate that (16) is slack for (z_2, D_2) if $\gamma_b = \gamma_{b1}$. Since π_f is strictly decreasing in γ_b and $\pi_b(D_2^* s), s|\gamma_{b2}) < \pi_b(D_1^* s), s|\gamma_{b1})$, it follows that (14) is also slack for (z_2, D_2) if $\gamma_b = \gamma_{b1}$. Then as in lemma 7 we can show that the optimal contract has $z < z_2$.

3. Let $D(z)$ be defined as the value of the debt such that the incentive constraint (16) binds. Mean-preserving spread of $W|r$ reduces $D(z)$. Then at $\{(z, D_2(z)) : z \leq z_1\}$ the break-even constraint does not hold. Then by the same argument as in (1), $z_2 > z_1$.

A.2 Proofs of results in Section 6

Proof (Proof of Proposition 12) First, we make a technical point.

Claim $\frac{F(z|r)-F(z|s)}{F(z|s)}$ is strictly decreasing in z.

Proof of claim Taking logs and differentiating we see that the claim is implied by $\frac{f(z|r)}{F(z|r)} < \frac{f(z|s)}{F(z|s)}$ for $z < z_b$. Proving this inequality is done in the same way as for lemma A.3. This proves the claim.

Suppose that the optimal contract for some $m_2 > 0$ is with a covenant: $(\tilde{z}_2, D_2)$. Consider $m_1 \in (0, m_2)$. Define $\tilde{z}_1$ by $(1-m_1)F(\tilde{z}_1|s) = (1-m_2)F(\tilde{z}_2|s)$. We have that $\tilde{z}_1 \in (z_a, \tilde{z}_2)$ and is unique. Then

$$\begin{aligned}(1-m_1)[F(\tilde{z}_1|r) - F(\tilde{z}_1|s)] &= \frac{(1-m_2)F(\tilde{z}_2|s)}{(1-m_1)F(z_1|s)}[(1-m_1)F(\tilde{z}_1|r) - F(\tilde{z}_1|s)] \\ &= (1-m_2)F(\tilde{z}_2|s)\frac{F(\tilde{z}_1|r) - F(\tilde{z}_1|s)}{F(\tilde{z}_1|s} \\ &> (1-m_2)F(\tilde{z}_2|s)\frac{F(\tilde{z}_2|r) - F(\tilde{z}_2|s)}{F(\tilde{z}_1|s} \\ &= (1-m_2)[F(\tilde{z}_2|r) - F(\tilde{z}_2|s),]\end{aligned}$$

where the strict inequality follows from the claim. Then the contract $(\tilde{z}_1, D_2)$ satisfies the constraints for m_1, gives the same payoff as the contract for m_2, and the incentive constraint is slack. By the same arguments as in lemma A.2, we show that there exist $(\tilde{z}_1', D_1)$ that gives strictly higher payoff and $\tilde{z}_1' < \tilde{z}_1$. This fact and the proposition hypothesis that the optimal contract has a covenant for some $m > 0$ proves all the claims of the proposition except that $\bar{m} < 1$.

Finally to show that $\bar{m} < 1$, it is sufficient to show that for all m large enough the constraints cannot be satisfied. As in lemma A.2, we can show that the incentive and break-even constraint imply:

$$R - \frac{1-(1-m)F(\tilde{z}|s)}{(1-m)(F(\tilde{z}|r) - F(\tilde{z}|s))}x \geq I + F(\tilde{z}|s)c. \tag{A.24}$$

If $(\tilde{z}, D)$ satisfy the constraints, then (A.24) is satisfied. We have

$$\frac{1-(1-m)F(\tilde{z}|s)}{(1-m)(F(\tilde{z}|r) - F(\tilde{z}|s))} \geq \frac{m}{1-m}\frac{1}{(F(\tilde{z}|r) - F(\tilde{z}|s))} \geq \frac{m}{1-m}\frac{1}{A},$$

where $A = \max_z F(\tilde{z}|r) - F(\tilde{z}|s)$ and $A > 0$. Then

$$\max_{\tilde{z}} \left\{ R - \frac{1-(1-m)F(\tilde{z}|s)}{(1-m)(F(\tilde{z}|r) - F(\tilde{z}|s))}x \right\} \leq R - \frac{m}{1-m}\frac{1}{A}x.$$

But the last term converges to $-\infty$ as $m \to 1$, which proves that the condition (A.24) cannot be satisfied for large enough m.

B Alternative Renegotiation Assumption

In this appendix, we will explore an alternative specification for the renegotiation process. In the main body of the paper, we assume that after signing a contract with the bank, the firm cannot borrow from other banks, or refinance the due loan with other lenders. As a result, the bank obtains monopoly power in the event of a covenant violation. Here we relax this assumption. We show that the main results of the paper remain unchanged.

We assume that in case of a covenant violation, the original lender has the right to demand repayment of the face value of the loan. The firm can contract with another lender, as long as it repays the original loan D and the renegotiation cost c[19]: in other words the loan can be refinanced. There is a mass of outsider banks which are perfectly competitive. Since the cost to the lender y of action r is determined by the reduced probability of being repaid, then the cost of risk shifting is borne by the ultimate holder of the loan (that is the bank that refinanced the loan if it was refinanced.) Finally, we assume that the firm type x is known.

When the principal cannot commit, the optimal equilibrium involves mixed strategies (see, for example,Bester and Strausz (2001)). Similarly, we will consider the more general case of mixed strategies: the firm chooses the probability p of action $a = r$.

Since the firm may have risk-shifted, the outside bank, even if competitive, demands repayment larger than $D + c$. They offer the following payment to the firms:

$$D_r = D + c + Prob(a = r|z)y.$$

$D + c$ is the amount paid to the current lender and $Prob(a = r|z)y$ is the expected value of the loss of the bank from risk-taking. Since the renegotiation occurs after the signal z has been observed: (1) the signal z is informative of the probability that risk-taking had occurred and (2) it will be used to price the new payment.

$Prob(a = r|z)$ is given by Bayes theorem:

$$Prob(a = r|z) = \frac{\bar{p}f(z|r)}{(1-\bar{p})f(z|s) + \bar{p}f(z|r)},$$

where $\bar{p}$ is the (ex ante) belief that the firm risk-shifted. In equilibrium the outside bank's ex ante beliefs are correct, so $p = \bar{p}$.

Then the outside banks offer to swap existing debt with violated covenants for straight debt with no covenants[20] and the following face value:

$$D_r(z) = D + c + \frac{\bar{p}f(z|r)}{(1-\bar{p})f(z|s) + \bar{p}f(z|r)}y.$$

The existing debt holder is constrained by the outside banks when it renegotiates. So $D(z) \le D_r(z)$. We also know that $D(z) \le R$, since the bank cannot demand more than what the firm will eventually get. Then $D(z) \le \min\{D_r(z), R\}$. We will assume that $D_r(z) \le R$ always. Since the bank cannot commit to renegotiation behavior at the start, it will extract as much as possible, so $D(z) = D_r(z)$.

Then the firm's payoff for a given contract $(D, \tilde{z})$ and bank belief $\bar{p}$ is given by:

$$\pi(p) = R - D + px - p\int_0^{\tilde{z}}(D_r(z) - D)f(z|r)dz - (1-p)\int_0^{\tilde{z}}(D_r(z) - D)f(z|s)dz.$$

If $p = \bar{p} = 1$, the loan with covenants is dominated by the loan $(I + y, \emptyset)$. On the other extreme, suppose that $\bar{p} = 0$. In this case, the outside firms always offer $D_r = D + c$. Then the firm's payoff is given by:

$$\pi(p) = R - D + p(x - (F(\tilde{z}|r) - F(\tilde{z}|s))c) - F(\tilde{z}|r)c.$$

Since $x > c$, and $F(\tilde{z}|r) - F(\tilde{z}|s) < 1$ the firm has an incentive to set $p = 1$, that is risk-shift. But then this is a contradiction! So, if the outside banks believe that the firm does not risk-shift, then they will not pay attention to the signal z and will

[19] This assumption simplifies the mathematics but has no bearing on the results.

[20] In our model there is a single decision to risk-shift, so there is no need for covenants on the new debt. In actuality, the new lenders may be concerned with future risk-shifting, so they may demand covenants on the new debt.

offer low rates to switch; as a result the firm's payment does not depend on its action, so it will always risk-shift. Since the outside banks are rational, they do not entertain this belief in the first place.

So, in equilibrium $p \in (0,1)$. A player will randomize between different actions only if they give him the same payoff. So, $\pi(0) = \pi(p) = \pi(1)$. This implies that $\pi'(p) = 0$. It is straightforward to show that this is equivalent to:

$$x = \int_0^{\tilde{z}} \left[c + \frac{\bar{p} f(z|r)}{(1-\bar{p}) f(z|s) + \bar{p} f(z|r)} y \right] (f(z|r) - f(z|s)) dz. \tag{B.1}$$

This equation implicitly pins down p (and $\bar{p}$) as a function of $\tilde{z}$. The maximum of the expression on the right (over $\tilde{z}$ and $\bar{p}$) is $(c+y)(F(z^*|r) - F(z^*|s))$, where z^* is defined by $f(z^*|r) = f(z^*|s)$. Then a necessary condition for an equilibrium with covenants is that

$$x < (c+y)(F(z^*|r) - F(z^*|s)).$$

As long as the condition above is satisfied, there are some bounds $\underline{z}, \bar{z}$ such that $0 < \underline{z} < z^* < \bar{z} \le 1$, such that equation (B.1) has a unique solution $p(\tilde{z})$ that is decreasing on $[\underline{z}, z^*]$ and increasing on $[z^*, \bar{z}]$ and $p(\tilde{z}) \in [0,1]$.

Lemma B.1 *There exists some $\underline{z}, \bar{z}$, $\underline{z} < z^* < \bar{z} \le 1$, such that equation (B.1) has a unique solution $p(\tilde{z}) \in [0,1]$ and $p(\tilde{z})$ is continuous, decreasing on $[\underline{z}, z^*]$ and increasing on $(z^*, \bar{z}]$.*

Proof Let the right-hand side of equation (B.1) be denoted $h(\bar{p}, \tilde{z})$.

$$h'_p(\bar{p}, \tilde{z}) = \int_0^{\tilde{z}} \left[\frac{f(z|r) f(z|s)}{[(1-\bar{p}) f(z|s) + \bar{p} f(z|r)]^2} y \right] (f(z|r) - f(z|s)) dz.$$

It is straightforward to show that $h'_p(\bar{p}, \tilde{z})$ is minimized at $\bar{p} = 1$, so

$$h'_p(\bar{p}, \tilde{z}) \ge \int_0^{\tilde{z}} \left[\frac{f(z|s)}{f(z|r)} y \right] (f(z|r) - f(z|s)) dz, \forall \bar{p} \in [0,1], \tilde{z} \in [0,1].$$

The expression on the right is positive and strictly increasing on $(0, z^*)$ and strictly decreasing on $(z^*, 1]$. Let

$$\bar{z} = \sup \left\{ z \in [0,1] : \int_0^{\tilde{z}} \left[\frac{f(z|s)}{f(z|r)} y \right] (f(z|r) - f(z|s)) dz \ge 0 \right\}.$$

It is obvious that $\bar{z} > z^*$.

Set $\underline{z}$ by

$$\underline{z} = \inf \left\{ z \in [0,1] : \int_0^{\tilde{z}} \left[c + \frac{f(z|r)}{f(z|r)} y \right] (f(z|r) - f(z|s)) dz - x \ge 0 \right\}$$

Clearly, $0 < \underline{z}$ and the assumption we made implies that $\underline{z} < z^*$. Equation (B.1) has a unique solution $\bar{p} = 1$ at $\underline{z}$.

Since $h'_p(\bar{p}, \tilde{z}) > 0$ on $(0, \bar{z})$ and $h'_z(\bar{p}, \tilde{z}) > 0$ for $z \in (0, z^*)$ and $h'_z(\bar{p}, \tilde{z}) < 0$ for $z \in (z^*, \bar{z})$, the implicit function theorem is applicable and all the results follow from it.

Then the bank's break-even constraint is:

$$D + \int_0^{\tilde{z}} (D_r(z) - D - c)[p(\tilde{z}) f(z|r) + (1 - p(\tilde{z})) f(z|s)] dz \ge I + p(\tilde{z}) y. \tag{B.2}$$

The equilibrium contract maximizes the firm's payoff subject to break-even and incentive constraints. Since D can be adjusted up or down without affecting the incentive constraints, in equilibrium the break-even constraint holds with equality. Therefore, the firm's payoff is given by

$$OF(\tilde{z}) = R - I - [p(\tilde{z}) F(\tilde{z}|r) + (1 - p(\tilde{z})) F(\tilde{z}|s)] c - p(\tilde{z})(y - x). \tag{B.3}$$

Since $p(\tilde{z})$, $F(\tilde{z}|a)$ are increasing on $[z^*, \bar{z})$, $OF(z^*) > OF(\tilde{z})$ if $\tilde{z} > z^*$. So, if covenants are used, $\tilde{z} \in [\underline{z}, z^*]$.

Therefore, the mechanism outlined in the main body of the paper is still operative even in the presence of refinancing.

C Estimation of Restatement Frequency

Financial statements are noisy indicators of the firm's underlying state. A measure of this noise is the likelihood that financial statements are restated due to errors or irregularities in the accounting numbers. In this appendix, we compute the frequency of restatements in a comprehensive sample of US public firms.

Data and Analysis We select all US public firms with greater than $10 million in inflation-adjusted total assets (base year=2000) from the Compustat database. We further require that the included firm-years have a share code of 10 or 11 in the CRSP database to restrict our analysis to a clean sample of ordinary common shares. Specifically, this restriction excludes certificates, ADRs (American Depository Receipts), SBIs (Shares of Beneficial Interest), units, closed-end funds, REITs, etc. from our sample. This results in a sample of 90,454 firm years from 2000 to 2018.

We next merge this comprehensive sample of US public firms with the Audit Analytics Advanced Non-Reliance restatement database. In comparison to other restatement databases (for example, from the US General Accounting Office (GAO) or the Securities Exchange Commission (SEC)), the Audit Analytics (AA) database is not only more comprehensive in its coverage, but also more useful as it indicates the specific fiscal quarters and years affected by each restatement. Moreover, the AA database excludes technical restatements such as those after a merger, discontinued operation, or changes in accounting principles that are unrelated to noise or misreporting (Lobo and Zhao (2013)). Our sample begins in 2000 as Audit Analytics firm identifiers required to merge the Audit Analytics and the Compustat databases are unavailable prior to 2000.

We mark firm-years in the comprehensive, cleaned Compustat database that were disclosed to have misstated financial reports using the RES_BEGIN_DATE and RES_END_DATE variables provided for each restatement in Audit Analytics. Table C.2 presents an annual breakdown of restatement frequency amongst US public firms during our sample period.

Our sample contains 10,198 unique firms. Of these, a staggering 39.4% (4,018 firms) restate their financial statements at least once during our sample period from 2000 to 2018. This analysis illustrates that financial statements on which covenants are based maybe misstated, and thus less reliable, in a significant proportion of cases.

Year	**Restated Firms**	**Total Firms**	**Percent Restated**
2000	837	6,609	12.7%
2001	1,011	6,020	16.8%
2002	1,128	5,701	19.8%
2003	1,150	5,525	20.8%
2004	1,196	5,431	22.0%
2005	963	5,304	18.2%
2006	804	5,131	15.7%
2007	650	4,934	13.2%
2008	573	4,676	12.3%
2009	608	4,484	13.6%
2010	627	4,295	14.6%
2011	670	4,290	15.6%
2012	698	4,204	16.6%
2013	652	4,176	15.6%
2014	561	4,120	13.6%
2015	501	4,052	12.4%
2016	403	3,919	10.3%
2017	347	3,828	9.1%
2018	325	3,755	8.7%
Total	**13,704**	**90,454**	**15.2%**

Table C.2 Restatement frequency in US public firms by year

Additional evidence of noise in financial statements. We note that even this relatively high likelihood of restatements understates the true level of noise in financial statements. This is because not all accounting irregularities or errors are caught and reported, and firms have considerable discretion to manage earnings without violating accounting rules that would necessitate a restatement (Dechow et al. (2010)). Specifically, Dyck et al. (2021) estimate that two-thirds of corporate fraud cases go undetected and that on average 42% of large public firms are intentionally violating accounting rules at any point in the business cycle.

Second, the SEC in 2022 noted that a significant number of companies elected to not fully restate their financial statements even in the presence of material errors due to lenient interpretation of accounting rules by audit committees and auditors. Thus, the downtrend seen in the percentage of firms restating each year in the table above is overstated. Moreover, as the SEC tightens enforcement, one is likely to see an increase in the number of restatements going forward.

Third, firms have considerable discretion to manipulate earnings using accrual or real earnings management without violating accounting regulations (GAAP) (Dechow et al. (2010)).

Finally, Moody's believes that noise in financial statements is a critical issue for financial market participants as evidenced by their recent work on financial statement quality ratings (Zhao and Dwyer, 2019). These ratings reflect the likelihood of mistakes and fraudulent manipulation of accounting numbers and are designed to capture the 'decision-usefulness' of financial statements.

The aforementioned evidence is indicative of the potentially high degree of noise in the underlying contracting variable. It is important to note that despite the noise, financial covenants continue to be important features of loan contracts. In Appendix D, we provide an estimate of the frequency of financial maintenance covenants in loan contracts.

D Estimation of Financial Maintenance Covenant Frequency in Loan Contracts

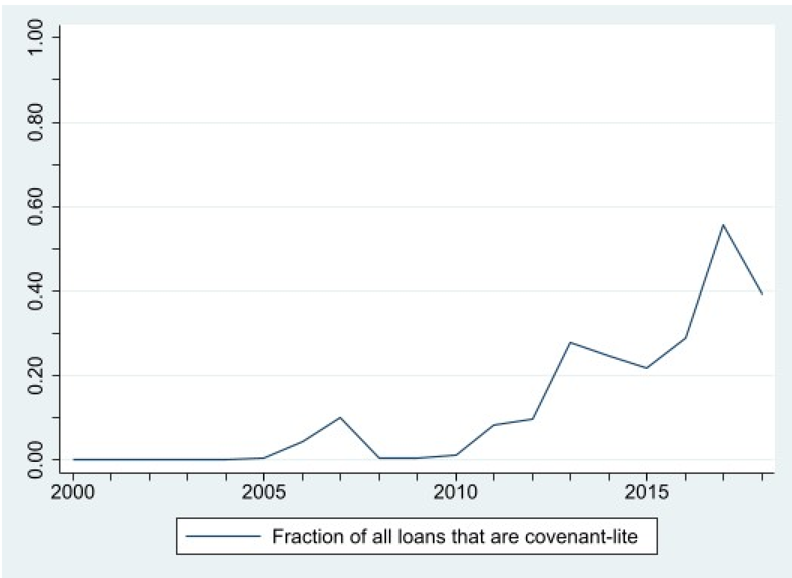
Corporate loans can be classified into two major types:

- Syndicated loans, in which multiple lenders (e.g., a group of banks or institutional investors) jointly lend to the firm. These loans are larger and often can be traded in the secondary market. Data on these loans is generally available from data vendors that track this market (e.g., Thomson Reuters Dealscan, Bloomberg, S&P Loan Pricing Corporation).
- Bilateral loans, in which a single bank lends to the firm. These loans are private contracts that are unlikely to be traded. As banks intend to keep these loans on their books (i.e., not sell them onwards to other investors or banks), data on these loans is generally unavailable.

In this Appendix, we first present our empirical findings on the frequency of financial maintenance covenants in syndicated loan contracts. Next, we discuss the current state of financial covenant use in bilateral (non-syndicated) loans.

Financial covenant use in syndicated loans We estimate the percentage frequency of covenant-lite loans over time using corporate loan data from Thomson Reuters' Dealscan database. Our sample consists of U.S. dollar-denominated syndicated loans made to US-based corporate borrowers. Each observation in our sample corresponds to a loan package, where a loan package may consist of multiple loan facilities made to the firm. We follow Beyhaghi and Ehsani (2016) and focus our analysis on term loans. We classify a package as covenant-lite when it includes loans flagged as covenant-lite in Dealscan. We merge Dealscan data with quarterly Compustat and CRSP using the linking table from Chava and Roberts (2008).

The figure below shows the fraction of covenant-lite loans among all loans in the sample by year. We note that there are no covenant-lite loans in the Dealscan database prior to 2004. Covenant-lite loans emerged around 2005, and briefly disappeared for two years following the global financial crisis (2008-2010). The proportion of covenant-lite loans has increased after 2010 and has risen to account for 40-50% of all loans towards the end of our sample period. Our data indicates that financial maintenance covenants are present in about half of the syndicated loans issued in recent years. The syndicated loan market consists of both leveraged loans with high credit risk and investment-grade loans. We find that while covenant-lite loans dominate the leveraged loan market, they are almost absent in the investment-grade loan market.



Financial covenant use in non-syndicated loans While most empirical studies focus on large syndicated loans (due in part to the availability of data), a significant proportion of corporate loans occur in the non-syndicated loan market. The majority of bank credit consists of bilateral commercial and industrial loans made by banks directly to firms (without being part of a syndicate). According to the aggregated bank disclosure available from the Federal Reserve, the total size of the commercial and industrial loans rose to more than in the $2.7 trillion in 2022.[21] In order to get a sense of the frequency of financial covenant use in this traditional but opaque market, we spoke to senior bankers involved in loan origination at large, medium sized and small banks. They noted that almost all bilateral loans issued by the banks included financial maintenance covenants.

Therefore, when we consider the loan market in its aggregate, we find that despite the noise in financial statements on which they are based, financial maintenance covenants continue to be important features for a broad set of loan contracts.

21 https://www.federalreserve.gov/releases/h8/current/.