BIFURCATION BOUNDARIES IN ECONOMIC MODELS

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OUTLINE

- Contributions and motivation of the study
- Literature review
- Definitions and Descriptions of Bifurcation boundaries
- Empirical measurement techniques
- Solution framework
- Results and Conclusion
KEY CONTRIBUTIONS TO THE LITERATURE

- Expand on bifurcation analysis of economic models
- Provide empirical assessment techniques of bifurcation boundaries in economic models
- Provide technical ways to address the issue in economic modeling
MOTIVATION

- Economic modeling has moved from comparative statics to dynamics
- Many dynamical models exhibit nonlinear dynamics
- The core of dynamics is bifurcation theory (fundamental to systems theory)
- Parameter spaces in dynamic models are stratified into subsets, each subsets supporting a different kind of dynamic solution
• Dynamic econometrics inference is produced from simulations with parameters set only at point estimates instead of various settings within the confidence region

• Bifurcation boundaries damage several qualitative properties of dynamic models

• Stratification of confidence region into bifurcated subsets seriously damages robustness of dynamic inference
• To know whether the confidence region is crossed by bifurcation boundaries, one needs to locate those boundaries

• The uncertainty surrounding parameters leads to more uncertainty about bifurcation boundaries linked to the parameter space
LITERATURE REVIEW

- First study on bifurcation boundaries was conducted by Poincare (1885, 1892) for two-dimensional vector fields.
- Andronov (1929) formulated the first theorem on Hopf bifurcation and developed important tools for analysing nonlinear dynamical systems (also for two-dimensional vector fields).
- Later on, in 1942, Hopf developed the most famous general theorem on the existence of Hopf bifurcation – generalized to $n$ dimensional vector fields.
• Studies such as Torre (1977) for Keynesian systems and Benhabib et al (1979) for multi-sectoral neoclassical optimal growth models were among the first one to find Hopf bifurcation boundaries in economic models

• Later on, bifurcation boundaries were found in overlapping generation models; see Benhabib et al (1982, 1991), Aiyagari (1989), etc

• In 1985, Grandmont expanded findings – par. space of even the simplest classical general-equilibrium models are stratified into bifurcation regions

Also, Barnett et al have expanded on the work of Grandmont bringing policy relevance to his findings; see Barnett et al (2004, 2005).

Barnett and Duzhak (2008, 2009) have investigated bifurcation boundaries within the more recent class of New Keynesian models.
DEFINITIONS AND DESCRIPTIONS

- Bifurcation refers to changes in qualitative features of solution dynamics as parameter values change.

- When the system has no bifurcations boundaries, only changes in quantitative features of dynamic solutions are observed.

- Bifurcation boundaries bring changes in qualitative features of dynamics solutions such as monotonic convergence to damped convergence to a steady state.
• Close to bifurcation boundaries, quantitative features of the system become more sensitive to parameter changes

• The use of parameter changes to assess the impact of policy shifts using the model might lead to unreliable results
EXAMPLE

Let a continuous dynamic system (Barnett et al. 2008, Banerjee et al. 2012)

\[ x = f(x), \quad x \in \mathbb{R}^n \quad \text{and} \quad x_0 \quad \text{an equilibrium of the system} \]

\[ M: \text{the Jacobian matrix } \frac{df}{dx} \text{ evaluated at } x_0 \]

Let the numbers of eigenvalues of \( A \)

- with negative real: \( n_- \)
- with zero real: \( n_0 \)
- with positive real: \( n_+ \)
Definition 1

We have an hyperbolic equilibrium when \( n_0 \).

It means no eigenvalues on the imaginary axis or unit circle.

Or, let the following two dynamical systems

\[
\begin{align*}
\dot{x} &= f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m \\
\dot{y} &= g(y, \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m
\end{align*}
\]
Definition 2

We can say that (1) is topologically equivalent to (2) if

- there is existence of a homeomorphism of the parameter space
  \[ p : \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad \beta = p(\alpha) \]
  
- there is a parameter-dependent homeomorphism of the phase space
  \[ h_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad y = h_\alpha(x) \]
  which maps the system’s orbits at parameter values
  \[ \beta = p(\alpha) \]
  while preserving the time direction
Definition 3

- Presence of topologically non-equivalent phase portrait under variation of parameters is called bifurcation.
- At the non-hyperbolic points, sufficiently small perturbations of parameters lead to changes in structural stability.
Definition 4

A transcritical bifurcation occurs when a system has non-hyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point with additional transversality conditions.
Fig. 1 – Portrait phase of saddle-node bifurcation
Fig. 2 – Stable versus unstable cycle in Hopf bifurcation

(i) stable cycle

(ii) unstable cycle
THE MARSHALLIAN MACROECONOMIC MODEL

Let the reduced form of the one sector MMM (Ngoie & Zellner, 2010)

\[
\frac{\dot{S}_t}{S_t} = b \frac{\dot{S}_t}{S_t} + C_E \left( S_t - \pi_t^e \right) + c
\]

(3)

\[
c = \frac{A_t}{A_i} + \left\{ \alpha \left[ \gamma_h (1 - \theta) + \gamma + \gamma_S \right] / (1 + \gamma) + \beta \left[ \phi_h (1 - \theta) + \phi + \phi_S \right] / (1 + \phi) - \theta_h \right\} / \delta(1 - \theta)
\]

\[
..... + \alpha \sum_{j=1}^{d} \frac{\dot{z}_j}{z_j} / \delta(1 + \gamma) + \beta \sum_{j=1}^{n} \frac{\dot{v}_j}{v_j} / \delta(1 + \phi) + \sum_{j=1}^{m} \left[ \theta_j / \delta(1 - \theta) \right] \left( \begin{array}{c} \dot{X}_j \\ X_j \end{array} \right) [\alpha(\gamma + \gamma_S) / (1 + \gamma)
\]

..... + \beta(\phi + \phi_S) / (1 + \phi) - 1]
and

\[ b = \left\{ \frac{1 - \theta_S - \alpha[(1 - \theta)(1 - \gamma_S) + (1 - \theta_S)(\gamma + \gamma_S)]}{1 + \alpha} \right\} / \delta(1 - \theta) \]

The logistic equation can be expressed as

\[ \frac{dS}{dt} = k_1S \left( 1 - \left( \frac{k_2}{k_1} \right)S \right) \]  \hspace{1cm} (4)

With \( k_1 = \frac{g - C_E \pi^e}{1 - b} \) and \( k_2 = \frac{-C_E}{1 - b} \)
(4) has two equilibrium values $S = 0$ and $S = \frac{k_1}{k_2}$

- For constant parameters there are no cyclical movements
- For discrete lags there is mixed differential-difference equation that can produce cyclical solutions
Fig. 3 – Stability of equilibrium solutions
SOLUTION FRAMEWORK

MODELS WITH ANALYTICAL SOLUTIONS

1. Construct an acceptance region

- Optimize and obtain the model’s solution either from its reduced form or any other form
- Construct an A-R (Accept Reject) algorithm to reconstruct an acceptance region, i.e. a region without bifurcation boundaries – A-R algorithm will restrict the acceptance sample to be iid
- To remove that restriction, I also construct a M-H (Metropolis Hastings) algorithm
The Accept-Reject algorithm

- The algorithm is used to generate the required parameter space only with draws that pass the set conditions.
- This algorithm allows to generate a candidate space from virtually any type of distribution.
- Using the known functional form of the density of interest $f$ (the target density) as well as the conditions of acceptance for the parameters space.
- From $f$, I use a simpler density $g$ (the candidate density) to generate rv that satisfy my parameter space.
Constraints

- Compatible support for both densities \( f \) and \( g \), i.e. \( g(x) > 0 \) when \( f(x) > 0 \) and vice versa

- I use a constant \( C \) such as \( \frac{f(x)}{g(x)} \leq C \) for all \( x \)

- \( g(x) \) contains only draws that satisfy the ‘no bifurcation boundaries’ condition
To illustrate

X can be simulated as follows:

1. I generate $Y \sim g$ and independently, I also generate $U \sim \mathcal{U}_{[0,1]}$

2. If $U \leq \frac{1}{c} \frac{f(Y)}{g(Y)}$, then I set $X=Y$

3. Should the inequality not being satisfied, the algorithm discards $Y$ and $U$ and start all over again
Algorithm representation

1. Form $g$ with draws that match the conditions
2. Generate $Y \sim g$, $U \sim \mathcal{U}_{[0,1]}$;
3. Accept $X = Y$ if $U \leq \frac{1}{c} \frac{f(Y)}{g(Y)}$
4. Return to 2 otherwise
Proof that the method works

It can be proven that the cdf of the accepted sample,

\[ P(Y \leq x | U \leq f(Y)/\{Cg(Y)\}) \]

is exactly the same as the cdf of \(X\).

\[
P \left( Y \leq x \left| U \leq \frac{f(Y)}{\{Cg(Y)\}} \right. \right) = \frac{P(Y \leq x, U \leq \frac{f(Y)}{\{Cg(Y)\}})}{P(U \leq \frac{f(Y)}{\{Cg(Y)\}})}
\]
\[
\begin{align*}
\int_{-\infty}^{x} & \frac{f(y)}{\{Cg(y)\}} \, du_g(y) \, dy \\
= & \int_{-\infty}^{\infty} \frac{f(y)}{\{Cg(y)\}} \, du_g(y) \, dy \\
= & \int_{-\infty}^{x} \left[ \frac{f(y)}{\{Cg(y)\}} \right] g(y) \, dy \\
= & \int_{-\infty}^{\infty} \left[ \frac{f(y)}{\{Cg(y)\}} \right] g(y) \, dy
\end{align*}
\]
\[ \int_{-\infty}^{x} f(y) \, dy = \frac{\int_{-\infty}^{x} f(y) \, dy}{\int_{-\infty}^{\infty} f(y) \, dy} = P(X \leq x), \]

Which proves that the cdf of the accepted sample is the same as the cdf of $X$. 
• The impact of bifurcation boundaries can therefore be assessed looking at the acceptance probability
• The higher the probability, the less concerned we should be about bifurcation boundaries
• The A-R algorithm helps derive a uniform distribution
The Metropolis-Hastings Algorithm

- M-H algorithm is far more sophisticated than the A-R algorithm as it removes the \( iid \) restriction on the candidate sample.
- \( Y_t \) are generated independently but the resulting sample is not \( iid \).
- Probability of acceptance of \( Y_t \) depends on a Markov chain \( X^{(t)} \).
- The M-H is a straightforward generalization of the A-R method.
- The M-H involves repeated occurrences of the sample value since rejection of \( Y_t \) leads to repetition of \( X^{(t)} \) at time \( t+1 \).
• A-R acceptance steps require the calculation of the upper bound
\[ C \geq \sup_x f(x)/g(x). \]

This is not required by the M-H algorithm

• The M-H requires very little knowledge about \( f \)
HOW DOES IT WORK?

- Given a target density $f$, I build a markov kernel $K$ with stationary distribution $f$ and then generate a markov chain using this kernel.

- The limiting distribution of $(X^{(t)})$ is $f$ and integrals can be approximated according to the Ergodic Theorem.

- The key issue here is therefore to construct a kernel $K$ that is associated with an arbitrary density $f$. 
Given the target density $f$, the kernel is associated with working conditional density $q(y|x)$ that is easy to simulate. Also, $q$ can be almost arbitrary in that the only theoretical requirements are that the ratio $\frac{f(y)}{q(y|x)}$ is known up to a constant independent $x$ and that $q(\bullet|x)$ has enough dispersion to lead to an exploration of the entire support of $f$.

For every given $q$, I can construct a Metropolis_Hastings kernel such that $f$ is stationary distribution.
The algorithm

The M-H algorithm associated with the objective density $f$ and the conditional density $q$ produces a markov chain through the following transition kernel:

1. Generate $Y_t \sim q(y|\textbf{x}^{(t)})$

2. Take

$$X^{(t+1)} = \begin{cases} 
Y_t \ldots \text{with probability} \ldots \cdot \rho(x^{(t)}, Y_t), \\
x^{(t)} \ldots \text{with probability} \ldots 1 - \rho(x^{(t)}, Y_t),
\end{cases}$$

where

$$\rho(x, y) = \min \left\{ \frac{f(y)q(x|y)}{f(x)q(y|x)}, 1 \right\}$$
2. BAYESIAN ESTIMATION USING PRIOR INFORMATION

- I construct an informative prior (conjugate in case of the A-R algorithm) using information obtained from the algorithms.
- In the case of M-H, the prior can be improper.
- I associate the problem to Bernoulli trial where the success is attributed to draws within the acceptance region and the failure outside of the region.
• I make use of the A-R or M-H algorithm to build an informative beta prior using Bernoulli distribution with known probability of success.

• Compute the posterior odds between the two models: (1) a model using the new prior (the one accounting for bifurcation boundaries); and (2) a model using a similar prior that does not account for bifurcation boundaries.
The usual conjugate prior associated to the Bernoulli distribution is the Beta prior obtained from a beta \( B(\alpha, \beta) \) distribution. Considering an observation \( x \) such as

\[
x \sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1} \mathbb{I}_{[0,1]}(x),
\]

From this, we can extract a family of conjugate priors as

\[
\theta(\alpha, \beta) \propto \left\{ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \right\}^\lambda x_0^\alpha y_0^\beta,
\]

with the hyperparameters \( \lambda, x_0, y_0 \)
The posterior is as follows:

\[ \theta(\alpha, \beta | x) \propto \left\{ \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right\}^{\lambda + 1} \left[ xx_0 \right]^\alpha \left[ (1 - x)y_0 \right]^\beta, \]

The problem here is that difficulty of dealing with gamma functions make it impossible to simulate directly from \( \theta(\alpha, \beta | x) \). It is required to make use of a substitute distribution \( g(\alpha, \beta) \) and integrate using Monte Carlo methods.
3. CASE OF MIXTURE REPRESENTATIONS

The mixture distribution can be represented as follows.

\[ f(x) = \int_{\mathbb{Q}} g(x|y)p(y)dy \]

with \( \mathbb{Q} \) the auxiliary continuous space and \( g \) and \( p \), two standard distributions.

In order to generate a rv \( X \) using such a representation, I first generate a variable \( Y \) from the mixing distribution and then generate \( X \) from the selected conditional distribution.

For \( y \sim p(y) \) and \( X \sim f(x|y) \), then \( X \sim f(x) \) in continuous space.
## RESULTS

### Table 1 - Acceptance rates

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Acceptance rate (%)</th>
<th>Kolmogorov Sm. test</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-R</td>
<td>68</td>
<td>0.0203</td>
</tr>
<tr>
<td>M-H</td>
<td>71</td>
<td>0.0301</td>
</tr>
</tbody>
</table>
Fig. 4 – Representation of the posterior distribution $\theta(\alpha, \beta|x)$
**Posterior odds between models**

Model 1 (M1) – Model that ignores bifurcation boundaries (no distinction between acceptance and rejection in the prior)

Model 2 (M2) – Model accounting for bifurcation boundaries (information about acceptance region included in the prior)

\[ K = \frac{p(y|X, M_1)}{p(y|X, M_2)} \times \frac{\Pr(M_1)}{\Pr(M_2)} = 0.24 \]

A posterior odd of 0.24 (1:4.2) strongly support consideration of bifurcation boundaries.
Fig. 5 – Histogram of $10^6$ rv generated from the mixture representation along the probability function
CONCLUSION

- Bifurcation boundaries remain a treat to policy guidance provided by dynamic models and reduce the confidence region.
- This study unveils tools useful to redefine the confidence region in the presence of bifurcation boundaries.
- This research provides a solution framework allowing Bayesian inferences to provide estimates that control for bifurcation boundaries.